

CALDERÓN–ZYGmund ESTIMATES IN GENERALIZED ORLICZ SPACES

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ABSTRACT. We establish the $W^{2,\varphi(\cdot)}$ -solvability of the linear elliptic equations in non-divergence form under a suitable, essentially minimal, condition of the generalized Orlicz function $\varphi(\cdot) = \varphi(x, t)$, by deriving Calderón–Zygmund type estimates. The class of generalized Orlicz spaces we consider here contains as special cases classical Lebesgue and Orlicz spaces, as well as non-standard growth cases like variable exponent and double phase growth.

1. INTRODUCTION

In this paper, we study the well-posedness of the following second order linear equations in non-divergence form with the zero Dirichlet boundary condition:

$$(1.1) \quad \begin{cases} a_{ij}D_{ij}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in generalized Orlicz spaces. Note that for the classical Lebesgue spaces, this is a starting point of the L^p -regularity theory and a crucial corollary of the L^p boundedness of the Calderón–Zygmund operators. We refer to [13, 14] for L^p -regularity results concerning the problem (1.1). Recently, these results have been extended to Orlicz spaces [6, 52] and variable exponent Lebesgue spaces [7, 8, 11]. Therefore, it is natural to consider generalized Orlicz spaces which form a unifying framework covering the aforementioned as special cases.

Our context is the following (see the next section for details): $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is bounded and open, and the coefficient matrix $\mathbf{A} = (a_{ij}) : \Omega \rightarrow \mathbb{R}^{n^2}$ is bounded and symmetric, and satisfies the uniformly ellipticity condition (1.2). For a weak Φ -function $\varphi(\cdot) = \varphi(x, t) : \Omega \times [0, \infty) \rightarrow [0, \infty)$, $\varphi(\cdot) \in \Phi_w(\Omega)$, we show that there exists a unique (strong) solution $u \in W^{2,\varphi(\cdot)}(\Omega)$ of (1.1) satisfying the Calderón–Zygmund type estimate

$$\|u\|_{W^{2,\varphi(\cdot)}(\Omega)} = \|u\|_{L^{\varphi(\cdot)}(\Omega)} + \|Du\|_{L^{\varphi(\cdot)}(\Omega)} + \|D^2u\|_{L^{\varphi(\cdot)}(\Omega)} \leq c\|f\|_{L^{\varphi(\cdot)}(\Omega)}$$

with constant $c > 0$ independent of f and u , under regularity assumptions on $\varphi(\cdot)$, Ω and \mathbf{A} which are in some sense minimal. Here, $L^{\varphi(\cdot)}(\Omega)$ is the generalized Orlicz space generated by $\varphi(\cdot)$ and (strong) solution of (1.1) means that equation (1.1) holds almost everywhere in Ω with the zero boundary on $\partial\Omega$ in the trace sense.

The generalized Orlicz spaces, also called Musielak–Orlicz spaces, have been actively studied in recent years. The basic example of a variable exponent space was introduced by Orlicz [48], and a major synthesis is due to Musielak [41]. Generalized Orlicz spaces can be naturally drawn from (classical) Orlicz spaces that are well known and have been

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studied for a long period, see for instance the monograph [50] and related references. In other words, the Orlicz space is a special case of the generalized Orlicz spaces. Indeed, if $\varphi(\cdot)$ is independent of the x variable, i.e. $\varphi(x, t) \equiv \varphi(t)$, $L^{\varphi(\cdot)}$ becomes an Orlicz space. From this observation, we can roughly understand the generalized Orlicz spaces as variable versions of Orlicz spaces with respect to the space variable x .

In the fields of partial differential equations and the calculus of variations, there have been a lot of research interest in problems with non-standard growth (e.g. [1, 2, 10]). Consider the following non-autonomous functional

$$v \in W^{1,1} \mapsto \int F(x, Dv) dx.$$

Marcellini [38, 39] has considered functionals with (p, q) -growth, that is, F satisfies $|\xi|^p - 1 \lesssim F(x, \xi) \lesssim |\xi|^q + 1$, $q > p$. Moreover, Zhikov [55, 56] has studied model problems of anisotropic materials and the so-called Lavrentiev phenomenon. In particular, in [56], he provided significant model problems, for example,

$$F(x, \xi) \approx |\xi|^{p(x)}, \quad 1 < \inf p(\cdot) \leq \sup p(\cdot) < \infty,$$

and

$$F(x, \xi) \approx |\xi|^p + a(x)|\xi|^q, \quad 1 < p \leq q < \infty, \quad a(\cdot) \geq 0.$$

For the first case, so-called the variable exponent case, the exponent of $|\xi|$ is a function of the x -variable which is usually assumed to be continuous, and it describes various phenomena, for example electrorheological fluids [51] and image restoration [12, 31], whose growths continuously change with respect to the position. The second case is so-called the double phase problems, which describe for instance composite materials or mixtures. Hence this case has a phase transition which is certainly not continuous on the borderline between constituent materials. In a series of papers, Baroni, Colombo and Mingione [3, 5, 16, 17, 18] have studied regularity properties of minimizers of these problems, see also [9, 46]. Other researchers [15, 21] have considered the variant of the double phase functional,

$$F(x, t) \approx (t - 1)_-^p + a(x)(t - 1)_-^q,$$

with $(s)_- := \max\{s, 0\}$, which is degenerate for positive values of the gradient. Furthermore, minimizers of borderline functionals like

$$F(x, t) = t^{p(x)} \log(e + t) \quad \text{or} \quad F(x, t) = t^p + a(x)t^p \log(e + t)$$

have been recently studied, see for instance [4, 9, 24, 44, 45, 47]. We would like to stress that all of these special cases are covered by the assumptions utilized in this paper.

As interest in functionals or partial differential equations with non-standard growth increase, so does the concern about their underlying spaces. We point out that all function spaces concerning the functional we mention above can be understood as generalized Orlicz spaces, see Section 2. Specially, the variable exponent Lebesgue and Sobolev spaces that underlie problems with variable exponent growth have been widely studied for the last two decades, see for instance [19, 22]. In the research of variable exponent Lebesgue spaces there is an important continuity condition on $p(\cdot)$ called log-Hölder continuity which is stronger than the plain continuity, but weaker than Hölder continuity. Under this condition, basic and important results for Lebesgue and Sobolev spaces are extended to the variable case, for instance the boundedness of the maximal operator, Sobolev embeddings, Poincaré inequalities and so on. On the other hand, little analysis and research have been devoted to Lebesgue and Sobolev spaces having double phase growth conditions since those spaces are more delicate and difficult to analyze by the phase transition.

For these reasons, concern about generalized Orlicz spaces has been also increasing, and we believe that most classical results in functional analysis can be obtained if we impose a suitable condition on the generating function $\varphi(\cdot)$. We refer to recent results [20, 23, 28, 30, 33, 34, 35, 36, 37, 42, 43, 53] for the analysis of generalized Orlicz spaces. In particular, in [30, 32] the authors have presented a natural and weaker condition on $\varphi(\cdot)$ which implies the conditions for the specific examples we mentioned above.

Next we introduce our main result. From now on, we assume that the coefficient $\mathbf{A} = (a_{ij}) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is a measurable, bounded and matrix-valued function on \mathbb{R}^n with the symmetry condition $a_{ij} = a_{ji}$. In addition, $\mathbf{A} = (a_{ij})$ is supposed to be *uniformly elliptic* with the ellipticity constant $\Lambda \geq 1$, that is,

$$(1.2) \quad \Lambda^{-1}|\xi|^2 \leq \langle \mathbf{A}(x)\xi, \xi \rangle \leq \Lambda|\xi|^2 \quad \text{for } \forall \xi \in \mathbb{R}^n \text{ and a.e. } x \in \mathbb{R}^n.$$

The following is our principal assumption on \mathbf{A} .

Definition 1.1. For $\delta, R > 0$, we say that the coefficient matrix $\mathbf{A} = (a_{ij})$ is (δ, R) -*vanishing* if

$$[\mathbf{A}]_R := \sup_{0 < r \leq R} \sup_{x \in \mathbb{R}^n} \int_{B_r(x)} |\mathbf{A}(y) - \overline{\mathbf{A}}_{B_r(x)}| \, dy \leq \delta.$$

Clearly, if \mathbf{A} is in BMO (Bounded Mean Oscillation) with $[A]_{BMO} \leq \delta$, then it is (δ, R) -vanishing for every $R > 0$, and if \mathbf{A} is in VMO (Vanishing Mean Oscillation), then for each $\delta > 0$ it is (δ, R) -vanishing for some $R > 0$.

We now state the main theorem of this paper. Notation and conditions of $\varphi(\cdot)$ will be introduced in the next section.

Theorem 1.2 (Main Theorem). *Let Ω be a bounded $C^{1,1}$ domain, and $\varphi \in \Phi_w(\Omega)$ satisfy (A0), (aInc) $_p$, (aDec) $_q^\infty$, and (A1- φ^-) for some $1 < p \leq q < \infty$. There exists a small $\delta = \delta(n, \Lambda, p, q, L, \beta, \Omega) > 0$ such that if \mathbf{A} is (δ, R) -vanishing for some $R > 0$ and $f \in L^{\varphi(\cdot)}(\Omega)$, then the problem (1.1) has a unique strong solution $u \in W^{2, \varphi(\cdot)}(\Omega) \cap W_0^{1, \varphi(\cdot)}(\Omega)$ with*

$$(1.3) \quad \|u\|_{W^{2, \varphi(\cdot)}(\Omega)} \leq c \|f\|_{L^{\varphi(\cdot)}(\Omega)},$$

for some positive constant $c = c(n, \Lambda, p, q, L, \beta, \Omega, R) > 0$.

Using the linearity of the equation (1.1), we can directly obtain the following result from the above theorem.

Corollary 1.3. *Let Ω be a bounded $C^{1,1}$ -domain, and $\varphi \in \Phi_w(\Omega)$ satisfy (A0), (aInc) $_p$, (aDec) $_q^\infty$, and (A1- φ^-) for some $1 < p \leq q < \infty$. There exists a small $\delta = \delta(n, \Lambda, p, q, L, \beta, \Omega) > 0$ such that if \mathbf{A} is (δ, R) -vanishing for some $R > 0$, $f \in L^{\varphi(\cdot)}(\Omega)$ and $g \in W^{2, \varphi(\cdot)}(\Omega)$, then the problem*

$$\begin{cases} a_{ij} D_{ij} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in W^{2, \varphi(\cdot)}(\Omega)$ with $u - g \in W_0^{1, \varphi(\cdot)}(\Omega)$, and we have the estimate

$$\|u\|_{W^{2, \varphi(\cdot)}(\Omega)} \leq c (\|f\|_{L^{\varphi(\cdot)}(\Omega)} + \|g\|_{W^{2, \varphi(\cdot)}(\Omega)}),$$

for some positive constant $c = c(n, \Lambda, p, q, L, \beta, \Omega, R) > 0$.

Finally, we would like to discuss the approach used in this paper. As a matter of fact, if we obtain a Calderón–Zygmund type result, that is the boundedness of the Calderón–Zygmund operators in the generalized Orlicz spaces, our result might follow in a natural way in the context of the harmonic analysis. However, this remains still an unsolved

problem. Hence, we do not use the classical approach of harmonic analysis tools. Instead, we use an approach based on a perturbation argument which is broadly used in the field of partial differential equations. In particular, we adopt the approach introduced by Mingione in [2] see also [40] for its origin. We also note that a similar method has been used in the variable exponent case, see [7, 11]. However, we realized that the generalized Orlicz case is not a direct generalization, and needs much delicate analysis, since the class of generalized Orlicz space considered here contains not only the variable exponent spaces but also spaces describing the double phase phenomena.

The remaining part of the paper is organized as follow. In Section 2, we introduce basic notation, generalized Orlicz spaces, our main assumption on $\varphi(\cdot)$ and basic comparison estimates in the classical Lebesgue spaces. In Section 3, we derive *a priori* localized Hessian estimates which are the main part of this paper. In Section 4, we complete the proof our main result, Theorem 1.2. Finally, we end the paper briefly sketching the Hessian estimates in the classical Orlicz spaces in Appendix A.

2. PRELIMINARIES

We recall some standard notation and definitions. For a point $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^n$, let $B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$, $B_r^+(x) := B_r(x) \cap \{x_n > 0\}$ and $B_r^-(x) := \{y' \in \mathbb{R}^{n-1} : |x' - y'| < r\}$. For the sake of simplicity, we write $B_r = B_r(0)$ and $B_r^+ := B_r^+(0)$. We also denote $T_r(x) = B_r(x) \cap \{x_n = 0\}$ and $T_r = B_r \cap \{x_n = 0\}$. For a vector-valued function $\mathbf{f} : U \rightarrow \mathbb{R}^N$, where U is a bounded domain in \mathbb{R}^n and $N \geq 1$, we denote $\bar{\mathbf{f}}_U$ by the integral average of \mathbf{f} on U , that is,

$$\bar{\mathbf{f}}_U = \int_U \mathbf{f}(x) dx = \frac{1}{|U|} \int_U \mathbf{f}(x) dx.$$

Finally, we say function f defined in \mathbb{R} is L -almost increasing (resp. decreasing) for some $L \geq 1$ if $f(s) \leq Lf(t)$ (resp. $f(t) \leq Lf(s)$) for every $s < t$.

We record a lemma which is useful when estimating generalized Orlicz functions.

Lemma 2.1. *Let $f \geq 0$ be measurable, $\varepsilon > 0$ and $0 < r < 1 < s$. There exists $c_\varepsilon > 0$ such that*

$$\int_\Omega f dx \leq \varepsilon \left(\int_\Omega f^s dx \right)^{\frac{1}{s}} + c_\varepsilon \left(\int_\Omega f^r dx \right)^{\frac{1}{r}}.$$

Proof. Let $\theta \in (0, 1)$ satisfy $1 = \frac{\theta}{s} + \frac{1-\theta}{r}$. We apply Hölder's inequality with exponent $\frac{s}{\theta}$ to $f = f^\theta f^{1-\theta}$:

$$\int_\Omega f dx \leq \left(\int_\Omega f^s dx \right)^{\frac{\theta}{s}} \left(\int_\Omega f^r dx \right)^{\frac{s-\theta}{s}}.$$

Next we use Young's inequality $ab \leq \varepsilon a^{1/\theta} + c_{\varepsilon, s, r} b^{(1/\theta)^\theta}$:

$$\int_\Omega f dx \leq \varepsilon \left(\int_\Omega f^s dx \right)^{\frac{1}{s}} + c_\varepsilon \left(\int_\Omega f^r dx \right)^{\frac{s-\theta}{s(1-\theta)}}.$$

Since $\frac{s-\theta}{s(1-\theta)} = \frac{1}{r}$, this gives the claim. □

2.1. Generalized Φ -functions. Let us consider a function $\varphi = \varphi(x, t) : \Omega \times [0, \infty) \rightarrow [0, \infty)$. Then we say that φ is a *weak Φ -function*, $\varphi \in \Phi_w(\Omega)$, if the following hold:

- φ is measurable in the x -variable, and non-decreasing and left-continuous in the t -variable.

- For every $x \in \Omega$,

$$\varphi(x, 0) = \lim_{t \downarrow 0} \varphi(x, t) = 0 \quad \text{and} \quad \lim_{t \uparrow \infty} \varphi(x, t) = \infty.$$

- The map $(0, \infty) \ni t \mapsto \frac{\varphi(x, t)}{t}$ is L -almost increasing. Here, L is independent of the x -variable.

Furthermore, we say that φ is a Φ -function, $\varphi \in \Phi(\Omega)$, if $\varphi \in \Phi_w(\Omega)$ and it is convex in the t -variable. The subset of $\Phi(\Omega)$ of functions independent of x is denoted by Φ , similarly for $\Phi_w(\Omega)$ and Φ_w .

In addition, we sometimes suppose the assumptions on φ below. Before stating them, we define

$$\varphi_U^+ := \sup_{x \in U} \varphi(x, t) \quad \text{and} \quad \varphi_U^- := \inf_{x \in U} \varphi(x, t), \quad U \subset \Omega.$$

When $U = \Omega$, we abbreviate $\varphi^\pm(t) := \varphi_\Omega^\pm(t)$.

(A0) There exists $\beta \in (0, 1)$ such that $\varphi^+(\beta) \leq 1 \leq \varphi^-(1/\beta)$.

(aInc) $_p$ There exists $p > 0$ and $L \geq 1$ such that the map $(0, \infty) \ni t \mapsto \frac{\varphi(x, t)}{t^p}$ is L -almost increasing for every $x \in \Omega$.

(aDec) $_q^\infty$ There exists $q > 0$ and $L \geq 1$ such that the map $(\beta, \infty) \ni t \mapsto \frac{\varphi(x, t)}{t^q}$ is L -almost decreasing for every $x \in \Omega$.

Note that we only need the almost decreasing condition “near infinity”, i.e. for large values of t .

We note that the three assumptions above are quite natural. For example if $\varphi(x, t) = a(x)t^{p(x)}$, (A0) condition implies $a(\cdot) \approx 1$, (aInc) $_p$ condition implies $p \leq p(\cdot)$, and (aDec) $_q^\infty$ implies $p(\cdot) \leq q$. Moreover, we deduce from conditions (A0), (aInc) $_p$ and (aDec) $_q^\infty$ that $L^{-1}\beta^p t^p \leq \varphi(x, t)$ for all $t > \beta^{-1}$ and $\varphi(x, t) \leq L\beta^{-q}t^q$ for all $t > \beta$, which imply that

$$(2.1) \quad L^{-1}(\beta^p t^p - 1) \leq \varphi(x, t) \leq L\beta^{-q}t^q + 1 \quad \text{for all } t > 0,$$

so that φ has (p, q) -growth, and that for $\alpha > 1$,

$$(2.2) \quad \alpha^p L^{-1} \varphi(x, t) \leq \varphi(x, \alpha t) \leq \alpha^q L (\varphi(x, t) + 1) \quad \text{for all } t > 0.$$

Here, if $t > \beta$ in particular, then we also have

$$(2.3) \quad \varphi(x, \alpha t) \leq \alpha^q L \varphi(x, t).$$

Two Φ -functions are equivalent $\varphi \simeq \psi$, if there exists $L \geq 1$ such that $\varphi(x, \frac{t}{L}) \leq \psi(x, t) \leq \varphi(x, Lt)$ for almost every x . The next lemma shows that a weak Φ function can be upgraded.

Lemma 2.2 (Lemma 3.1, [33]). *Let $\varphi \in \Phi_w(\Omega)$. Then there exists $\psi \in \Phi(\Omega)$ with $\varphi \simeq \psi$.*

We record one lemma which will be useful later on. Essentially, it says that (aInc) $_p$ gives a Jensen-type inequality.

Lemma 2.3. *Let $\varphi \in \Phi_w$ satisfy (aInc) $_p$, $p \geq 1$. Then there exists $c > 0$ such that*

$$\varphi \left(c \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}} \right) \leq \int_{\Omega} \varphi(|f|) dx.$$

Proof. Since φ satisfies (aInc) $_p$, the map $t \mapsto \varphi(t^{1/p})$ satisfies (aInc) $_1$. Thus by Lemma 2.2 there exists a convex function ζ such that $\zeta(\frac{t}{L}) \leq \varphi(t^{1/p}) \leq \zeta(Lt)$. By Jensen’s inequality,

$$\varphi \left(L^{-1-\frac{1}{p}} \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}} \right) \leq \zeta \left(L^{-p} \int_{\Omega} |f|^p dx \right) \leq \int_{\Omega} \zeta(L^{-p} |f|^p) dx \leq \int_{\Omega} \varphi(|f|) dx. \quad \square$$

Now, let us introduce a key assumption on $\varphi(\cdot)$ to obtain the Calderón–Zygmund estimates for $\varphi(\cdot)$, as well as the (A1) assumption from earlier papers.

(A1- φ^-) There exists $\beta \in (0, 1)$ such that for every ball $B \subset \Omega$

$$\varphi_B^+(\beta t) \leq \varphi_B^-(t) \quad \text{for all } t \in [1, (\varphi^-)^{-1}(|B|^{-1})]$$

(A1) There exists $\beta \in (0, 1)$ such that for every ball $B \subset \Omega$

$$\varphi_B^+(\beta t) \leq \varphi_B^-(t) \quad \text{for all } t \in [1, (\varphi_B^-)^{-1}(|B|^{-1})]$$

The latter condition plays a main role in the study of generalized Orlicz spaces, and the class of Φ -functions satisfying (A1) or (A1- φ^-) contains important examples (see the table below).

In this paper, we consider the (A1- φ^-) condition. Comparing the (A1) condition, we replace φ_B^- by φ^- in the interval of t , hence the (A1- φ^-) condition is stronger than the (A1) condition. However, all important examples, namely Orlicz growth, variable exponent growth and double phase growth, also satisfy this stronger condition. The next table contains an interpretation of the assumptions for four Φ -functions. Concerning (A1), we note that both C^{\log} and $C_p^{\frac{n}{p}(q-p)}$ are known to be the optimal moduli of continuity to ensure the boundedness of the maximal operator [4, 16, 33], but see also [49].

$\varphi(x, t)$	(A0)	(A1)	(A1- φ^-)	(aInc)	(aDec) $^\infty$
$t^{p(x)}a(x)$	$a \approx 1$	$p \in C^{\log}$	$p \in C^{\log}$	$p^- > 1$	$p^+ < \infty$
$t^{p(x)}\log(e+t)$	true	$p \in C^{\log}$	$p \in C^{\log}$	$p^- > 1$	$p^+ < \infty$
$t^p + a(x)t^q$	$a \in L^\infty$	$a \in C_p^{\frac{n}{p}(q-p)}$	$a \in C_p^{\frac{n}{p}(q-p)}$	$p > 1$	$q < \infty$
$t^p + a(x)t^p \log(e+t)$	$a \in L^\infty$	$a \in C^{\log}$	$a \in C^{\log}$	$p > 1$	$p < \infty$

We also point out that Gwiazda, Skrzypczak and Zatorska–Goldstein [26, 27] have recently studied existence of solutions in the Musielak–Orlicz setting with the regularity assumption

$$(2.4) \quad \varphi_B^+(t) \leq c(1 + t^{-\frac{a}{\log(b \operatorname{diam} B)}})\varphi_B^-(t)$$

for all $t > 0$ and some $a, b, c > 0$ and balls B of diameter at most δ . If (A0) holds, then it follows from concavity of $(\varphi^-)^{-1}$ that $\frac{\log(\varphi^-)^{-1}(|B|^{-1})}{\log \operatorname{diam} B} \leq c$. Thus the condition implies (A1- φ^-). Furthermore, (2.4) does not hold in the double phase situation, so (A1- φ^-) is strictly more general.

2.2. Generalized Orlicz spaces. Now, we introduce the *generalized Orlicz spaces* (also known as Musielak–Orlicz spaces). For $\varphi \in \Phi_w(\Omega)$, we define

$$L^{\varphi(\cdot)}(\Omega) := \left\{ g \in L^0(\Omega) : \lim_{\lambda \rightarrow 0} \varrho_{\varphi(\cdot)}(\lambda g) = 0 \right\},$$

where $\varrho_{\varphi(\cdot)}$ is the *modular of φ* ,

$$\varrho_{\varphi(\cdot)}(g) := \int_{\Omega} \varphi(x, |g|) \, dx,$$

with the (quasi)norm

$$\|g\|_{\varphi(\cdot)} := \|g\|_{L^{\varphi(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \varrho_{\varphi(\cdot)}\left(\frac{g}{\lambda}\right) \leq 1 \right\}.$$

If φ is independent of the x -variable, i.e., $\varphi(x, t) \equiv \varphi(t)$, we simply write $L^{\varphi}(\Omega) = L^{\varphi(\cdot)}(\Omega)$. We observe that

$$\|g\|_{\varphi(\cdot)} \leq 1 \text{ if and only if } \varrho_{\varphi(\cdot)}(g) \leq 1,$$

which is called the norm-modular *unit ball property*. Also, we have the following relation between norm and modular:

$$(2.5) \quad \min \left\{ \varrho_{\varphi(\cdot)}(g)^{\frac{1}{p}}, \varrho_{\varphi(\cdot)}(g)^{\frac{1}{q}} \right\} \leq \|g\|_{\varphi(\cdot)} \leq \max \left\{ \varrho_{\varphi(\cdot)}(g)^{\frac{1}{p}}, \varrho_{\varphi(\cdot)}(g)^{\frac{1}{q}} \right\}.$$

In addition, we define the *generalized Orlicz–Sobolev space* $W^{k,\varphi(\cdot)}(\Omega)$, $k \in \mathbb{N}$, by

$$W^{k,\varphi(\cdot)}(\Omega) := \{g \in W^{k,1}(\Omega) : \|g\|_{k,\varphi(\cdot)} < \infty\},$$

where

$$\|g\|_{k,\varphi(\cdot)} = \|g\|_{W^{k,\varphi(\cdot)}(\Omega)} := \sum_{0 \leq |\alpha| \leq k} \|D^\alpha g\|_{L^{\varphi(\cdot)}(\Omega)}.$$

As in the case of classical Sobolev spaces, $W_0^{1,\varphi(\cdot)}(\Omega)$ is taken to be the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi(\cdot)}(\Omega)$.

Let us prove a simple version of the compact embedding theorem which we need later. The proof follows [22, Section 8.4].

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $\varphi \in \Phi_w(\mathbb{R}^n)$ and suppose that (A0), (A1), (aInc) $_p$ and (aDec) $_q^\infty$ hold. Then*

$$W_0^{1,\varphi(\cdot)}(\Omega) \hookrightarrow L^{\varphi(\cdot)}(\Omega).$$

Moreover, if Ω is a $W^{1,\varphi(\cdot)}$ -extension domain, that is, there exists a bounded linear operator $E : W^{1,\varphi(\cdot)}(\Omega) \rightarrow W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ such that $Eu|_\Omega = u$, then

$$W^{1,\varphi(\cdot)}(\Omega) \hookrightarrow L^{\varphi(\cdot)}(\Omega).$$

Proof. Let $\psi(x, t) := \varphi(x, t) + t$. When $t \geq \frac{1}{\beta}$, $t \leq \varphi(1, \frac{1}{\beta})t \leq \frac{L}{\beta}\varphi(x, t)$. Thus

$$\varphi(x, t) \leq \psi(x, t) \leq (1 + \frac{L}{\beta})\varphi(x, t) + \frac{1}{\beta}\chi_\Omega(x).$$

Since Ω is bounded, it follows from [22, Theorem 2.8.1] that $L^{\varphi(\cdot)} = L^{\psi(\cdot)}$. Furthermore, a calculation shows that ψ satisfies the almost increasing condition in (aDec) $^\infty$ for all values of t , which means that ψ is doubling. Thus it suffices to establish the claims for φ doubling.

Let G be the standard mollifier and for $\varepsilon > 0$ let G_ε denote L^1 -scaling, i.e. $G_\varepsilon(t) = \varepsilon^{-n}G(\frac{t}{\varepsilon})$. For every $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^n)$ and almost every x we have (see, e.g., [22, Lemma 8.4.1])

$$|u * G_\varepsilon(x) - u(x)| \leq \varepsilon \int_0^1 |\nabla u| * G_{t\varepsilon}(x) dt.$$

Moreover, $|\nabla u| * G_{t\varepsilon} \leq cM(|\nabla u|)$ (e.g. [22, Lemma 4.6.3]). It follows from [30] that the maximal operator is bounded. Let $\psi \in \Phi(\mathbb{R}^n)$ with $\varphi \simeq \psi$. Then $\|\cdot\|_{\psi(\cdot)}$ is convex and we get by Jensen's inequality that

$$\begin{aligned} \|u * G_\varepsilon - u\|_{\varphi(\cdot)} &\approx \|u * G_\varepsilon - u\|_{\psi(\cdot)} \\ &\leq c\varepsilon \int_0^1 \| |\nabla u| * G_{t\varepsilon} \|_{\psi(\cdot)} dt \leq c\varepsilon \int_0^1 \|M(|\nabla u|)\|_{\psi(\cdot)} dt \leq c\varepsilon \|\nabla u\|_{\varphi(\cdot)}. \end{aligned}$$

Let $\{u_l\}_{l=1}^\infty$ be a weakly convergent sequence in $W_0^{1,\varphi(\cdot)}(\Omega)$. We extend the functions to \mathbb{R}^n be zero. We may assume that the weak limit is 0, and we aim to show that $u_l \rightarrow 0$ in $L^{\varphi(\cdot)}$ strongly. Denote $K := \sup_l \{\|u_l\|_{1,\varphi(\cdot)}\}$. By the previous calculation,

$$\|u_l\|_{\varphi(\cdot)} \leq \|u_l * G_\varepsilon - u_l\|_{\varphi(\cdot)} + \|u_l * G_\varepsilon\|_{\varphi(\cdot)} \leq c\varepsilon \|\nabla u_l\|_{\varphi(\cdot)} + \|u_l * G_\varepsilon\|_{\varphi(\cdot)}.$$

For fixed ε , $u_l * G_\varepsilon \rightarrow 0$ point-wise as $l \rightarrow \infty$, by definition of weak convergence. Let $\Omega_\varepsilon := \{x \in \mathbb{R}^n : d(x, \Omega) < \varepsilon\}$. By Hölder's inequality

$$|u_l * G_\varepsilon| \leq 2\|u_l\|_{\varphi(\cdot)}\|G_\varepsilon(x - \cdot)\|_{\varphi'(\cdot)} \leq c(\varepsilon, \varphi)K\chi_{\Omega_\varepsilon}.$$

Inequality (2.1) implies that $\rho_{\varphi(\cdot)}(\alpha\chi_{\Omega_\varepsilon}) < \infty$ for every $\alpha > 0$ and so it follows from dominated convergence that $\|u_l * G_\varepsilon\|_{\varphi(\cdot)} \rightarrow 0$ for fixed ε . Then our previous estimate gives

$$\limsup_{l \rightarrow \infty} \|u_l\|_{\varphi(\cdot)} \leq c\varepsilon K.$$

Thus $u_l \rightarrow 0$ follows as $\varepsilon \rightarrow 0$.

Now, we prove our second assertion. Let $U \ni \Omega$ be a bounded open set, and $\eta \in C_0^\infty(U)$ satisfy $\eta = 1$ in Ω . Then we have

$$\|(Eu)\eta\|_{L^{\varphi(\cdot)}(U)} \leq c\|u\|_{W^{1,\varphi(\cdot)}(\Omega)}$$

and

$$\|D[(Eu)\eta]\|_{L^{\varphi(\cdot)}(U)} \leq \|D(Eu)\|_{L^{\varphi(\cdot)}(U)} + c\|Eu\|_{L^{\varphi(\cdot)}(U)} \leq c\|u\|_{W^{1,\varphi(\cdot)}(\Omega)}.$$

These imply that if $\{u_l\}_{l=1}^\infty$ is weakly convergent in $W^{1,\varphi(\cdot)}(\Omega)$, then $\{(Eu_l)\eta\}_{l=1}^\infty$ is also weakly convergent in $W^{1,\varphi(\cdot)}(U)$. Therefore, applying the results above to $(Eu_l)\eta \in W_0^{1,\varphi(\cdot)}(U)$, we obtain the second assertion. \square

Remark 2.5. By standard reflection and covering arguments, we see that Lipschitz domains are $W^{1,\varphi(\cdot)}$ -extension domains, and so are $C^{1,1}$ -domains. Since Lipschitz domains are quasiconvex, it follows from [29, Lemma 4.7.3] that $\varphi \in \Phi_w(\Omega)$ can be extended to $\Phi_w(\mathbb{R}^n)$ while preserving the assumptions of the previous proposition.

2.3. Comparison estimates. We finally introduce comparison results such that for a given solution of a linear equation under such smallness assumptions, we will find a solution of a homogeneous linear equation with constant coefficient whose Hessian is sufficiently closed to the Hessian of the given one in an L^γ sense. These results were obtained in [8, Section 4] for linear parabolic equations, and we present here their elliptic counter-parts which can be obtained in the same argument. Therefore, we state them without proofs. We start with interior comparison estimates.

Lemma 2.6. *Let $\gamma > 1$ and $\mathbf{B} = (b_{ij}) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ satisfy the uniform ellipticity condition (1.2). For any $\varepsilon \in (0, 1)$, there is a small $\delta = \delta(\varepsilon, n, \Lambda, \gamma) > 0$ such that the following holds: if \mathbf{B} is $(\delta, 4)$ -vanishing and if $w \in W^{2,\gamma}(B_4)$ is a solution of*

$$b_{ij}D_{ij}w = g \quad \text{in } B_4, \quad \text{with} \quad \int_{B_4} |D^2w|^\gamma dx \leq 1 \quad \text{and} \quad \int_{B_4} |g|^\gamma dx \leq \delta,$$

then there exist a constant matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$ with $|\overline{\mathbf{B}}_{B_4} - \tilde{\mathbf{B}}| \leq \varepsilon$ and a solution $v \in W^{2,\gamma}(B_4)$ of

$$\tilde{b}_{ij}D_{ij}v = 0 \quad \text{in } B_4, \quad \text{with} \quad \int_{B_4} |D^2v|^\gamma dx \leq 1,$$

such that

$$\int_{B_1} |D^2(w - v)|^\gamma dx \leq \varepsilon \quad \text{and} \quad \|D^2v\|_{L^\infty(B_1)} \leq c$$

for some $c = c(n, \Lambda, \gamma) > 0$.

We next state comparison estimates on half balls B_4^+ . Note that we ascribe boundary values only on the flat part of the boundary, T_4 .

Lemma 2.7. *Let $\gamma > 1$ and $\mathbf{B} = (b_{ij}) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ satisfy the uniformly elliptic condition (1.2). For any $\varepsilon \in (0, 1)$, there is small $\delta = \delta(\varepsilon, n, \Lambda, \gamma) > 0$ such that if \mathbf{B} is $(\delta, 4)$ -vanishing and if $w \in W^{2,\gamma}(B_4^+)$ is a solution of*

$$\begin{cases} b_{ij}D_{ij}w = g & \text{in } B_4^+ \\ w = 0 & \text{on } T_4 \end{cases} \quad \text{with} \quad \int_{B_4^+} |D^2w|^\gamma dx \leq 1 \quad \text{and} \quad \int_{B_4^+} |g|^\gamma dx \leq \delta,$$

then there exist a constant matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$ with $|\overline{\mathbf{B}}_{B_4^+} - \tilde{\mathbf{B}}| \leq \varepsilon$ and a solution $v \in W^{2,\gamma}(B_4^+)$ of

$$\begin{cases} \tilde{b}_{ij}D_{ij}v = 0 & \text{in } B_4^+ \\ v = 0 & \text{on } T_4 \end{cases} \quad \text{with} \quad \int_{B_4^+} |D^2v|^\gamma dx \leq 1,$$

such that

$$\int_{B_1^+} |D^2(u - v)|^\gamma dx \leq \varepsilon \quad \text{and} \quad \|D^2v\|_{L^\infty(B_1^+)} \leq c$$

for some $c = c(n, \Lambda, \gamma) > 0$.

3. LOCALIZED HESSIAN ESTIMATES

In this section, we establish *a priori* interior and boundary Hessian estimates. For the boundary estimates, we consider the flat boundary and the zero boundary condition. First, we consider the classical Orlicz functions $\varphi(t) \equiv \varphi(x, t)$, and state related Hessian estimates.

Theorem 3.1. *Let $\varphi \in \Phi_w$ and satisfy $(\text{aInc})_p$ and $(\text{aDec})_q^\infty$ for some $1 < p \leq q < \infty$ with constants L , and $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ satisfy the uniformly elliptic condition (1.2). Then there exist a small $\delta = \delta(n, \Lambda, p, q, L) \in (0, 1)$ and $c = c(n, \Lambda, p, q, L) > 0$ such that for $\rho > 0$,*

- (1) *if \mathbf{A} is $(\delta, 2\rho)$ -vanishing, $f \in L^\varphi(B_{2\rho})$ and $u \in W^{2,\varphi}(B_{2\rho})$ is a solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_{2\rho},$$

then

$$\|D^2u\|_{L^\varphi(B_\rho)} \leq c \left(\|f\|_{L^\varphi(B_{4\rho})} + \rho^{-2} \|u\|_{L^p(B_{2\rho})} \right),$$

- (2) *if \mathbf{A} is $(\delta, 2\rho)$ -vanishing, $f \in L^\varphi(B_{2\rho}^+)$ and $u \in W^{2,\varphi}(B_{2\rho}^+)$ is a solution of*

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in } B_{2\rho}^+, \\ u = 0 & \text{on } T_{2\rho}, \end{cases}$$

then

$$\|D^2u\|_{L^\varphi(B_\rho^+)} \leq c \left(\|f\|_{L^\varphi(B_{2\rho}^+)} + \rho^{-2} \|u\|_{L^p(B_{2\rho}^+)} \right).$$

The Calderón–Zygmund estimates in the classical Orlicz spaces have been treated in many works, see for instance [6, 52, 54], from which one can easily deduce the previous result. However, for completeness, we shall record the sketch of its proof in Appendix A.

We next consider generalized Orlicz functions $\varphi = \varphi(x, t)$, and state the main result of this section.

Theorem 3.2. *Let $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfy (A0) , $(\text{A1-}\varphi^-)$, $(\text{aInc})_p$ and $(\text{aDec})_q^\infty$ for some $1 < p \leq q < \infty$ with constants $\beta \in (0, 1)$ and $L \geq 1$, and $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ satisfy the uniformly elliptic condition (1.2). There exist $\delta = \delta(n, \Lambda, p, q, \beta, L) > 0$ and $c = c(n, \Lambda, p, q, \beta, L) > 0$ such that for $\rho \in (0, \frac{1}{4}]$ the following hold.*

- (1) *Interior estimate:* If \mathbf{A} is $(\delta, 2\rho)$ -vanishing, $f \in L^{\varphi(\cdot)}(B_{2\rho})$ and $u \in W^{2,\varphi(\cdot)}(B_{2\rho})$ is a solution of

$$a_{ij}D_{ij}u = f \quad \text{in } B_{4\rho},$$

then

$$\|D^2u\|_{L^{\varphi(\cdot)}(B_\rho)} \leq c\rho^{n(\sqrt{p}-q)/p^{3/2}} \left(\|f\|_{L^{\varphi(\cdot)}(B_{4\rho})} + \rho^{-2}\|u\|_{L^{\varphi^-}(B_{4\rho})} \right),$$

- (2) *Boundary estimate:* If \mathbf{A} is $(\delta, 4\rho)$ -vanishing, $f \in L^{\varphi(\cdot)}(B_{4\rho}^+)$ and $u \in W^{2,\varphi(\cdot)}(B_{4\rho}^+)$ is a solution of

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in } B_{4\rho}^+, \\ u = 0 & \text{on } T_{4\rho}, \end{cases}$$

then

$$(3.1) \quad \|D^2u\|_{L^{\varphi(\cdot)}(B_\rho^+)} \leq c\rho^{n(\sqrt{p}-q)/p^{3/2}} \left(\|f\|_{L^{\varphi(\cdot)}(B_{4\rho}^+)} + \rho^{-2}\|u\|_{L^{\varphi^-}(B_{4\rho}^+)} \right).$$

We shall only prove Theorem 3.2(2). The proof of Theorem 3.2(1) is similar and in fact simpler.

Proof of Theorem 3.2(2). We divide the proof into five steps.

Step 1, Scaling: Let us consider

$$\tilde{u} = \frac{u}{\|f\|_{L^{\varphi(\cdot)}(B_{4\rho}^+)} + \rho^{-2}\|u\|_{L^{\varphi^-}(B_{4\rho}^+)}} \quad \text{and} \quad \tilde{f} = \frac{f}{\|f\|_{L^{\varphi(\cdot)}(B_{4\rho}^+)} + \rho^{-2}\|u\|_{L^{\varphi^-}(B_{4\rho}^+)}}$$

where u and f are given in Theorem 3.2(2). Then it is clear that \tilde{u} is a solution of

$$\begin{cases} a_{ij}D_{ij}\tilde{u} = \tilde{f} & \text{in } B_{4\rho}^+, \\ \tilde{u} = 0 & \text{on } T_{4\rho}, \end{cases}$$

with

$$(3.2) \quad \|\tilde{f}\|_{L^{\varphi(\cdot)}(B_{4\rho}^+)} \leq 1 \quad \text{and} \quad \|\tilde{u}\|_{L^{\varphi^-}(B_{4\rho}^+)} \leq \rho^2.$$

We will prove

$$(3.3) \quad \|D^2\tilde{u}\|_{L^{\varphi(\cdot)}(B_\rho^+)} \leq c\rho^{n(\sqrt{p}-q)/p^{3/2}},$$

which implies (3.1). For simplicity, we denote again \tilde{f} and \tilde{u} by f and u , and proceed to prove the previous inequality. In addition, in view of Theorem 3.1 and (3.2), we observe that $\|D^2u\|_{L^{\varphi^-}(B_{2\rho}^+)} \leq c$, which implies that

$$(3.4) \quad \int_{B_{2\rho}^+} \varphi^-(|D^2u|) dx \leq c.$$

Step 2, Covering via a stopping time argument: Let us denote

$$(3.5) \quad 1 < \nu := \sqrt{p} < p \leq q,$$

$$\psi(x, t) := [\varphi(x, t)]^{\frac{1}{\nu}} \quad \text{and} \quad \psi^-(t) := \psi_{B_{4\rho}^-}(t) = \inf_{x \in B_{4\rho}^-} \psi(x, t),$$

and fix any

$$1 \leq s_1 < s_2 \leq 2.$$

Furthermore, we define

$$(3.6) \quad \lambda_0 = \lambda_0(s_2) := \int_{B_{s_2\rho}^+} [\psi(x, |D^2u|) + \frac{1}{\delta}(\psi(x, |f|) + 1)] dx \geq 1,$$

where $\delta \in (0, 1)$ is a small constant which will be determined later, and super-level sets

$$(3.7) \quad E(\lambda) = E(\lambda, s_1) := \{x \in B_{s_1\rho}^+ : \psi(x, |D^2u(x)|) > \lambda\}, \quad \lambda > 0.$$

Now we fix λ with

$$(3.8) \quad \lambda \geq A\lambda_0 > 1, \quad \text{where } A := \left(\frac{240}{s_2 - s_1}\right)^n.$$

For each $y \in E(\lambda)$, we define a continuous function $G_y : (0, (s_2 - s_1)\rho] \rightarrow [0, \infty)$ by

$$G_y(\tau) := \int_{B_\tau^+(y)} \left[\psi(x, |D^2u|) + \frac{1}{\delta} \psi(x, |f|) \right] dx.$$

On one hand, since $B_\tau^+(y) \subset B_{s_2\rho}^+$ for all $0 < \tau \leq (s_2 - s_1)\rho$, (3.8) yields that for any $\tau \in \left[\frac{(s_2 - s_1)\rho}{120}, (s_2 - s_1)\rho\right]$,

$$\begin{aligned} G_y(\tau) &\leq c \left(\frac{2\rho}{\tau}\right)^n \int_{B_{s_2\rho}^+} \left[\psi(x, |D^2u|) + \frac{1}{\delta} \psi(x, |f|) \right] dx \\ &< \left(\frac{240}{s_2 - s_1}\right)^n \int_{B_{s_2\rho}^+} \left[\psi(x, |D^2u|) + \frac{1}{\delta} \psi(x, |f|) \right] dx \\ &< A\lambda_0 \leq \lambda. \end{aligned}$$

On the other hand, by Lebesgue's differentiation theorem we have

$$\lim_{\tau \downarrow 0} G_y(\tau) = \lim_{\tau \downarrow 0} \int_{B_\tau^+(y)} \left[\psi(x, |D^2u|) + \frac{1}{\delta} \psi(x, |f|) \right] dx > \lambda$$

for almost every $y \in E(\lambda)$. Therefore, we have from the continuity of G_y that for almost every $y \in E(\lambda)$, there exists $\tau_y \in \left(0, \frac{(s_2 - s_1)\rho}{120}\right)$ such that

$$(3.9) \quad G_y(\tau_y) = \lambda \quad \text{and} \quad G_y(\tau) < \lambda, \quad \text{for all } \tau \in (\tau_y, (s_2 - s_1)\rho].$$

In view of Vitali's covering lemma, one can find a disjoint family $\{B_{\tau_k}^+(y^k)\}_{k=1}^\infty$ with $y^k \in E(\lambda)$ and $\tau_k \in \left(0, \frac{(s_2 - s_1)\rho}{120}\right)$ such that

$$(3.10) \quad E(\lambda) \setminus N \subset \bigcup_{k=1}^\infty B_{5\tau_k}^+(y^k)$$

for some a Lebesgue measure zero set N .

Since $G_{y^k}(\tau_k) = \lambda$ by (3.9), we conclude that

$$\begin{aligned} |B_{\tau_k}^+(y^k)| &\leq \frac{1}{\lambda} \left(\int_{B_{\tau_k}^+(y^k) \cap \{\psi(\cdot, |D^2u|) > \frac{\lambda}{4}\}} \psi(x, |D^2u|) dx \right. \\ &\quad \left. + \int_{B_{\tau_k}^+(y^k) \cap \{\frac{\psi(\cdot, |f|)}{\delta} > \frac{\lambda}{4}\}} \frac{\psi(x, |f|)}{\delta} dx + \frac{\lambda}{2} |B_{\tau_k}^+(y^k)| \right), \end{aligned}$$

and hence, we obtain

$$(3.11) \quad |B_{\tau_k}^+(y^k)| \leq \frac{2}{\lambda} \left(\int_{B_{\tau_k}^+(y^k) \cap \{\psi(\cdot, |D^2u|) > \frac{\lambda}{4}\}} \psi(x, |D^2u|) dx \right. \\ \left. + \int_{B_{\tau_k}^+(y^k) \cap \{\frac{\psi(\cdot, |f|)}{\delta} > \frac{\lambda}{4}\}} \frac{\psi(x, |f|)}{\delta} dx \right).$$

Step 3, Comparison estimates: We use the notation and results from the previous step. For simplicity, we denote $B_k := B_{20\tau_k}(y^k)$. For $k = 1, 2, \dots$, there are two possible cases, namely

- (i) (Interior case) $B_k \subset B_{s_2\rho}^+$,
- (ii) (Boundary case) $B_k \not\subset B_{s_2\rho}^+$, which implies $B_k \cap T_{s_2\rho} \neq \emptyset$.

We first consider the case (i). From the second inequality in (3.9), we have

$$\int_{B_k} \psi(x, |D^2u|) dx \leq \lambda \quad \text{and} \quad \int_{B_k} \psi(x, |f|) dx \leq \delta\lambda.$$

Let us set

$$\varphi_k^-(t) := \inf_{x \in B_k} \varphi(x, t) \quad \text{and} \quad \varphi_k^+(t) := \sup_{x \in B_k} \varphi(x, t).$$

$$\psi_k^-(t) := \inf_{x \in B_k} \psi(x, t) \quad \text{and} \quad \psi_k^+(t) := \sup_{x \in B_k} \psi(x, t).$$

Since ψ_k^- satisfies $(\text{aInc})_\nu$, it follows from Lemma 2.3 and (2.2) that

$$\begin{aligned} \psi_k^- \left(\left(\int_{B_k} |D^2u|^\nu dx \right)^{\frac{1}{\nu}} \right) &\leq c \left(\int_{B_k} \psi_k^- (|D^2u|) dx + 1 \right) \\ &\leq c \left(\int_{B_k} \psi(x, |D^2u|) dx + 1 \right) \leq c\lambda. \end{aligned}$$

The same argument for φ^- , (2.2) and (3.4) give

$$\begin{aligned} \left(\int_{B_k} |D^2u|^\nu dx \right)^{\frac{1}{\nu}} &\leq c(\varphi^-)^{-1} \left(\int_{B_k} \varphi^- (|D^2u|) dx + 1 \right) \\ &\leq c(\varphi^-)^{-1} \left(|B_k|^{-1} \int_{B_{2\rho}^+} \varphi^- (|D^2u|) dx + 1 \right) \leq c_0(\varphi^-)^{-1} (|B_k|^{-1}) \end{aligned}$$

for some $c_0 \geq 1$. Therefore, applying the $(\text{A1-}\varphi^-)$ condition of φ and using the above results, we have

$$\begin{aligned} \varphi_k^+ \left(\left(\int_{B_k} |D^2u|^\nu dx \right)^{\frac{1}{\nu}} \right) &\leq c \left(\varphi_k^+ \left(\frac{1}{c_0} \left(\int_{B_k} |D^2u|^\nu dx \right)^{\frac{1}{\nu}} \right) + 1 \right) \\ &\leq c \left(\varphi_k^- \left(\left(\int_{B_k} |D^2u|^\nu dx \right)^{\frac{1}{\nu}} \right) + 1 \right) \\ &\leq c\lambda^\nu, \end{aligned}$$

which together with (2.2) yields that

$$\left(\int_{B_k} |D^2u|^\nu dx \right)^{\frac{1}{\nu}} \leq (\psi_k^+)^{-1}(c\lambda) \leq c_1(\psi_k^+)^{-1}(\lambda).$$

for some $c_1 = c_1(n, \Lambda, p, q, \beta, L) \geq 1$. In addition, in a similar way as above, we also have

$$\left(\int_{B_k} |f|^\nu dx \right)^{\frac{1}{\nu}} \leq c(\psi_k^+)^{-1}(\delta\lambda) \leq c_1\delta^{\frac{\nu}{q}}(\psi_k^+)^{-1}(\lambda).$$

Here, we have used (2.3) in the second inequality, and, if necessary, we choose $c_1 > 0$ sufficiently large.

Consequently, in view of Lemma 2.6 with

$$w(x) = \frac{u(y_k + 5\tau_k x)}{c_1(\psi_k^+)^{-1}(\lambda)(5\tau_k)^2} \quad \text{and} \quad g(x) = \frac{f(y_k + 5\tau_k x)}{c_1(\psi_k^+)^{-1}(\lambda)},$$

for any $\varepsilon > 0$ one can find v_k , $k = 1, 2, \dots$, such that

$$(3.12) \quad \int_{B_k} |D^2 u - D^2 v_k|^\nu dx \leq \varepsilon [c_1(\psi_k^+)^{-1}(\lambda)]^\nu$$

and

$$(3.13) \quad \|D^2 v_k\|_{L^\infty(B_k)} \leq c_2(\psi_k^+)^{-1}(\lambda)$$

for some $c_2 = c_2(n, \Lambda, p, q, \beta, L) \geq 1$, with sufficiently small $\delta = \delta(n, \Lambda, p, q, \beta, L, \varepsilon) > 0$.

Next, we examine the boundary case, i.e. case (ii). We write

$$\tilde{y}^k := (y^{k'}, 0), \quad \text{where } y^k = (y_1^k, \dots, y_n^k) = (y^{k'}, y_n^k).$$

Then, since $|y^k - \tilde{y}^k| < 20\tau_k$, $120\tau_k \leq (s_2 - s_1)\rho \leq \rho_0$ and $y^k \in B_{s_1\rho}$, we have

$$B_{5\tau_k}(y^k) \subset B_{25\tau_k}^+(\tilde{y}^k) \quad \text{and} \quad B_{100\tau_k}^+(\tilde{y}^k) \subset B_{120\tau_k}^+(y^k) \subset B_{s_2\rho}^+.$$

Therefore, by the same argument that we used in the case (i), with Lemma 2.7 instead of Lemma 2.6, we obtain

$$\left(\int_{B_{100\tau_k}^+(\tilde{y}^k)} |D^2 u|^\nu dx \right)^{\frac{1}{\nu}} \leq c_3(\psi_k^+)^{-1}(\lambda)$$

and

$$\left(\int_{B_{100\tau_k}^+(\tilde{y}^k)} |f|^\nu dx \right)^{\frac{1}{\nu}} \leq c_3 \delta^{\frac{\nu}{q}} (\psi_k^+)^{-1}(\lambda)$$

for some $c_3 = c_3(n, \Lambda, p, q, \beta, L) \geq 1$, so that one can find v_k such that

$$(3.14) \quad \begin{aligned} \int_{B_{5\tau_k}^+(y^k)} |D^2 u - D^2 v_k|^\nu dx &\leq 5^n \int_{B_{25\tau_k}^+(\tilde{y}^k)} |D^2 u - D^2 v_k|^\nu dx \\ &\leq 5^n \varepsilon [c_3(\psi_k^+)^{-1}(\lambda)]^\nu \end{aligned}$$

and

$$(3.15) \quad \|D^2 v_k\|_{L^\infty(B_{5\tau_k}^+(y^k))} \leq \|D^2 v_k\|_{L^\infty(B_{25\tau_k}^+(\tilde{y}^k))} \leq c_4(\psi_k^+)^{-1}(\lambda)$$

for some $c_4 = c_4(n, \Lambda, p, q, \beta, L) \geq 1$, with sufficiently small $\delta = \delta(n, \Lambda, p, q, \beta, L, \varepsilon) > 0$.

Step 4, Upper-level set estimates: In this step we estimate the measure of $E(K\lambda)$ defined in (3.7) when

$$(3.16) \quad K := (2 \max\{c_2, c_4\})^{\frac{q}{\nu}} L^{\frac{1}{\nu}} > 1$$

and $\lambda \geq \lambda_0$, $1 \leq s_1 < s_2 \leq 2$, and $c_2, c_4 \geq 1$ are given in (3.13) and (3.15), respectively. Recall the notation in the previous steps. Note that by (3.10),

$$E(K\lambda) \setminus N \subset E(\lambda) \setminus N \subset \bigcup_{k=1}^{\infty} B_{5\tau_k}^+(y^k).$$

Now, since $(\psi_k^+)^{-1}(t) = (\varphi_k^+)^{-1}(t^\nu)$, in view of (2.3) and (3.16), we have

$$\frac{1}{2}(\psi_k^+)^{-1}(K\lambda) = \frac{1}{2}(\varphi_k^+)^{-1}((2 \max\{c_2, c_4\})^q L \lambda^\nu) \geq \max\{c_2, c_4\}(\psi_k^+)^{-1}(\lambda).$$

Then it follows from (3.13) and (3.15) that

$$|D^2 v_k(x)| \leq \frac{1}{2}(\psi_k^+)^{-1}(K\lambda).$$

With this inequality for the last step, we have

$$\psi(x, |D^2 u(x)|) > K\lambda \Rightarrow |D^2 u(x)| > (\psi_k^+)^{-1}(K\lambda) \Rightarrow |D^2 u(x) - D^2 v_k(x)| > \frac{1}{2}(\psi_k^+)^{-1}(K\lambda)$$

for $x \in B_{5\tau_k}^+(y_k)$. Hence we conclude that

$$\begin{aligned} & |\{x \in B_{5\tau_k}^+(y^k) : \psi(x, |D^2 u(x)|) > K\lambda\}| \\ & \leq |\{x \in B_{5\tau_k}^+(y^k) : |D^2 u(x) - D^2 v_k(x)|^\nu > 2^{-\nu}[(\psi_k^+)^{-1}(K\lambda)]^\nu\}| \\ & \leq \frac{2^\nu}{[(\psi_k^+)^{-1}(K\lambda)]^\nu} \int_{B_{5\tau_k}^+(y^k)} |D^2 u - D^2 v_k|^\nu dx \\ & \leq c\varepsilon |B_{5\tau_k}^+(y^k)| \leq c\varepsilon |B_{\tau_k}^+(y^k)|, \end{aligned}$$

where the last line follows from (3.12) and (3.14). With this estimate and (3.11), we find that

(3.17)

$$\begin{aligned} |E(K\lambda)| & \leq \sum_{k=1}^{\infty} |\{x \in B_{5\tau_k}^+(y^k) : \psi(x, |D^2 u(x)|) > K\lambda\}| \\ & \leq \frac{c\varepsilon}{\lambda} \sum_{k=1}^{\infty} \left(\int_{B_{\tau_k}^+(y^k) \cap \{\psi(\cdot, |D^2 u|) > \frac{\lambda}{4}\}} \psi(x, |D^2 u|) dx \right. \\ & \quad \left. + \int_{B_{\tau_k}^+(y^k) \cap \{\frac{\psi(\cdot, |f|)}{\delta} > \frac{\lambda}{4}\}} \frac{\psi(x, |f|)}{\delta} dx \right) \\ & \leq \frac{c\varepsilon}{\lambda} \left(\int_{B_{s_2\rho}^+ \cap \{\psi(\cdot, |D^2 u|) > \frac{\lambda}{4}\}} \psi(x, |D^2 u|) dx + \int_{B_{s_2\rho}^+ \cap \{\frac{\psi(\cdot, |f|)}{\delta} > \frac{\lambda}{4}\}} \frac{\psi(x, |f|)}{\delta} dx \right), \end{aligned}$$

where we used that the integration sets are disjoint in the last step.

Step 5, Hessian estimates: In the final step, we derive the estimate (3.3). Then the proof is complete. Fix any $1 \leq s_1 < s_2 \leq 2$. Then applying (3.7), we obtain

$$\begin{aligned} (3.18) \quad \int_{B_{s_1\rho}^+} \varphi(x, |D^2 u|) dx & = \nu K^\nu \int_0^\infty \lambda^{\nu-1} |E(K\lambda)| d\lambda \\ & \leq (KA\lambda_0)^\nu |B_{s_1\rho}^+| + \nu K^\nu \int_{A\lambda_0}^\infty \lambda^{\nu-1} |E(K\lambda)| d\lambda \\ & =: I_1 + I_2. \end{aligned}$$

We first estimate I_1 . Recalling the definitions of λ_0, A, K (see (3.6), (3.8), (3.16)) and using Hölder's inequality, we have

$$\begin{aligned} (3.19) \quad I_1 & \leq \frac{c|B_{2\rho}^+|}{(s_2 - s_1)^{n\nu}} \left(\int_{B_{s_2\rho}^+} [\psi(x, |D^2 u|) + \frac{1}{\delta}\psi(x, |f|) + 1] dx \right)^\nu \\ & \leq \frac{c|B_{2\rho}^+|}{(s_2 - s_1)^{n\nu}} \left\{ \left(\int_{B_{s_2\rho}^+} \psi(x, |D^2 u|) dx \right)^\nu + \frac{1}{\delta^\nu} \int_{B_{s_2\rho}^+} \varphi(x, |f|) dx + 1 \right\}. \end{aligned}$$

We next use the Young type inequality Lemma 2.1 with $s = \nu$ and $r = \frac{\nu^2}{q}$ and the fact that $\varphi = \psi^\nu$, in order to get for $\varepsilon_1 > 0$,

$$\begin{aligned} & \frac{c|B_{2\rho}^+|}{(s_2 - s_1)^{n\nu}} \left(\int_{B_{s_2\rho}^+} \psi(x, |D^2u|) dx \right)^\nu \\ & \leq \varepsilon_1 \int_{B_{s_2\rho}^+} \varphi(x, |D^2u|) dx + \frac{c(\varepsilon_1)}{(s_2 - s_1)^d} \left(\int_{B_{s_2\rho}^+} \varphi(x, |D^2u|)^{\frac{\nu}{q}} dx \right)^{\frac{q}{\nu^2}}, \end{aligned}$$

where the constant d depends on n, ν, q . From (2.1) and (3.4) we conclude that

$$\int_{B_{s_2\rho}^+} [\varphi(x, |D^2u|)]^{\frac{\nu}{q}} dx \leq c \int_{B_{s_2\rho}^+} [|D^2u|^\nu + 1] dx \leq \frac{c}{|B_\rho^+|}.$$

Therefore, inserting this inequality and the first inequality in (3.2) into (3.19), we have

$$(3.20) \quad I_1 \leq c_5\varepsilon_1 \int_{B_{s_2\rho}^+} \varphi(x, |D^2u|) dx + c(\varepsilon_1) \frac{|B_\rho^+|^{1-\frac{q}{\nu}} + \delta^{-\nu} + 1}{(s_2 - s_1)^{d\nu}}$$

for some $c_5 = c_5(n, \Lambda, p, q, \beta, L) > 0$ and $c(\varepsilon_1) = c(n, \Lambda, p, q, \beta, L, \varepsilon_1) > 0$.

We next estimate I_2 . Applying (3.17), we have

$$\begin{aligned} I_2 & \leq c\varepsilon\nu K^\nu \left(\int_0^\infty \lambda^{\nu-2} \int_{B_{s_2\rho}^+ \cap \{\psi(x, |D^2u|) > \frac{\lambda}{4}\}} \psi(x, |D^2u|) dx d\lambda \right. \\ & \quad \left. + \int_0^\infty \lambda^{\nu-2} \int_{B_{s_2\rho}^+ \cap \{\frac{\psi(x, |f|)}{\delta} > \frac{\lambda}{4}\}} \frac{\psi(x, |f|)}{\delta} dx d\lambda \right) \\ & \leq c\varepsilon \left(\int_{B_{s_2\rho}^+} \varphi(x, |D^2u|) dx + \int_{B_{s_2\rho}^+} \frac{\varphi(x, |f|)}{\delta^\nu} dx \right). \end{aligned}$$

Here we have used the identity

$$\int_U |h|^\alpha dx = (\alpha - 1) \int_0^\infty \lambda^{\alpha-2} \int_{U \cap \{|h| > \lambda\}} |h| dx d\lambda \quad \text{for } \alpha > 1,$$

which easily follows from Fubini's theorem. Therefore by the first inequality in (3.2), we have

$$(3.21) \quad I_2 \leq c_6\varepsilon \int_{B_{s_2\rho}^+} \varphi(x, |D^2u|) dx + \frac{c}{\delta^\nu}$$

for some $c_6 = c_6(n, \Lambda, p, q, \beta, L) > 0$. Inserting (3.20) and (3.21) into (3.18), we obtain

$$\int_{B_{s_1\rho}^+} \varphi(x, |D^2u|) dx \leq (c_5\varepsilon_1 + c_6\varepsilon) \int_{B_{s_2\rho}^+} \varphi(x, |D^2u|) dx + c(\varepsilon_1) \frac{|B_\rho^+|^{1-\frac{q}{\nu}} + \delta^{-\nu} + 1}{(s_2 - s_1)^{d\nu}} + \frac{c}{\delta^\nu}.$$

At this stage, we choose $\varepsilon, \varepsilon_1 \in (0, 1)$, depending on $n, \Lambda, p, q, \beta, L$, so small that

$$c_5\varepsilon_1 + c_6\varepsilon \leq \frac{1}{2},$$

and hence $\delta = \delta(n, \Lambda, p, q, \beta, L)$ is also determined. Then we have

$$\int_{B_{s_1\rho}^+} \varphi(x, |D^2u|) dx \leq \frac{1}{2} \int_{B_{s_2\rho}^+} \varphi(x, |D^2u|) dx + \frac{c\rho^{n(1-\frac{q}{\nu})}}{(s_2 - s_1)^{d\nu}} + c.$$

Here, c is independent of the choices of $1 \leq s_1 < s_2 \leq 2$. Finally, applying Lemma 3.3 below to $h(t) = \int_{B_{t\rho}^+} \varphi(x, |D^2u|) dx$, $(a, b, \theta) = (1, 2, \frac{1}{2})$ and $(s_1, s_2) = (1, 2)$ in (3.22) and using $\rho \leq 1$, we obtain

$$\int_{B_\rho^+} \varphi(x, |D^2u|) dx \leq c\rho^{n(1-\frac{q}{\nu})},$$

which together with (2.5) and (3.5) yield (3.3). \square

Lemma 3.3 (e.g., Lemma 6.1, p. 191, [25]). *Let $h : [a, b] \rightarrow \mathbb{R}$ be a bounded nonnegative function and suppose that for any s_1, s_2 with $0 < a \leq s_1 < s_2 \leq b$,*

$$h(s_1) \leq \theta h(s_2) + \frac{\sigma_1}{(s_2 - s_1)^\beta} + \sigma_2,$$

where $\sigma_1, \sigma_2 \geq 0, \beta > 0$ and $0 \leq \theta < 1$. Then we have

$$(3.22) \quad h(s_1) \leq c \left(\frac{\sigma_1}{(s_2 - s_1)^\beta} + \sigma_2 \right)$$

for some constant $c = c(\beta, \theta) > 0$.

4. GLOBAL ESTIMATES: PROOF OF THEOREM 1.2

We now prove our main result, Theorem 1.2. Hereafter, we denote by c a universal constant depending only on $n, \Lambda, p, q, L, \beta, R$ and Ω .

Proof of Theorem 1.2. We first suppose that

$$u \in W^{2, \varphi(\cdot)}(\Omega)$$

is a solution of (1.1), and then prove the estimate (1.3). Note that the existence and the uniqueness of the solution of (1.1) in $W^{2, \varphi(\cdot)}$ will be shown later. To prove the estimate (1.3), we use standard covering and flattening method, and adopt the results derived in the previous section.

Fix any point $x^0 \in \partial\Omega$. For convenience, we shall assume $x^0 = 0$. Since Ω is a $C^{1,1}$ domain, one can find $r > 0$ and $C^{1,1}$ -function $\mu = \mu(x') : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, after rotating of the coordinate system, such that $D_{x'}\mu(0) = 0$, $\|D_{x'}^2\mu\|_{L^\infty(\mathbb{R}^{n-1})} < \infty$, and

$$(4.1) \quad \Omega \cap B_r(0) = \{x \in \Omega : x_n > \mu(x')\} \cap B_r(0).$$

Note that (4.1) holds for all $\tilde{r} < r$ instead of r , hence (4.1) is satisfied for any sufficiently small r with $r \leq R$. We next flatten out the boundary near the origin by the change of coordinates. Consider the invertible map $\zeta = (\zeta^1, \dots, \zeta^n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\zeta^i(x) := x_i - \delta_{in}\mu(x')$$

and its inverse $\theta := \zeta^{-1}$, where δ_{ij} is the Kronecker delta. Then we see that

$$D\zeta(x) = (D_j\zeta^i(x)) \quad \text{with} \quad D_j\zeta^i(x) = \begin{cases} \delta_{ij} & 1 \leq i \leq n-1, 1 \leq j \leq n, \\ -D_j\mu(x') & i = n, 1 \leq j \leq n-1, \\ 1 & i = j = n \end{cases}$$

and

$$D\theta(y) = D\theta(\zeta(x)) = [D\zeta(x)]^{-1} \quad \text{with} \quad D_j\theta^i(y) = \begin{cases} \delta_{ij} & 1 \leq i \leq n-1, 1 \leq j \leq n, \\ D_j\mu(x') & i = n, 1 \leq j \leq n-1, \\ 1 & i = j = n. \end{cases}$$

Notice that $\det(D\zeta) = \det(D\theta) = 1$.

Furthermore, set

$$\tilde{\varphi}(y, t) := \varphi(\theta(y), t).$$

Then $\tilde{\varphi}$ satisfies (A0), (aInc) $_p$ and (aDec) $_q^\infty$. Let us now show that $\tilde{\varphi}$ satisfies the (A1- φ^-) condition, where the relevant constant $\beta \in (0, 1)$ might be changed depending on the structure constants of φ and $\|D\theta\|_{L^\infty}$. We first observe that $\tilde{\varphi}^- = \varphi^-$. Consider any ball $B = B_r(y^0)$ and any $t \in [1, (\tilde{\varphi}^-)^{-1}(|B|^{-1})]$. Set $B' := B_{r'}(\theta(y^0))$ with $r' = \sup\{|\theta(y) - \theta(y^0)| : y \in B\}$. Then we see that $r' \leq \|D\theta\|_{L^\infty} r$ and $\theta(B) \subset B'$, hence

$$\tilde{\varphi}_B^+(t) = \sup_{y \in B} \varphi(\theta(y), t) \leq \varphi_{B'}^+(t) \quad \text{and} \quad \tilde{\varphi}_B^-(t) = \inf_{y \in B} \varphi(\theta(y), t) \geq \varphi_{B'}^-(t).$$

If $\|D\theta\|_{L^\infty} \leq 1$, then $(\tilde{\varphi}^-)^{-1}(|B|^{-1}) \leq (\varphi^-)^{-1}(|B'|^{-1})$ and so, by the (A1- φ^-) condition of φ , we have

$$\tilde{\varphi}_B^+(\beta t) \leq \varphi_{B'}^+(\beta t) \leq \varphi_{B'}^-(t) \leq \tilde{\varphi}_B^-(t).$$

On the other hand, if $\|D\theta\|_{L^\infty} > 1$, then applying (2.3) we have

$$\varphi^- \left([L\|D\theta\|_{L^\infty}^n]^{-\frac{1}{q}} t \right) \leq \|D\theta\|_{L^\infty}^{-n} \varphi^-(t) = \|D\theta\|_{L^\infty}^{-n} \tilde{\varphi}^-(t) \leq \|D\theta\|_{L^\infty}^{-n} |B|^{-1} \leq |B'|^{-1}$$

and so, by the (A1- φ^-) condition of φ ,

$$\begin{aligned} \tilde{\varphi}_B^+ \left(\beta [L\|D\theta\|_{L^\infty}^n]^{-\frac{1}{q}} t \right) &\leq \varphi_{B'}^+ \left(\beta \max \left\{ [L\|D\theta\|_{L^\infty}^n]^{-\frac{1}{q}} t, 1 \right\} \right) \\ &\leq \varphi_{B'}^- \left(\max \left\{ [L\|D\theta\|_{L^\infty}^n]^{-\frac{1}{q}} t, 1 \right\} \right) \\ &\leq \tilde{\varphi}_B^-(\max\{t, 1\}) = \tilde{\varphi}_B^-(t). \end{aligned}$$

We next consider $\rho \leq \rho_0 = \rho_0(r, \mu)$ such that $4\rho_0 \leq 1$, $4\rho_0\|D\theta\|_{L^\infty} \leq R$ and $B_{4\rho_0}^+ \subset \zeta(\Omega \cap B_r(0))$. Denoting

$$\begin{aligned} \tilde{u}(y) &:= u(\theta(y)), \quad \tilde{f}(y) := f(\theta(y)) - a_{ij}(\theta(y)) \zeta_{x_i x_j}^l(\theta(y)) D_{y_l} \tilde{u}, \\ (\tilde{a}_{lm}(y)) &= \tilde{\mathbf{A}}(y) := [D\zeta(\theta(y))] \mathbf{A}(\theta(y)) [D\zeta(\theta(y))]^t, \end{aligned}$$

we see that \tilde{u} is in $W^{2, \tilde{\varphi}(\cdot)}(B_{4\rho}^+)$ and a solution of

$$\begin{cases} \tilde{a}_{lm} D_{y_l y_m} \tilde{u} = \tilde{f} & \text{in } B_{4\rho}^+, \\ \tilde{u} = 0 & \text{on } T_{4\rho}, \end{cases}$$

Clearly, $\tilde{\mathbf{A}}$ is uniformly elliptic with the ellipticity constant Λ . Moreover, for $B_r(y_0)$ with $x_0 = \theta(y_0)$ and $r' = \sup_{y \in B_r(x_0)} |\theta(y) - \theta(y_0)| \leq \|D\theta\|_{L^\infty} r$,

$$\begin{aligned} &\int_{B_r(y_0)} \left| \tilde{\mathbf{A}}(y) - (\tilde{\mathbf{A}})_{B_r(y_0)} \right| dy \\ &\leq 2 \int_{B_r(y_0)} \left| [D\zeta(\theta(y))] \mathbf{A}(\theta(y)) [D\zeta(\theta(y))]^t - [D\zeta(\theta(y))] (\mathbf{A})_{B_r(x_0)} [D\zeta(\theta(y))]^t \right| dy \\ &\quad + \int_{B_r(y_0)} \int_{B_r(y_0)} \left| [D\zeta(\theta(y)) - D\zeta(\theta(\tilde{y}))] (\mathbf{A})_{B_r(x_0)} [D\zeta(\theta(y))]^t \right| dy d\tilde{y} \\ &\quad + \int_{B_r(y_0)} \int_{B_r(y_0)} \left| [D\zeta(\theta(\tilde{y}))] (\mathbf{A})_{B_r(x_0)} [D\zeta(\theta(y)) - D\zeta(\theta(\tilde{y}))]^t \right| dy d\tilde{y} \\ &\leq c(\|D\mu\|_{L^\infty}) \frac{|B_{r'}|}{|B_r|} \int_{B_r(x_0)} \left| \mathbf{A}(x) - (\mathbf{A})_{B_r(x_0)} \right| dy + c(n, \Lambda, \|\mu\|_{C^{1,1}}) r \\ &\leq c(n, \Lambda, \|\mu\|_{C^{1,1}}) (\delta + r), \end{aligned}$$

and

$$\|\tilde{f}\|_{L^{\tilde{\varphi}(\cdot)}(B_{4\rho}^+)} \leq c(n, \Lambda, \|\mu\|_{C^{1,1}}) \left(\|f(\theta(y))\|_{L^{\tilde{\varphi}(\cdot)}(B_{4\rho}^+)} + \|D\tilde{u}\|_{L^{\tilde{\varphi}(\cdot)}(B_{4\rho}^+)} \right).$$

Hence, in view of Theorem 3.2 (2), we find that

$$\begin{aligned} \|D^2\tilde{u}\|_{L^{\tilde{\varphi}(\cdot)}(B_{4\rho}^+)} &\leq c \left(\|\tilde{f}\|_{L^{\tilde{\varphi}(\cdot)}(B_{4\rho}^+)} + \|\tilde{u}\|_{L^{\tilde{\varphi}^-}(B_{4\rho}^+)} \right) \\ &\leq c \left(\|f(\theta(y))\|_{L^{\tilde{\varphi}(\cdot)}(B_{4\rho}^+)} + \|D\tilde{u}\|_{L^{\tilde{\varphi}(\cdot)}(B_{4\rho}^+)} + \|\tilde{u}\|_{L^{\tilde{\varphi}^-}(B_{4\rho}^+)} \right), \end{aligned}$$

by taking $r = r(n, \Lambda, p, q, \beta, L, \mu) > 0$ and $\delta = \delta(n, \Lambda, p, q, \beta, L, \mu) > 0$ sufficiently small. Consequently, returning to the original coordinate system and removing our temporally assumption that $x^0 = 0$, we obtain

$$(4.2) \quad \begin{aligned} \|D^2u\|_{L^{\varphi(\cdot)}(V_{x^0})} &\leq c \left(\|f\|_{L^{\varphi(\cdot)}(U_{x^0})} + \|Du\|_{L^{\varphi(\cdot)}(U_{x^0})} + \|u\|_{L^{\varphi^-}(U_{x^0})} \right) \\ &\leq c \left(\|f\|_{L^{\varphi(\cdot)}(\Omega)} + \|Du\|_{L^{\varphi(\cdot)}(\Omega)} + \|u\|_{L^{\varphi^-}(\Omega)} \right), \end{aligned}$$

where $V_{x^0} := \theta(B_\rho^+)$ and $U_{x^0} := \theta(B_{4\rho}^+)$.

Since $\partial\Omega$ is compact, it can be covered by a finite number of sets $V_{x^1}, V_{x^2}, \dots, V_{x^N}$ for some points $x^j \in \partial\Omega$, $j = 1, 2, \dots, N$, as above. In addition, in view of the interior estimate: Theorem 3.2(1), we infer that

$$(4.3) \quad \|D^2u\|_{L^{\varphi(\cdot)}(V)} \leq c \left(\|f\|_{L^{\varphi(\cdot)}(\Omega)} + \|u\|_{L^{\varphi^-}(\Omega)} \right)$$

for some open set $V \subset\subset \Omega$ satisfying $\Omega = V \cup \left(\bigcup_{j=1}^N V_{x^j} \right)$. By combining the estimates (4.2), when $x^0 = x^1, \dots, x^N$, with (4.3), we conclude that

$$\|D^2u\|_{L^{\varphi(\cdot)}(\Omega)} \leq c \left(\|f\|_{L^{\varphi(\cdot)}(\Omega)} + \|Du\|_{L^{\varphi(\cdot)}(\Omega)} + \|u\|_{L^{\varphi^-}(\Omega)} \right),$$

from which we have

$$(4.4) \quad \|u\|_{W^{2,\varphi(\cdot)}(\Omega)} \leq c \left(\|f\|_{L^{\varphi(\cdot)}(\Omega)} + \|u\|_{W^{1,\varphi(\cdot)}(\Omega)} \right).$$

We claim that $\|u\|_{W^{1,\varphi(\cdot)}(\Omega)}$ can be dropped from the right-hand side in the previous estimate, which would give (1.3). To prove this, we argue by contradiction. Suppose that (1.3) is not true. Then there exist sequences $\{\varphi_l\}_{l=1}^\infty$ satisfying (A0), (aInc) $_p$, (aDec) $_q^\infty$, and (A1- φ^-) with the same parameters (p, q, L and β) for all l , $\{u_l\}_{l=1}^\infty$ and $\{f_l\}_{l=1}^\infty$ such that u_l is a solution of

$$(4.5) \quad \begin{cases} a_{ij}D_{ij}u_l = f_l & \text{in } \Omega, \\ u_l = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$(4.6) \quad \|u_l\|_{W^{2,\varphi_l(\cdot)}(\Omega)} > l \|f_l\|_{L^{\varphi_l(\cdot)}(\Omega)},$$

for every $l \geq 1$. Without loss of generality, we may assume $\|u_l\|_{W^{2,\varphi_l(\cdot)}(\Omega)} = 1$, and then (4.6) implies

$$\|f_l\|_{L^{\varphi_l(\cdot)}(\Omega)} < \frac{1}{l}.$$

By [29, Lemma 3.7.7], $L^{\varphi_l(\cdot)}(\Omega) \hookrightarrow L^p(\Omega)$ with embedding constant independent of l . Thus $\{u_l\}_{l=1}^\infty$ is bounded in $W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$, there exist a subsequence, which we still denote by $\{u_l\}_{l=1}^\infty$, and a function $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\begin{cases} u_l \rightharpoonup u_0 \text{ (weakly)} & \text{in } W^{2,p}(\Omega) \\ u_l \rightarrow u_0 \text{ (strongly)} & \text{in } W^{1,p}(\Omega) \end{cases} \quad \text{as } l \rightarrow \infty.$$

Then, in view of the classical Calderón–Zygmund estimates for the equations (4.5), we see that

$$\|u_0\|_{W^{2,p}(\Omega)} \leq c \liminf_{l \rightarrow 0} \|f_l\|_{L^p(\Omega)} = 0,$$

which means that $u_0 = 0$, and hence, $\varphi_l(\cdot, |u_l|), \varphi_l(\cdot, |Du_l|) \rightarrow 0$ a.e. in Ω (up to subsequence). By [29, Proposition 6.3.10], there exists c depending on $\|u_l\|_{W^{1,\varphi_l(\cdot)}(\Omega)}$ such that

$$\int_{\Omega} \varphi(x, |u_l - (u_l)_{\Omega}|)^{n'} dx \leq c \quad \text{and} \quad \int_{\Omega} \varphi(x, |D_j u_l - (D_j u_l)_{\Omega}|)^{n'} dx \leq c.$$

Therefore, $\{\varphi_l(\cdot, |u_l|)\}_{l=1}^{\infty}$ and $\{\varphi_l(\cdot, |Du_l|)\}_{l=1}^{\infty}$ are equi-integrable. In view of Vitali's convergence theorem, we have

$$\lim_{l \rightarrow \infty} \int_{\Omega} \varphi_l(x, |u_l|) dx = \lim_{l \rightarrow \infty} \int_{\Omega} \varphi_l(x, |Du_l|) dx = 0,$$

and so $\lim_{l \rightarrow \infty} \|u_l\|_{W^{1,\varphi_l(\cdot)}(\Omega)} = 0$. However, it follows from (4.4) that

$$1 \leq c(\|f_l\|_{L^{\varphi_l(\cdot)}(\Omega)} + \|u_l\|_{W^{1,\varphi_l(\cdot)}(\Omega)}) \rightarrow 0 \quad \text{as } l \rightarrow \infty,$$

which is a contradiction. Therefore, the counter-assumption was false and we have established (1.3).

It remains to prove the existence and the uniqueness of the $W^{2,\varphi(\cdot)}$ -solution of (1.1). We first show existence. Let $\{\mathbf{A}^l\}_{l=1}^{\infty} = \{(a_{ij}^l)\}_{l=1}^{\infty}$ be a sequence of smooth and uniformly elliptic (with ellipticity constant 2Λ) matrix functions such that

$$(4.7) \quad a_{ij}^l \rightarrow a_{ij} \quad \text{in } L^t(\Omega) \quad \text{for each } 1 < t < \infty,$$

and $\{f_l\}_{l=1}^{\infty}$ be a sequence of smooth functions in $C_0^{\infty}(\Omega)$ satisfying

$$f_l \rightarrow f \quad \text{in } L^{\varphi(\cdot)}(\Omega) \quad \text{and} \quad \|f_l\|_{L^{\varphi(\cdot)}(\Omega)} \leq \|f\|_{L^{\varphi(\cdot)}(\Omega)} + 1.$$

By the classical $W^{2,p}$ regularity theory, see for instance [14], there exists a unique solution $u_l \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ of

$$\begin{cases} a_{ij}^l D_{ij} u_l = f_l & \text{in } \Omega, \\ u_l = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $u_l \in W^{2,\varphi(\cdot)}(\Omega) \cap W_0^{1,\varphi(\cdot)}(\Omega)$, by the *a priori* estimate (1.3) we have

$$\|u_l\|_{W^{2,\varphi(\cdot)}(\Omega)} \leq c \|f_l\|_{L^{\varphi(\cdot)}(\Omega)} \leq c(\|f\|_{L^{\varphi(\cdot)}(\Omega)} + 1).$$

Therefore, $\{u_l\}_{l=1}^{\infty}$ is bounded in $W^{2,\varphi(\cdot)}(\Omega)$, and there exist a subsequence, which is still denoted by $\{u_l\}_{l=1}^{\infty}$, and a function $u \in W^{2,\varphi(\cdot)}(\Omega)$ such that

$$(4.8) \quad u_l \rightharpoonup u \quad \text{weakly in } W^{2,\varphi(\cdot)}(\Omega).$$

In view of (4.7)–(4.8), we easily check that $u \in W^{2,\varphi(\cdot)}(\Omega)$ satisfies (1.1).

The uniqueness is a direct consequence of the linearity of our equation (1.1) and the uniqueness of the solution of

$$\begin{cases} a_{ij} D_{ij} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in $W^{2,p}(\Omega)$ space. Consequently, we have completed the proof of Theorem 1.2. \square

APPENDIX A. PROOF OF THEOREM 3.1

We present the proof of Theorem 3.1. Since the proof is similar to the one of Theorem 3.2, we shall only show (1) of Theorem 3.1 and omit repeated computations in the proof.

Proof of (1) of Theorem 3.1.

As in Section 3, it suffices to show that

$$(A.1) \quad \int_{B_\rho} \varphi(|Du|) dx \leq c$$

for some $c = c(n, \Lambda, p, q, L) > 0$, under the assumptions that

$$(A.2) \quad \|f\|_{L^\varphi(B_{4\rho})} \leq 1 \quad \text{and} \quad \|u\|_{L^\varphi(B_{4\rho})} \leq \rho^2.$$

We shall denote $\nu = \sqrt{p}$ and $\psi(t) := \varphi(t^{1/\nu})$. Then we see that ψ satisfies (aInc) $_\nu$ and (aDec) $_{q/\nu}^\infty$. Here, in view of Lemma 2.2, we may assume without loss of generality that ψ is convex.

Fix any $1 \leq s_1 < s_2 \leq 2$, and define

$$(A.3) \quad \lambda_0 := \int_{B_{2\rho}} \left[|D^2u|^\nu + \frac{1}{\delta} |f|^\nu \right] dx + 1,$$

where $\delta \in (0, 1)$ will be determined later. Using Vitali's covering lemma and Lebesgue differentiation theorem, for each

$$(A.4) \quad \lambda > A\lambda_0, \quad \text{where } A := \left(\frac{40}{s_2 - s_1} \right)^n.$$

there exists the family of mutually disjoint balls $B_i := B_{r_i}(x_i)$ with $B_{20r_i}(x_i) \subset B_{s_2\rho}$, $i = 1, 2, \dots$, such that

$$(A.5) \quad \{x \in B_{s_1\rho} : |D^2u|^\nu > \lambda\} \setminus N \subset \bigcup_{i=1}^{\infty} 5B_i$$

for some a Lebesgue measure zero set N ,

$$(A.6) \quad \int_{B_i} \left[|D^2u|^\nu + \frac{1}{\delta} |f|^\nu \right] dx = \lambda$$

and

$$(A.7) \quad \int_{20B_i} \left[|D^2u|^\nu + \frac{1}{\delta} |f|^\nu \right] dx \leq \lambda,$$

where we define $sB_i := B_{sr_i}(x_i)$. We notice that (A.6) implies

$$|B_i| \leq \frac{2}{\lambda} \left(\int_{B_i \cap \{|D^2u|^\nu > \frac{\lambda}{4}\}} |D^2u|^\nu dx + \int_{B_i \cap \{\frac{|f|^\nu}{\delta} > \frac{\lambda}{4}\}} \frac{|f|^\nu}{\delta} dx \right)$$

and (A.7) implies

$$\int_{20B_i} |D^2u|^\nu \leq \lambda \quad \text{and} \quad \int_{20B_i} |f|^\nu dx \leq \delta\lambda.$$

Therefore, for any $\varepsilon \in (0, 1)$, there exist a small $\delta(n, \Lambda, p, \varepsilon) \in (0, 1)$ and $v_i \in C^{1,1}(5B_i)$ such that

$$\int_{5B_i} |D^2u - D^2v_i|^\nu dx \leq \varepsilon\lambda \quad \text{and} \quad \|D^2v_i\|_{L^\nu(5B_i)}^\nu \leq c_7\lambda.$$

for some $c_7 = c_7(n, \Lambda, p) \geq 1$.

Let $K := 2^{\nu-1}c_7$. Then from (A.5) we see that for $\lambda > \frac{A}{(s_2-s_1)^n}\lambda_0$,

$$\{x \in B_{s_1\rho} : |D^2u|^\nu > K\lambda\} \setminus N \subset \bigcup_{i=1}^{\infty} 5B_i.$$

Applying the previous results, we see that

$$\begin{aligned} & |\{x \in B_{s_1\rho} : |D^2u|^\nu > K\lambda\}| \\ & \leq \sum_{i=1}^{\infty} (|\{x \in 5B_i : |g - D^2v_i|^\nu > c_7\lambda\}| + |\{x \in 5B_i : |D^2v_i|^\nu > c_7\lambda\}|) \\ & \leq \sum_{i=1}^{\infty} \frac{5^n\varepsilon}{c_7}|B_i| \leq \frac{c\varepsilon}{\lambda} \left(\int_{B_{s_2\rho} \cap \{|D^2u|^\nu > \frac{\lambda}{4}\}} |D^2u|^\nu dx + \int_{B_{s_2\rho} \cap \{|\frac{f|^\nu}{\delta} > \frac{\lambda}{4}\}} \frac{|f|^\nu}{\delta} dx \right), \end{aligned}$$

which implies

$$\begin{aligned} \int_{B_{s_1\rho}} \varphi(|Du|^2) dx &= \int_0^\infty |\{x \in B_{s_1\rho} : |D^2u|^\nu > \lambda\}| d[\psi(\lambda)] \\ &= |B_{2\rho}| \int_0^{KA\lambda_0} 1 d[\psi(\lambda)] + \int_{KA\lambda_0}^\infty |\{x \in B_{s_1\rho} : |D^2u|^\nu > \lambda\}| d[\psi(\lambda)] \\ &\leq c\psi(KA\lambda_0)|B_\rho| + \int_{A\lambda_0}^\infty |\{x \in B_{s_1\rho} : |D^2u|^\nu > K\lambda\}| d[\psi(K\lambda)] \\ &\leq c\psi(A\lambda_0)|B_\rho| + c\varepsilon \int_0^\infty \int_{B_{s_2\rho} \cap \{|D^2u|^\nu > \frac{\lambda}{4}\}} \frac{|D^2u|^\nu}{\lambda} dx d[\psi(K\lambda)] \\ &\quad + c\varepsilon \int_0^\infty \int_{B_{s_2\rho} \cap \{|\frac{f|^\nu}{\delta} > \frac{\lambda}{4}\}} \frac{|f|^\nu}{\delta\lambda} dx d[\psi(K\lambda)] \\ &=: I_3 + I_4 + I_5. \end{aligned}$$

Now, we estimate I_3 , I_4 and I_5 .

Estimation of I_3 : Recalling the definitions of λ_0 and A , see (A.3) and (A.4), and using Jensen's inequality and (A.2), we have

$$\begin{aligned} I_3 &\leq \frac{c|B_\rho|}{(s_2-s_1)^{qn/\nu}} \psi \left(\int_{B_{2\rho}} [|D^2u|^\nu + \frac{1}{\delta}|f|^\nu] dx \right) \\ &\leq \frac{c|B_\rho|}{(s_2-s_1)^{qn/\nu}} \psi \left(\int_{B_{4\rho}} \left[\left(\frac{|u|}{\rho^2} \right)^\nu + \left(1 + \frac{1}{\delta}\right) |f|^\nu \right] dx \right) \\ &\leq \frac{c(\delta)}{(s_2-s_1)^{qn/\nu}} \left(\int_{B_{4\rho}} \left[\varphi \left(\frac{|u|}{\rho^2} \right) + \varphi(|f|) \right] dx \right) \leq \frac{c(\delta)}{(s_2-s_1)^{qn/\nu}}. \end{aligned}$$

Estimation of I_4 : By Fubini's theorem, integration by parts, (2.2) and the (aInc) $_\nu$ condition of ψ ,

$$\begin{aligned}
I_4 &= c\varepsilon \int_{B_{s_2\rho}} |D^2u|^\nu \int_0^{4|D^2u|^\nu} \frac{d[\psi(K\lambda)]}{\lambda} dx \\
&= c\varepsilon \int_{B_{s_2\rho}} |D^2u|^\nu \left[\frac{\psi(4K|D^2u|^\nu)}{4K|D^2u|^\nu} + 2 \int_0^{4|D^2u|^\nu} \frac{\psi(K\lambda)}{\lambda^2} d\lambda \right] dx \\
&\leq c\varepsilon \int_{B_{s_2\rho}} \left[(4K)^{q-1} L(\psi(|D^2u|^\nu)) + 1 \right] + \frac{\psi(|D^2u|^\nu)}{|D^2u|^{\nu^2-\nu}} \int_0^{4|D^2u|^\nu} \lambda^{\nu-2} d\lambda \Big] dx \\
&\leq c\varepsilon \int_{B_{s_1\rho}} \varphi(|D^2u|) dx + c.
\end{aligned}$$

Taking $\varepsilon = \varepsilon(n, \Lambda, p, q, L) \in (0, 1)$ sufficiently small so that $c\varepsilon = \frac{1}{2}$, hence $\delta = \delta(n, \Lambda, p, q, L)$ is also determined, we have

$$I_4 \leq \frac{1}{2} \int_{B_{s_2r}} \varphi(|D^2u|) dx + c,$$

Estimation of I_5 : In the same way above, replacing $|D^2u|^\nu$ by $\frac{1}{\delta}|f|^\nu$, we have

$$I_5 \leq c \int_{B_{s_2\rho}} \varphi(|f|) dx \leq c \int_{B_{4\rho}} \varphi(|f|) dx \leq c.$$

In the last inequality above, we have used (A.2).

Therefore, combining the previous results, we have

$$\int_{B_{s_1\rho}} \varphi(|Du|^2) dx \leq \frac{1}{2} \int_{B_{s_2\rho}} \varphi(|D^2u|) dx + \frac{c}{(s_2 - s_1)^{qn/\nu}} + c.$$

Here the constants c are independent of the choices of $1 \leq s_1 < s_2 \leq 2$.

Finally, Applying Lemma 3.3 we get

$$\int_{B_\rho} \varphi(|Du|^2) dx \leq c,$$

which proves (A.1). □

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