GROMOV HYPERBOLICITY OF DENJOY DOMAINS THROUGH FUNDAMENTAL DOMAINS

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1. Introduction

In the 1980s Mikhail Gromov introduced a notion of abstract hyperbolic spaces, which have thereafter been studied and developed by many authors. This theory is a good way to understand the important connections between graphs and Potential Theory on Riemannian manifolds (see e.g. [4, 9, 11, 22, 23, 24, 39]). Initially, the research was mainly centered on hyperbolic group theory, but lately researchers have shown an increasing interest in more direct studies of spaces endowed with metrics used in geometric function theory.

One of the primary questions is naturally whether a metric space (X, d) is hyperbolic in the sense of Gromov or not. The most classical examples are metric trees, the classical Poincaré hyperbolic metric developed in the unit disk and, more generally, simply connected complete Riemannian manifolds with sectional curvature $K \le -k^2 < 0$.

However, it is not easy to determine whether a given space is Gromov hyperbolic or not. In recent years several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Klein-Hilbert metric (see [7, 25]) is Gromov hyperbolic (under particular conditions on the domain of definition); the Gehring-Osgood j-metric (see [17]) is Gromov hyperbolic; and the Vuorinen j-metric (see [17]) is not Gromov hyperbolic except in the punctured space. Also, in [26] the hyperbolicity of the conformal modulus metric μ and the related so-called Ferrand metric λ^*, is studied.

The Gromov hyperbolicity of the quasihyperbolic metric has also recently been a topic of interest. In [8], Bonk, Heinonen and Koskela found necessary and sufficient conditions for when a planar domain D endowed with the quasihyperbolic metric is Gromov hyperbolic. This was extended by Balogh and Buckley, [5]: they found two different necessary and sufficient conditions which work in Euclidean spaces of all dimensions and also in metric spaces under some conditions.

Since the Poincaré metric is also the metric giving rise to what is commonly known as the hyperbolic metric when speaking about open domains in the complex plane or in Riemann surfaces, it could be expected that there is a connection between the notions of hyperbolicity. For simply connected subdomains Ω of the complex plane, it follows directly from the Riemann mapping theorem that the metric space (Ω, h_Ω) is in fact Gromov hyperbolic. However, as soon as simple connectedness is omitted, there is no immediate answer to whether the space Ω is hyperbolic or not. The question has lately been studied in [3, 18, 20, 19, 21, 27], [29]–[38] and [40].

In the current paper our main aim is to study the Gromov hyperbolicity of Denjoy domains (i.e., plane domains Ω with $\partial\Omega \subset \mathbb{R}$) with the Poincaré metric. This kind of surfaces are becoming more and more important in geometric function theory, since, on the one hand, they are a very general type of Riemann surfaces, and, on the other hand, they are more manageable due to symmetry. For instance, Garnett and

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Jones have proved in [12] the Corona Theorem for Denjoy domains, and Alvarez, Pestana and Rodríguez in [2] obtained a characterization of Denjoy domains which satisfy a linear isoperimetric inequality (see also [1, 14, 28, 35]).

The Gromov hyperbolicity of Denjoy domains with the Poincaré and quasihyperbolic metrics has been studied previously in [18], [20] and [21] in terms of the Euclidean size of the boundary of the Denjoy domain. The same topic, for the Poincaré metric only, has been dealt with in [3] and [33], but from a geometric point of view; the criteria so obtained involve the lengths of some kind of closed geodesics.

In this paper we adopt a completely different viewpoint and derive criteria to guarantee hyperbolicity for Denjoy domains in terms of fundamental domains. This point of view allows us to improve some known results (see Corollary 3.12). Our goal is to find criteria (easily applicable in practical cases) which allow us to decide when a Denjoy domain is either hyperbolic or not. Theorem 3.17 provides both sufficient conditions and necessary conditions in order to guarantee the hyperbolicity; we also obtain a not so simply stated characterization of hyperbolicity (see Theorem 3.15). Furthermore, Theorem 3.19 characterizes the hyperbolicity for a special kind of domains.

2. Background and notation

We denote by X a geodesic metric space. By d_X and L_X we shall denote, respectively, the distance and the length in the metric of X. From now on, when there is no possible confusion, we will not write the subindex X.

We denote by $\Re z$ and $\Im z$ the real and imaginary part of z, respectively. We define $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$.

We denote by Ω a Denjoy domain with its Poincaré metric.

Finally, we denote by c and c_i, positive constants which can assume different values in different theorems.

We denote by $\mathbb{H}$ the upper half plane, $\{z \in \mathbb{C} : \ \Im z > 0\}$ and by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} : \ |z| < 1\}$. Recall that a domain $\Omega \subset \mathbb{C}$ is said to be of *non-exceptional* if it has at least two finite boundary points. The universal cover of such domain is the unit disk $\mathbb{D}$. In Ω we can define the Poincaré metric, i.e. the metric obtained by projecting the metric $ds = 2|dz|/(1-|z|^2)$ of the unit disk by any universal covering map $\pi : \mathbb{D} \longrightarrow \Omega$. Equivalently, we can project the metric $ds = |dz|/\Im z$ of the the upper half plane $\mathbb{H}$. Therefore, any simply connected subset of Ω is isometric to a subset of $\mathbb{D}$. With this metric, Ω is a geodesically complete Riemannian manifold with constant curvature -1; in particular, Ω is a geodesic metric space. The *Poincaré metric* is natural and useful in complex analysis; for instance, any holomorphic function between two domains is Lipschitz with constant 1, when we consider the respective Poincaré metrics.

We denote by λ_Ω the density of the hyperbolic metric in Ω. It is well known that for all domains $\Omega_1 \subseteq \Omega_2$ we have $\lambda_{\Omega_1}(z) \geq \lambda_{\Omega_2}(z)$ for every $z \in \Omega_1$.

A *Denjoy domain* $\Omega \subset \mathbb{C}$ is a domain whose boundary is contained in the real axis. Since $\Omega \cap \mathbb{R}$ is an open set contained in $\mathbb{R}$, it is the union of pairwise disjoint open intervals; as each interval contains a rational number, this union is countable. Hence, we can write $\Omega \cap \mathbb{R} = \cup_{n\in\Lambda}(a_n, b_n)$, where Λ is a countable index set, $\{(a_n, b_n)\}_{n\in\Lambda}$ are pairwise disjoint, and it is possible to have $a_{n_1} = -\infty$ for some $n_1 \in \Lambda$ and/or $b_{n_2} = \infty$ for some $n_2 \in \Lambda$.

In order to study Gromov hyperbolicity, we consider the case where Λ is countably infinite, since if Λ is finite then it is easy to see that Ω is Gromov hyperbolic by [18, Proposition 3.6] or [36, Proposition 3.2].

As we mentioned in the introduction of this paper, Denjoy domains are becoming more and more interesting in Geometric Function Theory (see e.g. [1, 2, 12, 14, 28, 35]).

{def:geo}
Definition 2.1. If $\gamma : [a, b] \longrightarrow X$ is a continuous curve in a metric space (X, d), the *length* of γ is

$$L(\gamma) := \sup \Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in [a, b]$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by xy any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but convenient as well).

{def:Rips

Definition 2.2. If X is a geodesic metric space and J is a polygon whose sides are $J_1, J_2, \ldots, J_n$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leq \delta$. We say that a polygon is *geodesic* if all of its sides are geodesics. The space X is *δ-thin* (or *δ-hyperbolic*) if every geodesic triangle in X is δ-thin.

Remark 2.3. If X is δ-thin, it is easy to check that every geodesic polygon with n sides is $(n-2)\delta$-thin.

Example 2.4.

(1) Every bounded metric space X is $(diam X)$-hyperbolic (see e.g. [13, p. 29]).
(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k < 0$ is hyperbolic (see e.g. [13, p. 52]).
(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [13, p. 29]).

We refer to [10, 13, 15, 16] for more background on Gromov hyperbolic spaces.
The following is a key tool in our proofs.

{t:caracI

Theorem 2.5 ([3, Theorem 5.1])**.** *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$, and for each $n \geq 1$, consider a fixed geodesic γ_n joining (a_0, b_0) with (a_n, b_n). Then, Ω is δ-hyperbolic if and only if there exists a constant c such that $d_\Omega(z, \mathbb{R}) \leq c$ for every $z \in \cup_n \gamma_n$.*

Furthermore, if Ω is δ-hyperbolic, then the constant c only depends on δ. If $d_\Omega(z, \mathbb{R}) \leq c$ for every $z \in \cup_n \gamma_n$, then Ω is δ-hyperbolic, with δ a constant which only depends on c.

3. Main results

Definition 3.1. Let $\{x_n\}_{n=1}^{\infty}, \{\rho_n\}_{n=1}^{\infty}$ be two sequences of positive numbers, such that $x_n > \rho_n$, $\forall n \geq 1$ and $(x_n - \rho_n, x_n + \rho_n) \cap (x_m - \rho_m, x_m + \rho_m) = \varnothing$ if $n \neq m$. We define geodesics in $\mathbb{H}$ given by $s_0 := \{z \in \mathbb{H} : \Re z = 0\}$ and $s_n := \{z \in \mathbb{H} : |z - x_n| = \rho_n\}$ for $n \geq 1$.

Definition 3.2. Let $\Omega \subset \mathbb{C}$ be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$ and $\pi : \mathbb{H} \to \Omega$ a universal covering map such that $\pi(s_n) = \pi(-\overline{s_n}) = (a_n, b_n)$ and $\pi(s_0) = (a_0, b_0)$ for some sequences $\{x_n\}_n, \{\rho_n\}_n$. We define a *symmetric fundamental domain* of Ω with parameters $\{x_n\}_n$ and $\{\rho_n\}_n$, and we denote it by $\hat{\Omega}$, as the set:

$$\hat{\Omega} := \mathbb{H} \setminus \bigcup_{n=1}^{\infty} \big(\{z \in \mathbb{C} : \, |z - x_n| < \rho_n\} \cup \{z \in \mathbb{C} : \, |z + x_n| < \rho_n\} \big).$$

Since the universal covering map is a local isometry, Theorem 2.5 can be reworded as follows:

{t:caract

Theorem 3.3. *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain. Let us consider a symmetric fundamental domain corresponding to Ω, determined by the geodesics $\{s_n\}_n$. For each $n \geq 1$ consider a fixed geodesic g_n joining s_0 with s_n. Then, Ω is hyperbolic if and only if there exists a constant c such that $d_{\mathbb{H}}(z, \cup_{n\geq 0}\, s_n) \leq c$ for every $z \in \cup_n g_n$.*

{l:sup}

Lemma 3.4. *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain. Let us consider a symmetric fundamental domain corresponding to Ω, determined by the geodesics $\{s_n\}_n$. For each $n \geq 1$ consider a fixed geodesic g_n that minimizes the distance between s_0 and an arbitrary fixed point in s_n. If*

$$D_0 := \sup \left\{ d_{\mathbb{H}}(z, \cup_{n\geq 0}\, s_n) : \, z \in \cup_n g_n \right\},$$
$$D_1 := \sup \left\{ d_{\mathbb{H}}(z, \cup_{n\geq 1}\, s_n) : \, z \in \cup_n g_n \right\},$$

then $D_0 \leq D_1 \leq 2\, D_0$.

Proof. The first inequality is trivial, so let us deal with the second one. If $D_0 = \infty$, then $D_0 = D_1 = \infty$. Assume now that $D_0 < \infty$. Let us fix $m \geq 1$ and $z \in g_m$. It suffices to show $d_{\mathbb{H}}(z, \cup_{n\geq 1} s_n) \leq 2\,D_0$. If $d_{\mathbb{H}}(z, s_0) > D_0$, then

$$d_{\mathbb{H}}(z, \cup_{n\geq 1} s_n) = d_{\mathbb{H}}(z, \cup_{n\geq 0} s_n) \leq D_0 \,.$$

If $d_{\mathbb{H}}(z, \cup_{n\geq 1} s_n) > D_0$ and $d_{\mathbb{H}}(z, s_0) \leq D_0$, let us fix $0 < \varepsilon < D_0$. Let z^* be the point in g_m at distance $D_0 + \varepsilon$ from s_0; then $d_{\mathbb{H}}(z, z^*) \leq D_0 + \varepsilon$ and $d_{\mathbb{H}}(z^*, s_0) > D_0$. Thus $d_{\mathbb{H}}(z^*, \cup_{n\geq 1} s_n) \leq D_0$, and consequently,

$$d_{\mathbb{H}}(z, \cup_{n\geq 1} s_n) \leq d_{\mathbb{H}}(z, z^*) + d_{\mathbb{H}}(z^*, \cup_{n\geq 1} s_n) \leq 2\,D_0 + \varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary, we conclude $d_{\mathbb{H}}(z, \cup_{n\geq 1} s_n) \leq 2\,D_0$. □

{l:pi3}

Lemma 3.5. *For every $z \in \mathbb{H}$ with $\arg z \in [\pi/3, \pi/2]$, we have $d_{\mathbb{H}}(z, s_0) \leq \operatorname{Ar}\tanh(1/2)$.*

Proof. By [6, p. 162], we know that $\tanh d_{\mathbb{H}}(z, s_0) = \cos\arg z$. Then, for any $z \in \mathbb{H}$ with $\arg z \in [\pi/3, \pi/2]$, we have $\tanh d_{\mathbb{H}}(z, s_0) \leq \cos \pi/3 = 1/2$, and the conclusion of the lemma holds. □

Theorem 3.3, and Lemmas 3.4 and 3.5 give directly the following result.

{t:caract3}

Theorem 3.6. *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain. Let us consider a symmetric fundamental domain corresponding to Ω, determined by the geodesics $\{s_n\}_n$. For each $n \geq 1$ consider a fixed geodesic g_n that minimizes the distance between s_0 and an arbitrary fixed point in s_n. Then, Ω is hyperbolic if and only if there exists a constant c such that $d_{\mathbb{H}}(z, \cup_{n\geq 1} s_n) \leq c$ for every $z \in \cup_n g_n$ with $\arg z \leq \pi/3$.*

:DenjoyNoHyp1}

Theorem 3.7. *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain and $\hat{\Omega}$ a symmetric fundamental domain with parameters $\{x_n\}_n$ and $\{\rho_n\}_n$ satisfying*

$$\lim_{n\to\infty} \frac{\rho_n}{x_n} = 0 \,.$$

If $\{x_n\}_n$ is unbounded, then Ω is not hyperbolic.

Proof. Seeking a contradiction, let us assume that Ω is δ-hyperbolic. By Theorem 3.3 there must exist a constant $c > 0$ which only depends on δ such that $d_{\mathbb{H}}(z, \cup_{n\geq 0} s_n) \leq c$ for every $z \in \cup_n g_n$. Note that c just depends on δ and not on the choice of the g_n's.

For this c, let us call $V_n \subset \hat{\Omega}$ the c-neighborhoods for s_n, with $n \geq 0$. It means that, if $n > 0$, V_n is contained in a disc with center belonging to the line $\Re z = x_n$ and whose highest point has imaginary part $h_n = \rho_n e^c$. Analogously, $V_0 := \{z \in \mathbb{H} : \Im(z) > |\Re(z)|/\sinh c\}$ (see e.g. [6, p. 162]).

Let us consider N large enough so that for every $n \geq N$ we have that $\rho_n/x_n < 2/(e^{2c} - 1)$. Then, $\rho_n^2(e^{2c}-1) - 2x_n\rho_n < 0$. Hence, $x_n^2 + \rho_n^2 e^{2c} < x_n^2 + \rho_n^2 + 2x_n\rho_n$, and it follows that $x_n < \sqrt{x_n^2 + \rho_n^2 e^{2c}} < x_n + \rho_n$. If g_n is the geodesic defined by $g_n := \{z \in \mathbb{H} : |z| = \sqrt{x_n^2 + \rho_n^2 e^{2c}}\}$, then g_n joins s_0 with s_n and intersects V_n at $x_n + ih_n$.

Let us define $\theta_n := \arg(x_n + ih_n)$. Hence,

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \arctan \frac{h_n}{x_n} = \lim_{n\to\infty} \arctan \frac{\rho_n e^{2c}}{x_n} = 0 \,.$$

Therefore, for every $n \geq N$, g_n is not contained in $\cup_{m=0}^{\infty} V_m$ and it is possible to find a point w in each of those g_n's such that $d_{\mathbb{H}}(w, \cup_{n=0}^{\infty} s_n) > c$.

This is the contradiction we were looking for, and therefore Ω is not hyperbolic. □

As a direct consequence of this theorem we have the two following results:

:DenjoyNoHyp2}

Corollary 3.8. *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain, and $\hat{\Omega}$ a symmetric fundamental domain with parameters $\{x_n\}_n, \{\rho_n\}_n$. If $\{x_n\}_n$ is unbounded and $\{\rho_n\}_n$ is bounded, then Ω is not hyperbolic.*

In order to prove the next result, we use the following lemma.

{l:Beardon}

Lemma 3.9 ([6, pp. 166 and 75]). *Let us consider two geodesics L_1 (with end-points z_1 and z_2) and L_2 (with end-points w_1 and w_2) in $\mathbb{H}$, occurring in the order z_1, w_1, w_2, z_2 around the circle at infinity. Then*

$$\tanh^2 \frac{d_{\mathbb{H}}(L_1, L_2)}{2} = \frac{1}{[z_1, w_1, w_2, z_2]} = \frac{(w_1 - z_1)(w_2 - z_2)}{(w_2 - z_1)(w_1 - z_2)} .$$

{def:Trai

Definition 3.10. A *train* is a Denjoy domain $\Omega \subset \mathbb{C}$ with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty} (a_n, b_n)$, such that $-\infty \le a_0$ and $b_n \le a_{n+1}$ for every n. A *flute surface* is a train with $b_n = a_{n+1}$ for every n.

We say that a curve in a train Ω is a *fundamental geodesic* if it is a simple closed geodesic which only intersects $\mathbb{R}$ in (a_0, b_0) and (a_n, b_n) for some $n > 0$; we denote by γ_n^* the fundamental geodesic corresponding to n and $2l_n := L_\Omega(\gamma_n^*)$.

Given a train Ω and $n \ge 0$, we denote by σ_n the simple closed geodesic which just intersects $\mathbb{R}$ in (a_n, b_n) and (a_{n+1}, b_{n+1}). If $b_n = a_{n+1}$, we define σ_n as the puncture at this point.

A *fundamental Y-piece* in a train Ω is the generalized Y-piece in Ω bounded by $\gamma_n^*, \gamma_{n+1}^*, \sigma_n$ for some $n > 0$; we denote by Y_n the fundamental Y-piece corresponding to n. A *fundamental hexagon* in a train Ω is the intersection $H_n := Y_n \cap \overline{\mathbb{H}}$ for some $n > 0$.

{p:length

Proposition 3.11. *The lengths of the fundamental geodesics can be explicitly calculated by means of the following expression:*

$$l_n = \operatorname{Ar}\cosh \frac{x_n}{\rho_n} .$$

Proof. By Lemma 3.9, since $l_n = d_{\mathbb{H}}(s_0, s_n)$, we deduce

$$\frac{\cosh l_n - 1}{\cosh l_n + 1} = \tanh^2 l_n/2 = \frac{x_n - \rho_n}{x_n + \rho_n} = \frac{x_n/\rho_n - 1}{x_n/\rho_n + 1} , \qquad \cosh l_n = \frac{x_n}{\rho_n} . \qquad \square$$

The following result improves [3, Corollary 5.13].

{c:serie}

Corollary 3.12. *If Ω is a train with $\sum_{n=1}^{\infty} e^{-l_n} = \infty$ and $\lim_{n\to\infty} l_n = \infty$, then Ω is not hyperbolic.*

Proof. We denote by α_n the length of $Y_n \cap (a_0, b_0)$ (the opposite side to $\sigma_n \cap \overline{\mathbb{H}}$ in the fundamental hexagon H_n).

Let us define $g_n = \{\sqrt{x_n^2 - \rho_n^2}\, e^{i\theta} : \theta \in (0, \pi)\}$. A computation gives that g_n meets s_n orthogonally; hence, g_n is the shortest geodesic in $\mathbb{H}$ joining s_0 with s_n, and $\pi(g_n) = \gamma_n^*$.

On the one hand, by [3, Lemma 5.5] we know that $e^{-l_n} + e^{-l_{n+1}} \le \alpha_n$. Since $\sqrt{x_n^2 - \rho_n^2} = \sqrt{x_1^2 - \rho_1^2}\, e^{\sum_{k=1}^{n-1} \alpha_k}$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$, we deduce that $\lim_{n\to\infty} \sqrt{x_n^2 - \rho_n^2} = \infty$, and hence $\lim_{n\to\infty} x_n = \infty$.

On the other hand, from the fact $\lim_{n\to\infty} l_n = \infty$ we can conclude that $\lim_{n\to\infty} \rho_n/x_n = 0$, by Proposition 3.11. Hence, Ω is not hyperbolic by Theorem 3.7. $\square$

{l:DistPu

Lemma 3.13. *Let us consider a point $z = re^{i\theta} \in \mathbb{H}$ and the geodesic $\gamma = \{x + iy \in \mathbb{H} : (x - x_0)^2 + y^2 = \rho^2\}$. Then*

$$d_{\mathbb{H}}(z, \gamma) = \operatorname{Ar}\sinh \frac{\left| r^2 + x_0^2 - \rho^2 - 2rx_0 \cos\theta \right|}{2r\rho \sin\theta} .$$

Proof. Let us consider the isometry T of $\mathbb{H}$:

$$T(w) = \frac{w - x_0 - \rho}{w - x_0 + \rho} .$$

Since $T(x_0 + \rho) = 0$ and $T(x_0 - \rho) = \infty$, $T(\gamma)$ is the imaginary axis. Furthermore,

$$T(z) = \frac{re^{i\theta} - x_0 - \rho}{re^{i\theta} - x_0 + \rho} \cdot \frac{re^{-i\theta} - x_0 + \rho}{re^{-i\theta} - x_0 + \rho} = \frac{r^2 + x_0^2 - \rho^2 - 2rx_0\cos\theta + 2ir\rho\sin\theta}{(r\cos\theta - x_0 + \rho)^2 + r^2 \sin^2\theta} .$$

Standard hyperbolic computations (see e.g. [6, p. 162]) give that

$$\sinh d_{\mathbb{H}}(z,\gamma) = \sinh d_{\mathbb{H}}(T(z),T(\gamma)) = \big|\operatorname{cotan}\big(\arg T(z)\big)\big| = \frac{\left|r^2+x_0^2-\rho^2-2rx_0\cos\theta\right|}{2r\rho\sin\theta}\,. \qquad \square$$

l:CaracDenjoy}

Lemma 3.14. *Let $\Omega\subset\mathbb{C}$ be a Denjoy domain. Let us consider a symmetric fundamental domain corresponding to Ω, with parameters $\{x_n\}_n$ and $\{\rho_n\}_n$. Then Ω is hyperbolic if and only if*

$$M := \sup_{n\ge1}\max_{\theta\in[\theta_n,\pi/3]}\min_{m\ge1}\Lambda_{nm}(\theta)<\infty\,,$$

where

$$\Lambda_{nm}(\theta) := \frac{\left|x_n^2+x_m^2-\rho_m^2-2x_nx_m\cos\theta\right|}{2x_n\rho_m\sin\theta}\,,$$

and

$$\theta_n := \arccos\Big(1-\frac{\rho_n^2}{2x_n^2}\Big)\,.$$

Proof. Lemma 3.13 gives that

$$B_{nm}(\theta) := d_{\mathbb{H}}(x_ne^{i\theta},s_m) = \operatorname{Ar}\sinh\frac{\left|x_n^2+x_m^2-\rho_m^2-2x_nx_m\cos\theta\right|}{2x_n\rho_m\sin\theta}\,.$$

Consequently,

$$\sinh d_{\mathbb{H}}\Big(x_ne^{i\theta},\bigcup_{m\ge1}s_m\Big) = \min_{m\ge1}\sinh B_{nm}(\theta) = \min_{m\ge1}\Lambda_{nm}(\theta)\,.$$

Let us denote by g_n' the geodesic $g_n' := \{x_ne^{i\theta}\in\mathbb{H}:\,\theta\in[\theta_n,\pi/2]\}$, joining s_0 with s_n (a direct computation gives that $x_ne^{i\theta_n}\in s_n$). The geodesic s_n in $\mathbb{H}$ corresponds to (a_n,b_n) in Ω, and g_n' in $\mathbb{H}$ corresponds to a geodesic joining (a_0,b_0) with (a_n,b_n) in Ω. By Theorem 3.6, Ω is hyperbolic if and only if

$$\sinh\Big(\sup_{n\ge1}\max_{\theta\in[\theta_n,\pi/3]}\min_{m\ge1}d_{\mathbb{H}}\big(x_ne^{i\theta},s_m\big)\Big) = \sup_{n\ge1}\max_{\theta\in[\theta_n,\pi/3]}\sinh d_{\mathbb{H}}\Big(x_ne^{i\theta},\bigcup_{m\ge1}s_m\Big)<\infty\,,$$

since sinh is an increasing continuous function. We can write max and min instead of sup and inf, respectively, since $d_{\mathbb{H}}\big(z,\cup_{m\ge1}s_m\big)$ is attained (recall that $\cup_{m\ge1}s_m$ is a closed set in $\mathbb{H}$) and, in fact, $d_{\mathbb{H}}\big(z,\cup_{m\ge1}s_m\big)$ is a continuous function in z. This fact finishes the proof. $\square$

t:CaracDenjoy}

Theorem 3.15. *Let $\Omega\subset\mathbb{C}$ be a Denjoy domain. Let us consider a symmetric fundamental domain corresponding to Ω, with parameters $\{x_n\}_n$ and $\{\rho_n\}_n$. Then Ω is hyperbolic if and only if*

$$K := \sup_{n\ge1}\max_{\alpha\in[0,1]}\min_{m\ge1}\left\{\frac{(x_n-x_m)^2-\rho_m^2}{x_n\rho_m}\Big(\frac{x_n}{\rho_n}\Big)^{\alpha}+\frac{x_m}{\rho_m}\Big(\frac{\rho_n}{x_n}\Big)^{\alpha}\right\}<\infty.$$

Furthermore, if Ω is δ-hyperbolic, then K is bounded by a constant which only depends on δ; if $K<\infty$, then Ω is δ-hyperbolic, with δ a constant which only depends on K.

Remark 3.16. We write $f\approx g$ if there exists a positive constant c with $c^{-1}f\le g\le c\,f$.

Proof. The proof is based on showing that the constant K in this theorem is comparable to M in Lemma 3.14.

Let us consider the function $\Lambda_{mn}(\theta)$ from the previous lemma. We start with the case $m\ne n$. Since $|x_m-x_n|\ge\rho_n+\rho_m$, we see that

$$x_n^2+x_m^2-\rho_m^2-2x_nx_m\cos\theta = \underbrace{(x_n-x_m)^2-\rho_m^2}_{=:a}+\underbrace{2x_nx_m}_{=:b}(1-\cos\theta)>0.$$

Therefore, the absolute value signs in the definition of the function Λ_{mn} may be dropped. From the definition of θ_n, denoting $z:=\frac{\rho_n}{x_n}\in(0,1]$, we obtain

$$\cos\theta_n = 1-\tfrac12z^2\quad\text{and}\quad\sin\theta_n = z\sqrt{1-\tfrac14z^2}\,.$$

For $\theta \in [\theta_n, \pi/3]$, we have $1-\cos\theta \le 1/2$ and we can write $1-\cos\theta = \frac{1}{2}z^{2\alpha}$ for some $\alpha = \alpha(\theta) \in [0,1]$. We note that then $\sin\theta \approx z^{\alpha}$. Therefore,

$$\Lambda_{mn}(\theta) = \frac{a+b(1-\cos\theta)}{2x_n\rho_m\sin\theta} \approx \frac{a+\frac{1}{2}bz^{2\alpha}}{x_n\rho_m z^{\alpha}} = \frac{(x_n-x_m)^2-\rho_m^2}{x_n\rho_m}\Big(\frac{x_n}{\rho_n}\Big)^{\alpha} + \frac{x_m}{\rho_m}\Big(\frac{\rho_n}{x_n}\Big)^{\alpha}.$$

In the case $m=n$, we obtain

$$\begin{aligned}\Lambda_{nn}(\theta) &= \frac{-\rho_n^2+2x_n^2(1-\cos\theta)}{2x_n\rho_n\sin\theta} = \frac{-z+2z^{-1}(1-\cos\theta)}{2\sin\theta}\\ &= \frac{z^{2\alpha-1}-z}{2\sin\theta} \approx z^{\alpha-1}-z^{1-\alpha} = \frac{-\rho_n^2}{x_n\rho_n}\Big(\frac{x_n}{\rho_n}\Big)^{\alpha} + \frac{x_n}{\rho_n}\Big(\frac{\rho_n}{x_n}\Big)^{\alpha},\end{aligned}$$

with $\alpha = \alpha(\theta)$ and z as before.

Hence we find that

$$\sup_{n\ge1}\max_{\theta\in[\theta_n,\pi/3]}\min_{m\ge1}\Lambda_{nm}(\theta) \approx \sup_{n\ge1}\max_{\alpha\in[0,1]}\min_{m\ge1}\Big\{\frac{(x_n-x_m)^2-\rho_m^2}{x_n\rho_m}\Big(\frac{x_n}{\rho_n}\Big)^{\alpha} + \frac{x_m}{\rho_m}\Big(\frac{\rho_n}{x_n}\Big)^{\alpha}\Big\},$$

so the proof is complete. □

{t:CaracI

Theorem 3.17. *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain. Let us consider a symmetric fundamental domain corresponding to Ω, with parameters $\{x_n\}_n$ and $\{\rho_n\}_n$.*

(1) If Ω is hyperbolic, then there exists a constant K with the following property: for every n there exist m_1, m_2 such that

$$x_n^2 + x_{m_1}^2 - \rho_{m_1}^2 < Kx_n\rho_{m_1} \quad \text{and} \quad \big((x_n-x_{m_2})^2-\rho_{m_2}^2\big)\frac{x_n}{\rho_n} + \rho_n x_{m_2} < Kx_n\rho_{m_2}.$$

(2) If for every n there exists m such that

$$x_n^2 + x_m^2 - \rho_m^2 < Kx_n\rho_m \quad \text{and} \quad \big((x_n-x_m)^2-\rho_m^2\big)\frac{x_n}{\rho_n} + \rho_n x_m < Kx_n\rho_m,$$

for some fixed constant K, then Ω is hyperbolic.

Proof. The first claim is obtained directly from the previous theorem with the choices $\alpha = 0$ and $\alpha = 1$. The second claim follows from the theorem since the function $az^{-\alpha} + bz^{\alpha}$ is convex in α so that it suffices to obtain the boundedness at the end points, $\alpha = 0$ and $\alpha = 1$. □

A particular case of Theorem 3.17 is when we choose $m = n$ for each n. This gives the sufficient condition of the following corollary:

{c:lAcota

Corollary 3.18. *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain. Let us consider a symmetric fundamental domain corresponding to Ω, with parameters $\{x_n\}_n$ and $\{\rho_n\}_n$. If*

$$\liminf_{n\to\infty}\frac{\rho_n}{x_n} > 0\,,$$

then Ω is hyperbolic.

Theorem 3.7 and Corollary 3.18 give directly the following characterization.

{t:lAcota

Theorem 3.19. *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain. Let us consider a symmetric fundamental domain corresponding to Ω, with parameters $\{x_n\}_n$ and $\{\rho_n\}_n$. If $\{x_n\}_n$ is unbounded and $\lim_{n\to\infty}\rho_n/x_n$ exists, then Ω is hyperbolic if and only if*

$$\lim_{n\to\infty}\frac{\rho_n}{x_n} > 0\,.$$

4. Comparative results

Given two Denjoy domains Ω and Ω', it is useful to have some criteria relating their hyperbolicity. In this section we provide two of such criteria.

{t:comp1}

Theorem 4.1. *Let Ω, $\Omega' \subset \mathbb{C}$ be two Denjoy domains. Let us consider symmetric fundamental domains corresponding to Ω and Ω', with respective parameters $\{x_n\}_n$, $\{\rho_n\}_n$ and $\{x'_n\}_n$, $\{\rho'_n\}_n$, with $(x_n - \rho_n, x_n + \rho_n) \subseteq (x'_n - \rho'_n, x'_n + \rho'_n)$. If Ω is δ-hyperbolic, then Ω' is δ'-hyperbolic, with δ' a constant which just depends on δ.*

Proof. For each $n \ge 1$ consider a fixed geodesic g_n joining s_0 with s_n; we choose the geodesic g'_n as the subcurve of g_n that joins s'_0 and s'_n.

Since Ω is hyperbolic, by Theorem 3.3, there exists a constant c such that $d_{\mathbb{H}}(z, \cup_{n\ge 0} s_n) \le c$ for every $z \in \cup_n g_n$. Hence, $d_{\mathbb{H}}(z, \cup_{n\ge 0} s'_n) \le d_{\mathbb{H}}(z, \cup_{n\ge 0} s_n) \le c$ for every $z \in \cup_n g'_n \subseteq \cup_n g_n$. Then Ω' is hyperbolic by Theorem 3.3. ☐

{t:comp2}

Theorem 4.2. *Let Ω, $\Omega' \subset \mathbb{C}$ be two Denjoy domains. Let us consider symmetric fundamental domains corresponding to Ω and Ω', with respective parameters $\{x_n\}_n$, $\{\rho_n\}_n$ and $\{x'_n\}_n$, $\{\rho'_n\}_n$. Assume that there exist n_1, n_2 such that*

$$\cup_{n\ge n_1} (x_n - \rho_n, x_n + \rho_n) = \cup_{n\ge n_2} (x'_n - \rho'_n, x'_n + \rho'_n)\,,$$
$$d_{Eucl}\big(\cup_{1\le n<n_1} (x_n - \rho_n, x_n + \rho_n), \cup_{n\ge n_1} (x_n - \rho_n, x_n + \rho_n)\big) > 0\,,$$
$$d_{Eucl}\big(\cup_{1\le n<n_2} (x'_n - \rho'_n, x'_n + \rho'_n), \cup_{n\ge n_2} (x'_n - \rho'_n, x'_n + \rho'_n)\big) > 0\,.$$

Then Ω is hyperbolic if and only if Ω' is hyperbolic.

Proof. Note that it is sufficient to prove the result for $n_1 = 1$ and $n_2 = 2$, since the theorem can be easily deduced by repeating the argument a finite amount of times. Then

$$\cup_{n\ge 1} (x_n - \rho_n, x_n + \rho_n) = \cup_{n\ge 2} (x'_n - \rho'_n, x'_n + \rho'_n)\,,$$

and

$$d_{Eucl}\big((x'_1 - \rho'_1, x'_1 + \rho'_1), \cup_{n\ge 2} (x'_n - \rho'_n, x'_n + \rho'_n)\big) > 0\,. \tag{4.1}$$

Without loss of generality we can assume that $x'_{n+1} = x_n$ and $\rho'_{n+1} = \rho_n$. Then $s'_0 = s_0$ and $s'_{n+1} = s_n$ for every $n > 1$. For each $n \ge 1$ consider a fixed geodesic g_n joining s_0 with s_n. Let us choose now a geodesic g'_1 joining s'_0 and s'_1, and $g'_{n+1} = g_n$ for every $n > 1$.

Assume first that Ω is hyperbolic. By Theorem 3.3, there exists a constant c such that $d_{\mathbb{H}}(z, \cup_{n\ge 0} s_n) \le c$ for every $z \in \cup_n g_n$. Hence, $d_{\mathbb{H}}(z, \cup_{n\ge 0} s'_n) \le d_{\mathbb{H}}(z, \cup_{n\ge 0} s_n) \le c$ for every $z \in \cup_{n\ge 2} g'_n$. Furthermore, $d_{\mathbb{H}}(z, \cup_{n\ge 0} s'_n) \le L_{\mathbb{H}}(g'_1)$ for every $z \in g'_1$, and consequently,

$$d_{\mathbb{H}}(z, s'_0) \le d_{\mathbb{H}}(z, \cup_{n\ge 0} s'_n) \le \max\{c,\, L_{\mathbb{H}}(g'_1)\}\,,$$

for every $z \in \cup_{n\ge 2} g'_n$. Then Ω' is hyperbolic by Theorem 3.3.

Assume now that Ω' is hyperbolic. By Theorem 3.3, there exists a constant c' such that $d_{\mathbb{H}}(z, \cup_{n\ge 0} s'_n) \le c'$ for every $z \in \cup_n g'_n$. Let us define the "banana" $B := \{z \in \mathbb{H} :\, d_{\mathbb{H}}(z, s'_1) \le c'\}$. By (4.1), there exists N such that $B \cap g'_n = \varnothing$ for every $n > N$. Consequently, $d_{\mathbb{H}}(z, s'_1) > c'$ for every $z \in \cup_{n>N} g'_n$, and hence, $d_{\mathbb{H}}(z, \cup_{n\ge 0} s_n) = d_{\mathbb{H}}(z, \cup_{n\ge 0} s'_n) \le c'$ for every $z \in \cup_{n>N} g'_n$. Furthermore, $d_{\mathbb{H}}(z, \cup_{n\ge 0} s_n) \le d_{\mathbb{H}}(z, s'_0) \le \max_{1\le n\le N} L_{\mathbb{H}}(g_n)$ for every $z \in \cup_{1\le n\le N} g_n$, and consequently,

$$d_{\mathbb{H}}(z, \cup_{n\ge 0} s'_n) \le \max\{c,\, \max_{1\le n\le N} L_{\mathbb{H}}(g_n)\}\,,$$

for every $z \in \cup_n g_n$. Then Ω is hyperbolic by Theorem 3.3. ☐

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