FURTHER RESULTS ON VARIABLE EXPONENT TRACE SPACES

LARS DIENING*

Section of Applied Mathematics, Eckerstr. 1,
Freiburg University, 79104 Freiburg/Breisgau, Germany
E-mail: diening@mathematik.uni-freiburg.de

PETER HÄSTÖ†

Department of Mathematics and Statistics, P.O. Box 68,
FI-00014 University of Helsinki, Finland
E-mail: peter.hasto@helsinki.fi

Recently, the trace space of Sobolev functions with variable exponents has been characterized by the authors [L. Diening and P. Hästö: Variable exponent trace spaces, Preprint (2005)]. In this note we relax the assumptions on the exponent need for some basic results on trace spaces, like a characterization of zero boundary value spaces in terms of traces.

1. Introduction

From the point of boundary value problems it is very important to study the trace spaces of the natural energy space. Indeed, a partial differential equation is in many cases solvable if and only if the boundary values are in the corresponding trace space. In the case of electrorheological fluids[16] the energy space is a Sobolev space with variable exponent, namely $W^{1,p(\cdot)}$. These spaces are defined as follows: For an open set $\Omega \subset \mathbb{R}^m$ let $p\colon \Omega \to [1,\infty)$ be a measurable bounded function, called a variable exponent on Ω with $p^+ := \operatorname{ess\,sup} p(x) < \infty$. Further, let $p^- := \operatorname{ess\,inf} p(x)$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $f\colon \Omega \to \mathbb{R}^m$ for which the modular

$$\varrho_{L^{p(\cdot)}(\Omega)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx$$

is finite. Then $\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0\colon \varrho_{L^{p(\cdot)}(\Omega)}(f/\lambda) \le 1\}$ defines a norm on $L^{p(\cdot)}(\Omega)$. The space $W^{1,p(\cdot)}(\Omega)$ is the subspace of $L^{p(\cdot)}(\Omega)$ such that $|\nabla f| \in L^{p(\cdot)}(\Omega)$. The norm $\|f\|_{W^{1,p(\cdot)}(\Omega)} = \|f\|_{L^{p(\cdot)}(\Omega)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. For basic properties of $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ we refer to Kováčik and Rákosník[14] or Fan and Zhao[9].

*Supported by Landesstiftung Baden-Württemberg
†Supported by the Academy of Finland

We are interested in domains Ω with Lipschitz boundary but for the sake of simplicity we assume that Ω is just the halfspace $\mathbb{H} = \mathbb{R}^n \times (0, \infty)$. Corresponding results for Lipschitz domains can than be achieved via flattening of the boundary by local Bi-Lipschitz mappings. We write $\mathbb{R}^n$ instead of $\mathbb{R}^n \times \{0\}$ for the boundary of $\mathbb{H}$. The trace space of $W^{1,p(\cdot)}(\mathbb{H})$ is naturally defined to be the quotient space of the traces of functions from $W^{1,p(\cdot)}(\mathbb{H})$, i.e.

$$\|f\|_{\mathrm{Tr}\, W^{1,p(\cdot)}(\mathbb{H})} = \inf \big\{ \|F\|_{W^{1,p(\cdot)}(\mathbb{H})} : F \in W^{1,p(\cdot)}(\mathbb{H}) \text{ and } \mathrm{tr} F = f \big\}.$$

Note that the trace $\mathrm{Tr}\, F$ is well defined, since every $F \in W^{1,p(\cdot)}(\mathbb{H})$ is in $W^{1,1}_{\mathrm{loc}}(\overline{\mathbb{H}})$. Trace spaces of Sobolev spaces with variable exponents first appeared in the study of the Laplace equation $-\Delta u = f$ on the half space with $f \in L^{p(\cdot)}(\mathbb{H})$ and prescribed boundary values[7,8].

Although the definition above is the most natural one, it depends on the exponent p in the interior of the domain. Nevertheless, it was found by the authors[6] that if p is globally log-Hölder continuous, then the definition of the trace space depends only on the values of p on the boundary (see Proposition 1.1 below) — we say that the exponent p is *globally* log-*Hölder continuous* if there exist constants $c > 0$ and $p_\infty \in (1, \infty)$ such that for all points $|x - y| < \frac{1}{2}$ and all points z

$$|p(x) - p(y)| \le \frac{c}{\log(1/|x-y|)} \quad \text{and} \quad |p(z) - p_\infty| \le \frac{c}{\log(e + |z|)}$$

hold. Let us denote by $\mathcal{P}(\Omega)$ the class of globally log-Hölder continuous variable exponents p on $\Omega \subset \mathbb{R}^m$ with $1 < p^- \le p^+ < \infty$.

The log-Hölder condition appears quite naturally in the context of variable exponent spaces: For example, we know that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$ if $p \in \mathcal{P}(\Omega)$[3,4]. Global log-Hölder continuity is the best possible modulus of continuity to imply the boundedness of the maximal operator[3,15]. Note that the translation operators are not continuous on $L^{p(\cdot)}$, but if the maximal operator is bounded, it is at least possible to use the technique of mollifiers.

Let us summarize the results[6] on $\mathrm{Tr}\, W^{1,p(\cdot)}(\mathbb{R}^n)$ that we will need:

Theorem 1.1. *Let $p_1, p_2 \in \mathcal{P}(\mathbb{H})$ with $p_1|_{\mathbb{R}^n} = p_2|_{\mathbb{R}^n}$. Then with equivalence of norms we have* $\mathrm{Tr}\, W^{1,p_1(\cdot)}(\mathbb{H}) = \mathrm{Tr}\, W^{1,p_2(\cdot)}(\mathbb{H})$.

This proposition is proved by the following useful extension theorem:

Proposition 1.1. *Let $p \in \mathcal{P}(\mathbb{R}^{n+1})$. Then there exists a bounded, linear extension operator $\mathcal{E} : W^{1,p(\cdot)}(\mathbb{H}) \to W^{1,p(\cdot)}(\mathbb{R}^{n+1})$.*

Proposition 1.2. *Let $X \subset \mathbb{R}^n$. If $p \in \mathcal{P}(X)$, then there exists an extension $\tilde{p} \in \mathcal{P}(\mathbb{R}^n)$.*

Remark 1.1. Due to Propositions 1.1 and 1.2 it is possible to define in some cases a trace space just by the knowledge of the values of p on the boundary $\mathbb{R}^n$. Indeed, if $p \in \mathcal{P}(\mathbb{R}^n)$ then we can extend p by Proposition 1.2 to some $q \in \mathcal{P}(\mathbb{H})$. It is now possible to consider the trace space $T := \operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$. Proposition 1.1 ensures that the definition of T does not depend on the extension q (up to isomorphism). Thus, it is possible to define the trace space $(\operatorname{Tr} W^{1,p(\cdot)})(\mathbb{R}^n) := \operatorname{Tr} W^{1,q(\cdot)}(\mathbb{H})$ for $p \in \mathcal{P}(\mathbb{R}^n)$, where q is an arbitrary extension of p with $q \in \mathcal{P}(\mathbb{H})$.

Although the definition of the trace space above is very natural it is not so useful for deciding if a function is a $W^{1,p(\cdot)}$-trace. For this purpose it is better to have an intrinsic norm, i.e. a norm only in terms of the values on the boundary. The following theorem[6] provides such an characterization for all globally log-Hölder continuous exponents:

Theorem 1.2. *Let $p \in \mathcal{P}(\mathbb{R}^n)$ and let $q \in \mathcal{P}(\overline{\mathbb{H}})$ be an arbitrary extension of p, i.e. $p(x) = q(x,0)$ for all $x \in \mathbb{R}^n$. Then the function f belongs to the trace space $(\operatorname{Tr} W^{1,p(\cdot)})(\mathbb{R}^n) \cong \operatorname{Tr} W^{1,q(\cdot)}(\mathbb{H})$ if and only if*

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx + \int_0^1 \int_{\mathbb{R}^n} \Big(\tfrac{1}{r} M^\sharp_{B^n(x,r)} f\Big)^{p(x)} dx\, dr < \infty,$$

where $M^\sharp_{B^n(x,r)}$ denotes the sharp operator

$$M^\sharp_{B^n(x,r)} f = \fint_{B^n(x,r)} \Big| f(y) - \fint_{B^n(x,r)} f(z)\, dz \Big| \, dy$$

and $B^n(x,t)$ denotes the n-dimensional ball with center x and radius t.

In this article we extend the results by the authors[6] and consider spaces with more general exponents. The standing assumption of Diening and Hästö[6] was that the exponent is globally log-Hölder continuous. In this article we work with the considerably weaker assumption that the exponent is such that smooth functions are dense in our Sobolev space. Note that smooth functions are certainly dense if the maximal operator is bounded.

The main result is a characterization of the variable exponent Sobolev functions with zero boundary values – we show that these are just the functions which have trace zero.

2. Trace spaces when continuous functions are dense

In order to work with classical derivatives in our proofs, we need to prove the density of smooth functions in our function space. In the variable exponent case this question is far from trivial, as convolutions cannot be used in general, see the articles[12,13,17,18]. So in this section we will simply assume that smooth functions are dense in the ambient space.

Notice the difference between the spaces $C_0^\infty(\overline{\mathbb{H}})$ and $C_0^\infty(\mathbb{H})$: in the former space functions simply have bounded support, in the latter the support of the function is bounded and disjoint from the boundary $\mathbb{R}^n$ of $\mathbb{H}$.

Theorem 2.1. *Suppose that $C_0^\infty(\overline{\mathbb{H}})$ is dense in $W^{1,p(\cdot)}(\mathbb{H})$. Then $C_0^\infty(\mathbb{R}^n)$ is dense in* $\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$.

Proof. Let $f \in \operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$, and let $F \in W^{1,p(\cdot)}(\mathbb{H})$ be such that $\operatorname{Tr} F = f$. Then if $\varphi_i \in C_0^\infty(\overline{\mathbb{H}})$ tend to F in $W^{1,p(\cdot)}(\mathbb{H})$, we see that $\varphi_i|_{\mathbb{R}^n} \to f$ in $\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$. □

Recall the definition of the Sobolev space of functions with zero boundary value: the space $W_0^{1,p(\cdot)}(\mathbb{H})$ is the completion of $C_0^\infty(\mathbb{H})$ in $W^{1,p(\cdot)}(\mathbb{H})$. (Other definitions are better, when smooth functions are not dense[10,11].) We next characterize $W_0^{1,p(\cdot)}(\mathbb{H})$ in terms of traces. For this we need to recall the definition of the Sobolev $p(\cdot)$-capacity: for $E \subset \mathbb{R}^n$ we define

$$\operatorname{cap}_{p(\cdot)}(E) = \inf_u \int_{\mathbb{R}^n} |u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\, dx,$$

where the infimum is taken over functions $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$, $0 \le u \le 1$, which equal 1 in an open set containing E. Basic properties of this capacity, like monotony and sub-additivity, were derived by Harjulehto, Hästö, Koskenoja and Varonen[11].

The next result was previously proven by the authors under the stronger assumption that p is log-Hölder continuous, see Diening and Hästö[6], Theorem 3.5. In that case the proof can be made much simpler using convolution.

Theorem 2.2. *Suppose that $p^+ < \infty$ and $C_0^\infty(\overline{\mathbb{H}})$ is dense in $W^{1,p(\cdot)}(\mathbb{H})$. Then $F \in W^{1,p(\cdot)}(\mathbb{H})$ belongs to $W_0^{1,p(\cdot)}(\mathbb{H})$ if and only if and* $\operatorname{Tr} F = 0$.

Proof. Let $F \in W^{1,p(\cdot)}(\mathbb{H})$ with $\operatorname{Tr} F = 0$. Multiplying F by a Lipschitz cut-off we see that it suffices to prove the claim for F with support in $B := B^{n+1}(0,r)$. Below, we will prove the claim for non-negative functions F. But the general claim follows from this, since we can write $F = F_+ - F_-$, where $F_+, F_- \in W^{1,p(\cdot)}(\mathbb{H})$ are non-negative functions with trace zero. We know that the first order Sobolev space is a lattice, i.e. if $F, G \in W^{1,p(\cdot)}(\mathbb{H})$, then $\max\{F,G\}, \min\{F,G\} \in W^{1,p(\cdot)}(\mathbb{H})$. Furthermore, $\min\{F,a\} \to F$ in $W^{1,p(\cdot)}(\mathbb{H})$ for a constant a tending to ∞. Thus we can also assume that F is bounded.

So let F be non-negative, bounded and with support in B, and fix $\varepsilon > 0$. By Harjulehto, Hästö, Koskenoja and Varonen[11] there exists a $p(\cdot)$-quasicontinuous function $F^* \in W^{1,p(\cdot)}(\mathbb{R}^{n+1})$ which equals F almost everywhere in $\mathbb{H}$ and is identically zero in $\mathbb{R}^{n+1} \setminus \mathbb{H}$. Recall that quasicontinuity (by definition) means that we can choose an open E such that $F^*|_{\mathbb{R}^{n+1}\setminus E}$ is continuous in $\mathbb{R}^{n+1} \setminus E$ and $\operatorname{cap}_{p(\cdot)}(E) < \varepsilon^{p^+ +1}$. Thus we can find a function $\varphi \in W^{1,p(\cdot)}(\mathbb{R}^{n+1})$, $0 \le \varphi \le 1$, which equals 1 on E and has $p(\cdot)$-modular at most $\varepsilon^{p^+ +1}$. Let $\psi\colon \mathbb{R}^{n+1} \to [0,1]$ be a $2/\varepsilon$-Lipschitz function, which equals 1 for $t < \varepsilon$ and 0 for $t \ge 2\varepsilon$. We denote the support of ψ by V.

Since $\overline{B}\setminus E$ is compact, we see (by continuity) that we can choose a neighborhood

$U \subset \mathbb{R}^{n+1}$ of $\mathbb{R}^n \times \{0\}$ such that $F^* < \varepsilon$ in $U \setminus E$. Then we define

$$\tilde{F} = (1 - \varphi\,\psi) \max\{F^* - \varepsilon, 0\}.$$

Since F^* is less than ε in $(U \setminus E) \cup (-\mathbb{H})$, and since $1 - \varphi\,\psi = 0$ in $E \cap \{(x,t) \colon t < \varepsilon\}$, we find that $\tilde{F} = 0$ in $U \cup (-\mathbb{H})$, which means that $\tilde{F}$ has compact support in $\mathbb{H}$. In the next estimate we think of $\tilde{F} - F$ as

$$\big(\max\{F^* - \varepsilon, 0\} - F\big) - \varphi\psi \max\{F^* - \varepsilon, 0\}.$$

Thus we find that

$$\begin{aligned}
\varrho_{1,p(\cdot)}(\tilde{F} - F) &= \int_B |\tilde{F}(z) - F(z)|^{p(z)} + |\nabla \tilde{F}(z) - \nabla F(z)|^{p(z)} dz \\
&\le 2^{p^+} |B|\, \varepsilon + 2^{p^+} \int_{\{F^* < \varepsilon\}} |\nabla F(z)|^{p(z)} dz \\
&\quad + 2^{p^+} \int_B \big|F(z)\, \varphi(z) \psi(z)\big|^{p(z)} + \big|\nabla\big(F(z)\, \varphi(z) \psi(z)\big)\big|^{p(z)} dz
\end{aligned}$$

The first two terms on the right-hand-side clearly tend to zero with ε (since $\nabla F = 0$ almost everywhere in the set $\{F = 0\}$). To estimate the rightmost integral, we note that $\psi \equiv 0$ in $B \setminus V$, and further use that $|\varphi| \le 1$ and $|\psi \le 1|$. Thus we have

$$\begin{aligned}
\varrho_{1,p(\cdot)}(F\,\varphi\,\psi) &\le 3^{p^+} \int_{B \cap V} \big(|F(z)|^{p(z)} + |\nabla F(z)|\big)^{p(z)} dz \\
&\quad + 3^{p^+} \|F\|_\infty^{p^+} \int_B \big(\|\psi\|_\infty |\nabla \varphi(z)|\big) + \|\nabla \psi\|_\infty |\varphi(z)|\big)^{p(z)} dz \\
&\le 3^{p^+} \int_{B \cap V} \big(|F(z)|^{p(z)} + |\nabla F(z)|\big)^{p(z)} dz + \left(\frac{6\,\|F\|_\infty}{\varepsilon}\right)^{p^+} \varrho_{1,p(\cdot)}(\varphi).
\end{aligned}$$

Since $\varrho_{1,p(\cdot)}(\varphi) \le \varepsilon^{p^+ + 1}$, we see that this upper bound goes to zero with ε.

Since ε was arbitrary, we have shown that F can be approximated by Sobolev functions with compact support in $\mathbb{H}$. So it remains to show that Sobolev functions with compact support in $\mathbb{H}$ can be approximated by functions in $C_0^\infty(\mathbb{H})$. But this is easy to do using a cut-off function ψ as before.

For the converse, if $F \in W_0^{1,p(\cdot)}(\mathbb{H})$, then, by definition, $F = \lim \varphi_i$ in $W^{1,p(\cdot)}(\mathbb{H})$, where $\varphi_i \in C_0^\infty(\mathbb{H})$. Since $\operatorname{Tr} \varphi_i = \varphi_i|_{\mathbb{R}^n} \equiv 0$, the claim follows by continuity of $\operatorname{Tr} : W^{1,p(\cdot)}(\mathbb{H}) \to \operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$. (Notice that the proof of the converse does not require the density of smooth functions.) □

References

1. R. Adams: *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press, New York–London, 1975.
2. C. Capone, D. Cruz-Uribe, and A. Fiorenza: The fractional maximal operator on variable L^p spaces, preprint (2004).
3. D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer: The maximal function on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 223–238; **29** (2004), 247–249.

4. L. Diening: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Math. Inequal. Appl.* **7** (2004), no. 2, 245–254.
5. L. Diening: Riesz potential and Sobolev embeddings of generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, *Math. Nachr.* **263** (2004), no. 1, 31–43.
6. L. Diening and P. Hästö: Variable exponent trace spaces, Preprint (2005).
7. L. Diening and M. Růžička: Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, part I, *J. Math. Anal. Appl.* **298** (2004), no. 2, 559–571.
8. L. Diening and M. Růžička: Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, part II, *J. Math. Anal. Appl.* **298** (2004), no. 2, 572–588.
9. X.-L. Fan and D. Zhao: On the spaces spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424–446.
10. P. Harjulehto, Variable exponent Sobolev spaces with zero boundary values, Preprint (2004).
11. P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen: Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$, *J. Funct. Spaces Appl.* **1** (2003), no. 1, 17–33.
12. P. Hästö: Counter-examples of regularity in variable exponent Sobolev spaces, *The p-Harmonic Equation and Recent Advances in Analysis* (Manhattan, KS, 2004), Contemp. Math. 367, Amer. Math. Soc., Providence, RI, 2005.
13. P. Hästö: On the density of smooth functions in variable exponent Sobolev space, preprint (2004). Available at `http://mathstat.helsinki.fi/analysis/varsobgroup/`.
14. O. Kováčik and J. Rákosník: On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41(116)** (1991), 592–618.
15. L. Pick and M. Růžička: An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded, *Expo. Math.* **19** (2001), 369–371.
16. M. Růžička: *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2000.
17. S. Samko: Denseness of $C_0^\infty(\mathbb{R}^n)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbb{R}^n)$, pp. 333–342 in *Direct and Inverse Problems of Mathematical Physics (Newark, DE, 1997)*, Int. Soc. Anal. Appl. Comput. 5, Kluwer Acad. Publ., Dordrecht, 2000.
18. V. V. Zhikov: Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR-Izv.* **29** (1987), no. 1, 33–66. [Translation of *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no. 4, 675–710, 877.]