VARIABLE EXPONENT SPACES ON METRIC MEASURE SPACES

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1. Introduction

The theory of Sobolev spaces on metric measure spaces has been developed by several researchers during the past ten or so years, see e.g. Heinonen[20]. For the existence of a viable theory, it turns out that one should assume that we are dealing with a metric measure space with doubling measure in which a certain Poincaré inequality holds (more on this later). There are many examples of metric measure spaces with a doubling measure supporting a Poincaré inequality. A. Björn[2], has collected the following examples: Unweighted and weighted (for example with Muckenhoupt-type weights) Euclidean spaces including, Riemannian manifolds with nonnegative Ricci curvature, graphs, and the Heisenberg group with the Lebesgue measure and a certain metric.

The study of variable exponent spaces has likewise gained impetus only during the last five years, see e.g. Diening, Hästö and Nekvinda[6] for a review of results in the Euclidean setting. Variable exponent spaces have been proposed e.g. for use in the analysis of certain fluids with complicated behavior.

In this survey we summarize recent results from the variable exponent, metric measure space setting, and some other closely related results. We will show that the variable exponent arises very naturally in the metric measure space setting e.g. when deriving optimal Sobolev embeddings. In the appendix we study a space which shows that the maximal operator can be bounded for piece-wise constant (but non-constant) exponents.

Definitions

By a *metric measure space* we mean a triple (X, d, μ), where X is a set, d is a metric on X and μ is a non-negative Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r \geq 0$ we denote by $B(x,r)$ the open ball centered at x with radius r. A metric measure space X or a measure μ is said to be *doubling* if there is a constant $C \geq 1$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x,r)) \tag{1}$$

for every open ball $B(x,r) \subset X$. The constant C in (1) is called the *doubling constant* of μ. The doubling property is equivalent to the following: there exist C and Q such that if $B(y,R)$ is an open ball in X, $x \in B(y,R)$ and $0 < r \leq R < \infty$, then

$$\frac{\mu(B(x,r))}{\mu(B(y,R))} \geq C_Q \left(\frac{r}{R}\right)^Q. \tag{2}$$

For example, in $\mathbb{R}^n$ with the Lebesgue measure (2) holds with Q equal to the dimension n.

We say that the measure μ is *lower Ahlfors Q–regular* if there exists a constant $C > 0$ such that $\mu(B) \geqslant C \operatorname{diam}(B)^Q$ for every ball $B \subset X$. The measure μ is *Ahlfors Q–regular* if $\mu(B) \approx \operatorname{diam}(B)^Q$ for every ball $B \subset X$. If X is a bounded doubling metric measure space (so that $\mu(X) < \infty$ and $\operatorname{diam}(X) < \infty$), then it is lower Ahlfors Q-regular.

2. Lebesgue spaces

We call a measurable function $p\colon X \to [1, \infty)$ a *variable exponent*. For $A \subset X$ we define $p_A^+ = \operatorname{ess\,sup}_{x\in A} p(x)$ and $p_A^- = \operatorname{ess\,inf}_{x\in A} p(x)$; we use the abbreviations $p^+ = p_X^+$ and $p^- = p_X^-$. For a μ–measurable function $u\colon X \to \mathbb{R}$ we define the *modular*

$$\varrho_{p(\cdot)}(u) = \int_X |u(y)|^{p(y)} d\mu(y)$$

and the *norm*

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0\colon \varrho_{p(\cdot)}(u/\lambda) \leq 1\}.$$

Sometimes we use the notation $\|u\|_{p(\cdot),X}$ when we also want to indicate in what metric space the norm is taken. The *variable exponent Lebesgue spaces on X*,

$L^{p(\cdot)}(X,d,\mu)$, consists of those μ–measurable functions $u\colon X\to\mathbb{R}$ for which there exists $\lambda>0$ such that $\varrho_{p(\cdot)}(\lambda u)<\infty$.

The following facts are easily proven, see Kováčik and Rákosník[22] for the Euclidean case and Harjulehto, Hästö & Pere[18], Section 3, for the metric space case:

- $\|\cdot\|_{p(\cdot)}$ is a norm;
- if $p^+<\infty$, then $\varrho_{p(\cdot)}(f_i)\to 0$ if and only if $\|f_i\|_{p(\cdot)}\to 0$;
- the Hölder inequality $\|fg\|_1\le C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}$ holds ($p>1$);
- the space $L^{p(\cdot)}(X)$ is a Banach space;
- if X is a locally compact doubling space and $p^+<\infty$, then continuous functions with compact support are dense in $L^{p(\cdot)}(X)$.

The following condition has emerged as the right one to guarantee regularity of variable exponent Lebesgue spaces in the Euclidean setting. We say that p is log-*Hölder continuous* if

$$|p(x)-p(y)|\le \frac{C}{-\log d(x,y)},$$

when $d(x,y)\le 1/2$. Following Diening, Lemma 3.2[4], it was shown in Lemma 3.6[18] that if p is log-Hölder continuous, and μ is lower Ahlfors Q-regular, then for all balls $B\subset X$ we have $\mu(B)^{p_B^- - p_B^+}\le C$.

It was shown by Diening[4] that the log-Hölder condition is sufficient for the local boundedness of the maximal operator. Recall that the maximal operator is defined by

$$Mu(x)=\sup_{r>0} ⨍_{B(x,r)} |u(x)|\,d\mu(x).$$

Moreover, an example by Pick and Růžička[25] shows that this is the best possible modulus of continuity under which this claim holds. It turns out that log-Hölder continuity is still sufficient for local boundedness in the metric measure spaces setting, however, it is no longer necessary in the modulus of continuity sense:

Theorem 2.1. *[Theorem 4.3[18]] Let X be a bounded doubling space. Suppose that p is* log-*Hölder continuous with $1<p^-\le p^+<\infty$. Then*

$$\|\mathcal{M}f\|_{p(\cdot)}\le C\|f\|_{p(\cdot)}.$$

To show that log-Hölder continuity is not necessary, the following example was given in Example 4.5[18]. Let $X_1=\{(x,0)\in\mathbb{R}^2: 0\le x<1/4\}$ and $X_2=\{(x,y)\in B(0,1/2)\colon x<0\}$ and define $(X,\mu)=(X_1,m_1)\cup(X_2,m_2)$, where m_i denotes the i-dimension Lebesgue measure. We set the exponent p equal to s on X_1 and to t on X_2 ($s,t>1$).

In Theorem 4.7[18], it was shown that the maximal operator is bounded in this space for certain values of s and t, but not for all. The situation is shown in Figure 1. Harjulehto, Hästö and Pere[18] were not able to settle the boundedness of the maximal operator for the critical case $s = t/(2-t)$ (for $t < 2$), the upper curve in the figure). In the appendix of this article we give a new simpler proof for the boundedness of the maximal operator, which also works in the critical case.

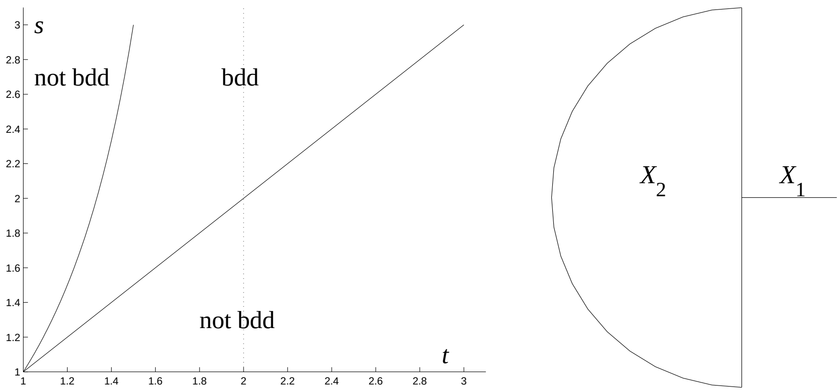

FIGURE 1. A sketch of the area where the maximal operator is bounded (left) and the space X (right).

It was pointed out by Lars Diening that the measure in this example is not doubling.

Open Problem 1. Is there an example of a Lebesgue space with doubling measure and piecewise constant exponent on which the maximal operator is bounded?

Variable dimension spaces will be defined below in Section 4. The dimension seems to play a fundamental role in the example above. Thus we may ask

Open Problem 2. Is it possible to derive an optimal modulus of continuity for the exponent in terms of the variability of the dimension in the metric measure space setting?

3. Hajłasz and Newtonian spaces

There are many different ways to define Sobolev spaces, with a fixed exponent, on metric measure spaces. In this chapter we discuss how two commanly used spaces, namely Hajłasz and Newtonian spaces, can be generalized to the variable exponent case. We restrict our attention to exponents with $1 < p^- \leq p^+ < \infty$.

Hajłasz spaces

We say that a $p(\cdot)$-integrable function u belongs to *Hajłasz space* $M^{1,p(\cdot)}(X)$ if there exists a non-negative $g \in L^{p(\cdot)}(X)$ such that

$$|u(x) - u(y)| \leqslant d(x,y)\,(g(x) + g(y)) \tag{3}$$

for μ-almost every $x, y \in X$. The function g is called a *Hajłasz gradient* of u. In $\mathbb{R}^n$ we may use $\mathcal{M}(|\nabla u|)$ as a Hajłasz gradient of $u \in W^{1,q}(\mathbb{R}^n)$, where $q > 1$ is a constant. We equip $M^{1,p(\cdot)}(X)$ with the norm

$$\|u\|_{M^{1,p(\cdot)}(X)} = \|u\|_{p(\cdot)} + \inf \|g\|_{p(\cdot)},$$

where the infimum is taken over all Hajłasz gradients of u. Following the original arguments of P. Hajłasz[12] it is easy to prove that $M^{1,p(\cdot)}(X)$ is a Banach space and Lipschitz continuous functions are dense. Integrating both sides of (3) over y in a bounded space X of finite measure and using Hölder's inequality we find that

$$|u(x) - u_X| \leqslant \operatorname{diam}(X)\Big(g(x) + \frac{C\|1\|_{p'(\cdot)}}{\mu(X)}\|g\|_{p(\cdot)}\Big),$$

where u_X denotes the integral average of u over X. This point-wise estimate leads easily to the *Poincaré inequality*

$$\|u - u_X\|_{p(\cdot)} \leqslant C\big(p^-, p^+, \mu(X)\big) \operatorname{diam}(X)\|g\|_{p(\cdot)}.$$

If the measure is atomless, then $L^{p(\cdot)}$ is reflexive. Using this and Mazur's lemma we find that for every $u \in M^{1,p(\cdot)}(X)$ there exists a unique Hajłasz gradient of u which minimizes the norm. By unique we mean that if g and g' are minimal Hajłasz gradients of u, then $\|g - g'\|_{L^{p(\cdot)}(X)} = 0$.

Newtonian spaces

A curve γ in X is a non-constant continuous map $\gamma : I \to X$, where $I = [a, b]$ is a closed interval in $\mathbb{R}$.

Let Γ be a family of rectifiable curves. We denote by $F(\Gamma)$ the set of all Borel measurable functions $\rho\colon X \to [0, \infty]$ such that

$$\int_\gamma \rho\, ds \geqslant 1$$

for every $\gamma \in \Gamma$, where ds represents integration with respect to path length. We define the $p(\cdot)$*-modulus of* Γ by

$$\mathrm{M}_{p(\cdot)}(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho(x)^{p(x)}\, d\mu(x).$$

If $F(\Gamma) = \emptyset$, then we set $\mathrm{M}_{p(\cdot)}(\Gamma) = \infty$. The arguments from $\mathbb{R}^n$ imply that the $p(\cdot)$-modulus is an outer measure on the space of all curves of X, for a proof see Lemma 2.1[17].

A non-negative Borel measurable function ρ on X is a $p(\cdot)$*-weak upper gradient* of u if there exists a family Γ of rectifiable curves with $\mathrm{M}_{p(\cdot)}(\Gamma) = 0$ and

$$|u(x) - u(y)| \leqslant \int_\gamma \rho\, ds$$

for every rectifiable curve $\gamma \notin \Gamma$ with endpoints x and y. In the Euclidean case we would use $|\nabla u|$ as the upper gradient.

The *Newtonian space* $N^{1,p(\cdot)}(X)$ is the collection of functions in $L^{p(\cdot)}(X)$ with a weak upper gradient in $L^{p(\cdot)}(X)$ equipped with the norm

$$\|u\|_{N^{1,p(\cdot)}(X)} = \|u\|_{p(\cdot)} + \inf \|\rho\|_{p(\cdot)},$$

where the infimum is taken over all weak upper gradients of u. The Newtonian space $N^{1,p(\cdot)}(X)$ is a Banach space, Theorem 3.4[19].

A metric measure space X is said to *support a* $(1,q)$*-Poincaré inequality* if there exists a constant $C > 0$ such that for all open balls B in X and all pairs of functions u and ρ defined on B the inequality

$$\fint_B |u - u_B|\, d\mu \leqslant C \operatorname{diam}(B) \left(\fint_B \rho^q \, d\mu \right)^{\frac{1}{q}}$$

holds whenever ρ is an upper gradient of u on B and u in integrable on B.

If $q > 1$ is a constant and the space X supports a $(1,q)$-Poincaré inequality, then the Newtonian space $N^{1,q}(X)$ is reflexive[3]. The same condition also implies that Lipschitz continuous functions are dense in Newtonian space. In Harjulehto, Hästö & Pere[19] it was shown that a $(1,1)$-Poincaré inequality implies the density of Lipschitz functions in Newtonian space.

Open Problem 3. Are Lipschitz continuous functions dense in $N^{1,p(\cdot)}(X)$ when the space X supports a $(1,p(\cdot))$-Poincaré inequality?

Open Problem 4. Show that $N^{1,p(\cdot)}(X)$ is reflexive, assuming either a $(1,1)$- or a $(1,p(\cdot)$-Poincaré inequality.

We end this chapter by studying when Hajłasz, Newtonian and classical Sobolev spaces agree. All the proof can be found in Harjulehto, Hästö & Pere[19].

Theorem 3.1. *We have the following relations between Hajłasz, Newtonian and classical Sobolev spaces:*

(1) $M^{1,p(\cdot)}(\mathbb{R}^n) \subset W^{1,p(\cdot)}(\mathbb{R}^n)$;
(2) If the maximal operator is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ *to itself, then* $M^{1,p(\cdot)}(\mathbb{R}^n) = W^{1,p(\cdot)}(\mathbb{R}^n)$;
(3) If $\Omega \subset \mathbb{R}^n$ *is open, then* $N^{1,p(\cdot)}(\Omega) \subset W^{1,p(\cdot)}(\Omega)$;
(4) If $\Omega \subset \mathbb{R}^n$ *is open,* $1 < p^- \leqslant p^+ < \infty$ *and* $C^1(\Omega)$ *is dense in* $W^{1,p(\cdot)}(\Omega)$, *then* $N^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega)$;
(5) $M^{1,p(\cdot)}(X) \subset N^{1,p(\cdot)}(X)$;
(6) If X *supports a* $(1,1)$*-Poincaré inequality and if the Hardy-Littlewood maximal operator is bounded from* $L^{p(\cdot)}(X)$ *to itself, then* $M^{1,p(\cdot)}(X) = N^{1,p(\cdot)}(X)$.

Recently P. Hajłasz[13] has generalized the definition of $M^{1,1}(\mathbb{R}^n)$ so that $M^{1,1}(\mathbb{R}^n) = W^{1,1}(\mathbb{R}^n)$. This raises the question whether the assumption of the boundedness of the maximal operator in (2), above, can be relaxed if this generalized definition is used.

4. Sobolev embeddings $p < Q$

Of the various methods for proving Sobolev inequalities in constant exponent spaces, the one based on the Riesz potential is best suited for the variable exponent case.

Let $\alpha > 0$ be fixed. We define the Riesz potential as

$$I_\alpha^\Omega |u|(x) = \int_\Omega \frac{|u(y)|\, d(x,y)^\alpha}{\mu\big(B(x,d(x,y))\big)}\, d\mu(y).$$

(For technical reasons we sometimes use a modified Riesz potential J_α^Ω from Hajłasz and Koskela[14] with the property that $I_\alpha^\Omega|u|(x) \le J_\alpha^\Omega u(x)$ for almost every $x \in X$ if the measure μ is doubling.) Riesz potentials in the variable exponent, Euclidean setting have to been studied for instance in [5,10,21,24].

It was shown recently in Harjulehto, Hästö and Latvala[16] that it is possible to study variable dimension, variable exponent spaces without any extra work due to the the variable dimension. In this context the exact definition of variable exponent is as follows. If $Q\colon X \to (0,\infty)$ is a BOUNDED function, then we say that μ is *Ahlfors $Q(\cdot)$-regular* if

$$\mu(B(x,r)) \approx r^{Q(x)}$$

for all $x \in X$ and $r \le \operatorname{diam} X$. As an example of a variable exponent space, a variable parameter von Koch curve was constructed in Section 3[16], see Figure 1.

Various embedding theorems for the modified Riesz potential were proven in Harjulehto, Hästö & Latvala[16]. We quote here only one representative and easily understandable result:

Theorem 4.1. *[Corollary 5.4[16]] Let μ be lower Ahlfors $Q(\cdot)$-regular and doubling in a bounded metric space X. Let p be log-Hölder continuous in X and let $1 < p^- \le p^+ < \infty$. If $1 < \inf \frac{Q(x)}{p(x)}$, then for each ball $B \subset X$ we have*

$$\|u - u_B\|_{L^{p^*(\cdot)}(B)} \le C\|g\|_{L^{p(\cdot)}(5B)}, \quad p^*(x) = \frac{Q(x)p(x)}{Q(x)-p(x)},$$

for every $u \in M^{1,p(\cdot)}(X)$.

The question of necessary condition for Sobolev embeddings remains largely open, both in the Euclidean and the metric space cases. The only known counterexample is due to Kováčik and Rákosník Example 3.2[22], which was slightly improved in Diening, Hästö & Nekvinda[6], where it was shown that there exists a continuous exponent p on a regular domain $\Omega \subset \mathbb{R}^2$ such that

$$W^{1,p(\cdot)}(\Omega) \not\hookrightarrow L^{p^*(\cdot)}(\Omega).$$

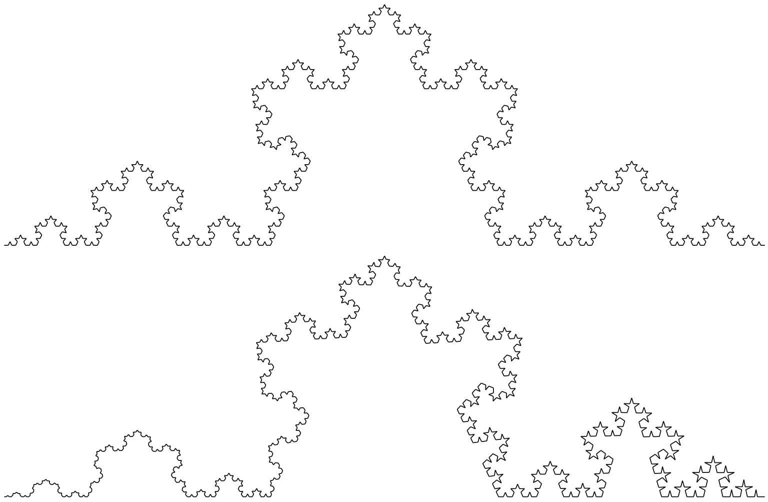

FIGURE 2. An ordinary Koch curve (upper) and a variable dimension Koch curve (lower).

The exponent in the this example was not uniformly continuous. Thus Diening, Hästö and Nekvinda[6] asked the following:

Open Problem 5. Are there counter-examples to the Sobolev embedding in regular domains for uniformly continuous exponents?

Open Problem 6. We saw before that the maximal operator features vastly different behavior in the metric space case as compared with the Euclidean one. Are similar examples relevant for the Sobolev embedding?

5. Exponential inequalities

First we consider a Trudinger-type inequality in the case $p(x) = Q(x)$. It turns out that the embeddings in this case are much the same as in the classical case. Then we consider the case when the exponent tends to the dimension at some point from above. Depending on the speed at which the exponent approaches the critical value, we get different embeddings. Section 5 deals with the case there $p(x) = Q$ at one point or on a sphere.

Trudinger-type embeddings

The introduction of variable dimension spaces allowed generalizations of the Trudinger embedding (i.e. $p = n$) to the variable exponent case. Recall that in the classical case the Trudinger embedding states that $W^{1,n}$ embeds into $\exp L^{n'}$, where the exponent $n' = n/(n-1)$ is the best possible. This generalizes as follows:

Theorem 5.1. *Let X be a bounded connected doubling space, and assume that μ is lower Ahlfors $Q(\cdot)$-regular, where Q is* log-Hölder *continuous and $1 < Q^- \le Q^+ < \infty$. Then there is a constant C_1, depending on X, such that*

$$\fint_B \exp\left(\frac{C_1|u(x) - u_B|}{\|g\|_{L^{Q(\cdot)}(5B)}}\right)^{Q'(x)} d\mu(x) \le 2$$

for every ball $B \subset X$ and for each $u \in M^{1,Q(\cdot)}(X)$ with Hajłasz gradient g.

The super-critical case

For fixed $x_0 \in X$, where X is a doubling metric measure spaces and Q is as in (2), let us consider an exponent $p(x)$ such that

$$p(x) \ge \frac{Q}{\alpha} \qquad \text{for } x \in B(x_0, r_0).$$

Set $B_0 = B(x_0, r_0)$ for simplicity. We further assume that

$$p(x) = \frac{Q}{\alpha} + \frac{a \log\left(\log(1/|x_0 - x|)\right)}{\log(1/|x_0 - x|)} + \frac{b}{\log(1/|x_0 - x|)} \tag{4}$$

for $x \in B_0$, where $0 < r_0 < 1/4$, $a \ge 0$ and b is a real number. If $a > 0$, then we can take r_0 so small that

$$p(x) > \frac{Q}{\alpha} \qquad \text{when } x \in B_0 \setminus \{x_0\}.$$

Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} < \infty$. In connection with Sobolev's imbedding theorem, the borderline value for a will be shown to be

$$a = \frac{Q - \alpha}{\alpha^2}.$$

Theorem 5.2. *[Theorems 5.6 and 5.9[11]] If $0 \le a < (Q - \alpha)/\alpha^2$, then there exist positive constants c_1 and c_2 such that*

$$\int_{B_0} \exp\left(c_1 (I_\alpha f(x))^{Q/(Q-\alpha-a\alpha^2)}\right) d\mu(x) \le c_2$$

for all nonnegative measurable functions f on B_0 with $\|f\|_{p(\cdot)} \le 1$.

If $a = (Q - \alpha)/\alpha^2$, then there exist positive constants c_1 and c_2 such that

$$\int_{B_0} \exp\left(\exp(c_1 (I_\alpha f(x))^{Q/(Q-\alpha)})\right) d\mu(x) \le c_2$$

for all nonnegative measurable functions f on B_0 with $\|f\|_{p(\cdot)} \le 1$.

Suppose next that $X = \mathbb{R}^n$ (and $Q = n$). Let φ be a non-increasing function the interval on $(0, \infty)$ and consider $p(x)$ satisfying

$$p(x) = \frac{n}{\alpha} + \frac{\log \varphi(|x_0 - x|)}{\log(1/|x_0 - x|)} \qquad \text{for } x \in B_0.$$

Theorem 5.3. *[Theorem 7.2[11]] Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. If*

$$\int_0^1 \varphi(t)^{-\alpha^2/(n-\alpha)} t^{-1} dt < \infty, \tag{5}$$

then $I_\alpha f$ is continuous on B_0 and

$$\lim_{x \to x_0} \Phi(|x - x_0|)^{-1} |I_\alpha f(x) - I_\alpha f(x_0)| = 0,$$

where $\Phi(r) = \left(\int_0^r \varphi(t)^{-\alpha^2/(n-\alpha)} t^{-1} dt \right)^{(n-\alpha)/n}$.

As a special case we consider $\varphi(r) = c(\log(1 + (1/r)))^a$ with $c > 0$ and $a \ge 0$. Then φ satisfies (5) if and only if $a > (n-\alpha)/\alpha^2$. Suppose that $p(\cdot)$ satisfies (4) for $a > (n-\alpha)/\alpha^2$ and let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} < \infty$. Then $I_\alpha f$ is continuous on B_0 and

$$\lim_{x \to x_0} (\log(1/|x - x_0|))^A |I_\alpha f(x) - I_\alpha f(x_0)| = 0,$$

where $A = (a\alpha^2 - (n-\alpha))/n$.

Futamura, Mizuta and Shimomura[11] also considered the case of a variable exponent p satisfying

$$p(x) = \frac{n}{\alpha} + \frac{a \log(\log(1/(1-|x|)))}{\log(1/(1-|x|))} + \frac{b}{\log(1/(1-|x|))}$$

in the unit ball of $\mathbb{R}^n$, where $a \ge 0$ and b is a real number. Assuming that $p(x) > \frac{n}{\alpha}$ when $x_n \neq 0$ they derived analogous results on exponential and double exponential integrability of Riesz potentials.

Generalizations

Let $p(r)$ be a continuous function on $[0, \infty)$ such that

$$p(r) = p_0 + \frac{a \log(\log(1/r))}{\log(1/r)} + \frac{b}{\log(1/r)}$$

for $0 < r < r_0 < 1/4$; set $p(0) = p_0$ and $p(r) = p(r_0)$ for $r > r_0$. Let K be a compact set in $\mathbb{R}^n$. Set

$$p(x) = p(\delta_K(x)),$$

where $\delta_K(x)$ denotes the distance of x from K.

(i) If $K = \{x_0\}$, then $p(\cdot)$ is nothing but the first case.
(ii) If $K = \partial B(0,1)$, then $p(\cdot)$ is nothing but the second case.

Open Problem 7. Under what conditions on the set K above do Sobolev inequalities still hold in the variable exponent spaces?

We can investigate the behavior of our spaces further by taking a closer look at the critical case, and defining the exponent $p(\cdot)$ by

$$p(x) = \frac{n}{\alpha} + \frac{b}{\log(1/|x_0 - x|)} + \frac{n-\alpha}{\alpha^2}\frac{\log(\log(1/|x_0 - x|))}{\log(1/|x_0 - x|)} + \frac{a\log(\log(\log(1/|x_0 - x|)))}{\log(1/|x_0 - x|)}.$$

Even more generally, we can consider an exponent with continuity modulus given by some function φ.

Open Problem 8. With p as before, is it possible to derive exponential integrability results similar to those in Edmunds, Gurka and Opic[7,8].

Appendix A. The boundedness of the maximal operator for a discontinuous exponent

In this appendix we consider the boundedness of the maximal operator on a certain metric measure space. The result is an improvement of a result by Harjulehto, Hästö and Pere[18].

Take positive integers ℓ and m, and set $n = \ell + m$. For a point $x \in \mathbb{R}^n$, we write $x = (x', x'')$, where $x' = (x_1, x_2, \dots, x_\ell) \in \mathbb{R}^\ell$ and $x'' = (x_{\ell+1}, x_{\ell+2}, \dots, x_n) \in \mathbb{R}^m$. Let

$$X_1 = \left\{x = (x', 0) \in \mathbb{R}^n : |x| < 1/4, x_1 > \tfrac{\sqrt{3}}{2}|x|\right\}$$

and

$$X_2 = \left\{x = (x', x'') \in \mathbb{R}^n : |x| < 1/2, x_1 < 0\right\}.$$

Consider the metric measure space (X, μ) given by

$$(X, \mu) = (X_1, H_\ell) \cup (X_2, H_n),$$

where H_i denotes the i-dimensional Lebesgue measure. For $s > 1$ and $t > 1$, set

$$p(x) = \begin{cases} s & \text{if } x \in X_1, \\ t & \text{if } x \in X_2. \end{cases}$$

Theorem A.1. *Consider the maximal operator on the space $L^{p(\cdot)}(X)$, where X and p are defined above.*

(i) If $t > s$, then $\mathcal{M}$ is not bounded.
(ii) If $s \ge t \ge n/m$, then $\mathcal{M}$ is bounded.
(iii) If $t < n/m$ and $t \le s \le \ell t/(n - mt)$, then $\mathcal{M}$ is bounded.
(iv) If $t < n/m$ and $s > \ell t/(n - mt)$, then $\mathcal{M}$ is not bounded.

To prove the theorem, we use the following result.

Lemma A.1. *In case $s \geq t$, $\mathcal{M}$ is bounded if and only if*

$$\|\mathcal{M}f_2\|_{s,X_1} \leq C\|f_2\|_{t,X_2} \tag{A.1}$$

for every $f \in L^{p(\cdot)}(X)$, where $f_2 = f\chi_{X_2}$.

Proof. By the classical boundedness of maximal functions in metric spaces and Hölder's inequality, we see that

$$\begin{aligned}\|\mathcal{M}f\|_{p(\cdot)} &\leq \|\mathcal{M}f_1\|_{s,X_1} + \|\mathcal{M}f_2\|_{s,X_1} + \|\mathcal{M}f\|_{t,X_2} \\ &\leq C\|f_1\|_{s,X} + \|\mathcal{M}f_2\|_{s,X_1} + C\|f\|_{t,X} \\ &\leq C(\|f_1\|_{p(\cdot)} + \|f\|_{p(\cdot)}) + \|\mathcal{M}f_2\|_{s,X_1},\end{aligned}$$

where $f_1 = f\chi_{X_1}$ and $f_2 = f\chi_{X_2}$. Hence (A.1) implies that $\mathcal{M}$ is bounded.

Conversely suppose $\mathcal{M}$ is bounded on $L^{p(\cdot)}(X)$. Then we have

$$\|\mathcal{M}f_2\|_{s,X_1} \leq \|\mathcal{M}f_2\|_{p(\cdot)} \leq C\|f_2\|_{p(\cdot)} = C\|f_2\|_{t,X_2},$$

which implies (A.1). □

Now we move on to the proof of the theorem.

Proof. To prove (i), consider the function

$$f(y) = \begin{cases} |y'|^{-2\ell/(s+t)} & \text{when } y \in X_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that

$$\int_X |f(y)|^{p(y)} d\mu(y) = \int_{X_1} |y'|^{-2\ell s/(s+t)} dH_\ell(y) < \infty$$

since $-2\ell s/(s+t) + \ell > 0$.

Let $\tilde{X}_2 = \{x \in X_2 : -x_1 > \frac{\sqrt{3}}{2}|x|\}$. If $x \in \tilde{X}_2$, then, letting $r = |x| + |x|^{n/\ell}$ and noting that $\mu(B(x,r)) \approx |x|^n$, we find that

$$\begin{aligned}\mathcal{M}f(x) &\geq \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap X_1} |f(y)|\, dH_\ell(y) \\ &\geq C|x|^{-n} \int_{B(0,|x|^{n/\ell})\cap X_1} |f(y)|\, dH_\ell(y) \\ &\geq C|x|^{-2n/(s+t)},\end{aligned}$$

so that

$$\int_X |\mathcal{M}f(x)|^{p(x)}\, d\mu(x) \geq C \int_{\tilde{X}_2} |x|^{-2nt/(s+t)} dH_n(x) = \infty$$

since $-2nt/(s+t) + n < 0$.

Next we prove (ii) and (iii). Take a ball $B(x,r)$ such that $x \in X_1$ and $B(x,r) \cap X_2 \neq \emptyset$. For $x \in X_1$, we have

$$\begin{aligned}
&\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f_2(y)|\, d\mu(y) \\
&\qquad = \frac{1}{H_\ell(B(x,r)\cap X_1) + H_n(B(x,r)\cap X_2)} \int_{B(x,r)\cap X_2} |f_2(y)|\, dH_n(y) \\
&\qquad \le \frac{C}{\min\{r^\ell, 1\}} \int_{B(x,r)\cap X_2} |f_2(y)|\, dH_n(y) \\
&\qquad \le C \int_{B(x,r)\cap X_2} |x-y|^{-\ell} |f_2(y)|\, dH_n(y),
\end{aligned}$$

which implies that

$$\mathcal{M} f_2(x) \le C \int_{X_2} |x-y|^{m-n} |f_2(y)|\, dH_n(y).$$

If $s \ge t$ when $t \ge n/m$, or $\ell t/(n-mt) \ge s \ge t$ when $t < n/m$, then we have by the Sobolev imbedding theorem (see Adams and Fournier[1])

$$\|\mathcal{M} f_2\|_{s,X_1} \le C \left(\int_{X_1} \left(\int_{X_2} |x-y|^{m-n} |f_2(y)|\, dH_n(y) \right)^s dH_\ell(x) \right)^{1/s} \le C \|f_2\|_{t,X_2},$$

which proves (ii) and (iii) by the aid of Lemma A.1.

To prove (iv), consider the function

$$g(x) = |x|^{-n/t} \left(\log (1/|x|)\right)^{-2/t} \chi_{X_2}.$$

Then

$$\begin{aligned}
\int_X g(y)^{p(y)} d\mu(y) &= \int_{X_2} |y|^{-n} \left(\log (1/|y|)\right)^{-2} dH_n(y) \\
&\le C \int_0^{1/2} r^{-1} \left(\log (1/r)\right)^{-2} dr < \infty,
\end{aligned}$$

so $g \in L^{p(\cdot)}(X)$. Furthermore, we have for $x \in X_1$

$$\begin{aligned}
\mathcal{M} g(x) &\ge \frac{1}{\mu(B(x,2|x|))} \int_{B(x,2|x|)} g(y)\, d\mu(y) \\
&\ge C|x|^{-\ell} \int_{B(0,|x|)\cap X_2} |y|^{-n/t} \left(\log (1/|y|)\right)^{-2/t} dH_n(y) \\
&\ge C|x|^{-\ell} \int_0^{|x|} r^{-n/t+n-1} \left(\log (1/r)\right)^{-2/t} dr \\
&\ge C|x|^{-n/t+n-\ell} \left(\log (1/|x|)\right)^{-2/t},
\end{aligned}$$

so that

$$\begin{aligned}
\int_{X_1} (\mathcal{M} g(x))^s\, dH_\ell(x) &\ge C \int_{X_1} |x|^{-sn/t+sm} \left(\log (1/|x|)\right)^{-2s/t} dH_\ell(x) \\
&\ge C \int_0^{1/4} r^{-sn/t+sm+\ell-1} \left(\log (1/r)\right)^{-2s/t} dr = \infty
\end{aligned}$$

since $-sn/t + sm + \ell < 0$. Hence $\mathcal{M}g \notin L^{p(\cdot)}(X)$. □

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