UNIFORMLY SEPARATED SETS AND GROMOV HYPERBOLICITY OF DOMAINS WITH THE QUASIHYPERBOLIC METRIC

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Abstract. In this article we prove comparative results on the Gromov hyperbolicity of plane domains equipped with the quasihyperbolic metric. By a comparative result we mean one which assumes hyperbolicity in one domain and obtains it in a different domain somehow related to the original one. We derive a characterization (simple to check in practical cases) of the Gromov hyperbolicity of a plane domain Ω^* obtained by deleting from the original domain Ω any uniformly separated union of compact sets. We present as well a result about stability of hyperbolicity.

1. Introduction

In the 1980s Mikhail Gromov, cf. [14], introduced a notion of abstract hyperbolic spaces, which have thereafter been studied and developed by many authors, e.g. [10, 11, 21, 33]. Initially, the research was mainly centered on hyperbolic group theory, but lately researchers have shown an increasing interest in more direct studies of spaces endowed with metrics used in geometric function theory, e.g. [5, 6, 8, 9, 15, 19, 20].

One of the primary questions is naturally whether a metric space (X, d) is hyperbolic in the sense of Gromov or not. The most classical examples, mentioned in every textbook on this topic, are metric trees, the classical Poincaré hyperbolic metric developed in the unit disk and, more generally, simply connected complete Riemannian manifolds with sectional curvature $K \leqslant -k^2 < 0$.

However, it is not easy to determine whether a given space is Gromov hyperbolic or not. In recent years several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Klein-Hilbert and Kobayashi metrics are Gromov hyperbolic (under particular conditions on the domain of definition, see [8, 19] and [5]); the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [15]). Also, in [20] the hyperbolicity of the conformal modulus metric μ and the related so-called Ferrand metric λ^*, is studied.

Since the Poincaré metric is also the metric giving rise to what is commonly known as the hyperbolic metric when speaking about open domains in the complex plane or in Riemann surfaces, it could be expected that there is a connection between the notions of hyperbolicity. For simply connected subdomains Ω of the complex plane, it follows directly from the Riemann mapping theorem that the metric space (Ω, h_Ω) is in fact Gromov hyperbolic. However, as soon as simply connectedness is omitted, there is no

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immediate answer to whether the space h_Ω is hyperbolic or not. The question has lately been studied in [3], [13], [16]–[18] and [23]–[31].

The related quasihyperbolic metric has also recently been a topic of interest regarding the question of Gromov hyperbolicity. In [9], Bonk, Heinonen and Koskela found necessary and sufficient conditions for when a planar domain D endowed with the quasihyperbolic metric is Gromov hyperbolic. This was extended by Balogh and Buckley, [6]: they found two different necessary and sufficient conditions which work in Euclidean spaces of all dimensions and also in metric spaces under some conditions.

A different approach is to try to connect the Gromov hyperbolicity of the hyperbolic and quasihyperbollic metric. One may ask whether there exists a domain such that one of these metrics is Gromov hyperbolic and the other is not. Since h_Ω and k_Ω are not in general quasi-isometric, it is natural to expect that this be the case. In [17] we studied this question in Denjoy domains and came to the surprising conclusion that in fact h_Ω is Gromov hyperbolic if and only if k_Ω is. Whether or not this is true in general domains remains an open question.

Earlier, we transfered results on the quasihyperbolic metric, which were inspired by [6, 9], to the hyperbolic metric in [16]. In this article we take the opposite approach: we prove for general domains with the quasihyperbolic metric results which were proved in [13] for the hyperbolic metric. Our results are built up following the methods of [13]; it is conceivable though not apparent that these results could be proved more easily using the characterization from [6]. However, we believe that using the same methods speaks more strongly to the question of whether there is a difference between the Gromov hyperbolicity of the two metrics.

The main aim in this paper is obtaining global results on hyperbolicity of general plane domains with their quasihyperbolic metric from local information. That was the idea that led us to identify some "ends" of a domain Ω^* with closed sets $\{E_n\}_n$ removed from an original domain Ω, in such a way that $\Omega^* = \Omega \setminus \cup_n E_n$.

Theorem 4.1 allows us, in many cases, to study the hyperbolicity of a domain in terms of the local hyperbolicity of its ends; this fact is a significant simplification in the study of the hyperbolicity. The theorem provides a necessary and sufficient condition. Besides, we have determined which are the relevant parameters in the hyperbolicity constant of Ω^*.

Theorem 4.2 allows one, in many cases, to forget isolated points and continua in the boundary in order to study the hyperbolicity of a domain; this fact is a significant simplification in the topology of the domain, and therefore makes the problem easier. The main advantage of this result is that it does not require to check the uniform hyperbolicity of $\{V_n^*\}_n$ as it was required in Theorem 4.1. The theorem gives also a necessary and sufficient condition. Theorem 4.4 is a simplified version of Theorem 4.2 if every E_n is an isolated point.

Proposition 3.8 is an important tool in the proof of Theorems 4.1, 4.2 and 4.4. It guarantees the hyperbolicity of domains of finite type, with hyperbolicity constants which only depend on simple metric restrictions. It is important by itself, since it can be also viewed as a result on uniform hyperbolicity and stability of the hyperbolicity. Corollary 4.8 is a further result on stability of hyperbolicity.

These results are similar to some theorems for the Poincaré metric (see [13, 25]), but have simpler statements in the case of the quasihyperbolic metric.

It is a remarkable fact that almost every constant appearing in the results of this paper depends just on a small number of parameters. This is a common place in the theory of hyperbolic spaces (see e.g. [12]) and is also typical of surfaces with curvature -1 (see e.g. the lemmas in [28] and [32], and Theorem 3.1 in [24]). In fact, this simple dependence is a crucial fact in the proof of Theorem 4.1.

2. Background

Definition 2.1. Let us fix a point w in a metric space (X, d). We define the *Gromov product* of $x, y \in X$ with respect to the point w as

$$(x|y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big) \geqslant 0\,.$$

We say that the metric space (X, d) is *δ-hyperbolic* ($\delta \geqslant 0$) if

$$(x|z)_w \geqslant \min\big\{(x|y)_w, (y|z)_w\big\} - \delta\,,$$

for every $x, y, z, w \in X$. We say that X is *hyperbolic* (in the Gromov sense) if the value of δ is not important. Usually we just say that d (instead of (X, d)) is hyperbolic.

Definition 2.2. If $\gamma : [a, b] \longrightarrow X$ is a continuous curve in a metric space (X, d), we can define the length of γ as

$$L(\gamma) := \sup\Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \,:\, a = t_0 < t_1 < \cdots < t_n = b \Big\}\,.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[x, y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected.

Definition 2.3. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. We say that T is *δ-thin* if for every $x \in [x_i, x_j]$ we have that $d(x, [x_j, x_k] \cup [x_k, x_i]) \leqslant \delta$. The space X is *δ-thin* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin.

A basic result is that hyperbolicity is equivalent to Rips condition:

Theorem 2.4 (p. 41, [12])**.** *Let us consider a geodesic metric space X.*

(1) *If X is δ-hyperbolic, then it is 4δ-thin.*
(2) *If X is δ-thin, then it is 4δ-hyperbolic.*

Examples of hyperbolic spaces:

(1) Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic (see e.g. [12, p. 29]).
(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [12, p. 52]).
(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [12, p. 29]).

We present now the class of maps which play the main role in the theory.

Definition 2.5. A function between two metric spaces $f : X \longrightarrow Y$ is a *quasi-isometry* if there are constants $a \geqslant 1$, $b \geqslant 0$ with

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leqslant d_Y(f(x_1), f(x_2)) \leqslant a\, d_X(x_1, x_2) + b\,, \qquad \text{for every } x_1, x_2 \in X.$$

A such function is called an (a, b)-*quasi-isometry.* We say that X is *quasi-isometric* to Y if there exists a quasi-isometry f from X to Y, with $f(X) = Y$.

Notice that a quasi-isometry can be discontinuous. Quasi-isometries are important since they preserve hyperbolicity:

Theorem 2.6 (p. 88, [12])**.** *Let us consider an (a, b)-quasi-isometry between two geodesic metric spaces $f : X \longrightarrow Y$. If Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which only depends on δ, a and b. Besides, if $f(X) = Y$, then X is hyperbolic if and only if Y is hyperbolic.*

Recall that a domain $\Omega \subset \mathbb{C}$ is said to be of *hyperbolic type* if it has at least two finite boundary points. The universal covering of such domain is the unit disk $\mathbb{D} = \{z \in \mathbb{C} :\ |z| < 1\}$. In Ω we can consider its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk $ds = 2|dz|/(1-|z|^2)$ or, equivalently, the Poincaré metric $ds = |dz|/\operatorname{Im} z$ of the the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} :\ \operatorname{Im} z > 0\}$. Therefore, any simply connected subset of Ω is isometric to a subset of $\mathbb{D}$. With this metric, Ω is a geodesically complete Riemannian manifold with constant curvature -1, and therefore Ω is a geodesic metric space. By h_Ω we denote the Poincaré distance in Ω or the Poincaré length of a curve in Ω.

The quasihyperbolic metric is the distance induced by the density $1/\delta_\Omega(z)$, where $\delta_\Omega(z) := d_{Eucl}(z, \partial\Omega)$ denotes euclidean distance. By k_Ω we denote the quasihyperbolic distance in Ω or the quasihyperbolic length of a curve in Ω. Euclidean distance or length will be denoted by the symbols d_{Eucl} or ℓ_{Eucl} respectively. An Euclidean ball with center p and radius r will be denoted by $B_{Eucl}(p, r)$. A quasihyperbolic ball with the same center and radius will be denoted by $B_{k_\Omega}(p, r)$.

If Ω_0 is an open subset of Ω, in Ω_0 we always consider its usual quasihyperbolic or Poincaré metric (independent of Ω). If D is a closed subset of Ω, we always consider in D the inner metric obtained by the restriction of the quasihyperbolic metric in Ω, that is

$$k_D(z, w) := k_{\Omega|D}(z, w) := \inf \big\{ k_\Omega(\gamma) :\ \gamma \subset D \text{ is a continuous curve joining } z \text{ and } w \big\} \geqslant k_\Omega(z, w)\,.$$

It is clear that $k_D(\gamma) = k_\Omega(\gamma)$ for every curve $\gamma \subset D$. We always require that ∂D is a union of pairwise disjoint Lipschitz curves; this fact guarantees that $(D, k_{\Omega|D})$ is a geodesic metric space.

We collect now some definitions and results that we will use later.

Definition 2.7. Let (X, d) be a metric space, and let $\{X_n\}_{n\in\Lambda} \subseteq X$ be a family of geodesic metric spaces such that $\eta_{nm} := X_n \cap X_m$ are compact sets. We say that $\{X_n\}_{n\in\Lambda}$ is a (k_1, k_2, k_3)-tree decomposition if it satisfies the following properties:

(*a*) If $\eta_{nm} \neq \varnothing$, then $X \setminus \eta_{nm}$ is not connected and a, b are in different connected components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$, with $m \neq n$.

(*b*) $\operatorname{diam}_{X_n}(\eta_{nm}) \leqslant k_1$ for every $m \neq n$, and there exists $A_n \subseteq \Lambda$, such that $\operatorname{diam}_{X_n}(\eta_{nm}) \leqslant k_2\, d_{X_n}(\eta_{nm}, \eta_{nk})$ if $m \neq k$ and $m, k \in A_n$, and $\sum_{m \notin A_n} \operatorname{diam}_{X_n}(\eta_{nm}) \leqslant k_3$.

We say that a claim holds quantitatively, if it holds for parameters depending only on other parameters. For instance, in the next theorem, "quantitatively" means that we can calculate δ from k_1, k_2, k_3 and k_4, or k_4 from k_1, k_2, k_3 and δ.

Theorem 2.8 (Theorem 2.9, [23]). *Let us consider a metric space X and a family of geodesic metric spaces $\{X_n\}_n \subseteq X$ which is a (k_1, k_2, k_3)-tree decomposition of X. Then X is δ-hyperbolic quantitatively if and only if there exists a constant k_4 such that X_n is k_4-hyperbolic for every n.*

Definition 2.9. A *normal neighborhood* of a subset F of a plane domain Ω is a compact set V such that $F \subset V \subset \Omega$, and ∂V is the union of non-trivial closed curves.

A set $E = \cup_n E_n$ in a plane domain Ω, with $\{E_n\}_n$ compact sets, is called (r, s)-*uniformly separated* in Ω if there exist normal neighborhoods $\{V_n\}_n$ of E_n such that $V_n \setminus E_n$ is connected, $k_\Omega(\partial V_n, E_n) \geqslant r$, $\ell_{k,\Omega}(\partial V_n) \leqslant s$ for every n, and $k_\Omega(V_n, V_m) \geqslant r$ for every $n \neq m$.

A set E in a plane domain Ω is called r-*uniformly separated* in Ω, if the balls $\{B_{k_\Omega}(p, r)\}_{p \in E}$ are pairwise disjoint. It is straightforward to check that every r-uniformly separated set is discrete.

Remark 2.10.

(1) From now on, whenever we deal with an (r, s)-uniformly separated set we assume that we have fixed a choice of normal neighborhoods $\{V_n\}_n$.
(2) Note that, since every non-trivial closed curve g satisfies $k_\Omega(g) \geqslant 2\pi$, ∂V_n is the union of at most $s/2\pi$ closed curves.

The following lemma is well-known. A proof can be found in for instance in [16, Lemma 4.3].

Lemma 2.11. *Let γ be a curve in a domain $D \subset \mathbb{R}^n$ from $a \in D$ with Euclidean length s. Then:*

$$k_D(\gamma) \geqslant \log\left(1 + \frac{s}{\delta_D(a)}\right).$$

3. Technical results

For any simple closed curve γ, we denote by $\operatorname{int}\gamma$ the set of points in the complex plane with index 1 with respect to γ when the curve is positively oriented, and by $\operatorname{ext}\gamma$ the set $\mathbb{C} \setminus (\gamma \cup \operatorname{int}\gamma)$.

Let V_n be a compact set in Ω whose boundary is a finite union of non-trivial closed curves. We denote by $\partial_j V_n$, for $j \geqslant 0$, each of the connected components of ∂V_n with the indexing such that, $\partial_0 V_n$ is the one that encloses all the others. We denote $\partial_{j,n}\Omega := \partial\Omega \cap \operatorname{int}\partial_j V_n$, for $j \geqslant 1$, and $\partial_{0,n}\Omega := \partial\Omega \cap \operatorname{ext}\partial_0 V_n$.

In what follows, whenever we talk about the distance between two sets, we assume that the statement holds whenever it makes sense, that is to say, when both sets are non-empty.

Lemma 3.1. *Let Ω be a plane domain and $E = \cup_n E_n$ an (r,s)-uniformly separated set in Ω. Then, for every n and every $j > 0$,*

$$2\, d_{Eucl}(\partial_j V_n, \partial_{j,n}\Omega) < \ell_{Eucl}(\partial_j V_n) \leqslant (e^s - 1)\, d_{Eucl}(\partial_j V_n, \partial_{j,n}\Omega).$$

The upper bound holds also in the case $j = 0$.

Proof. Let us define $\varepsilon_{j,n} := d_{Eucl}(\partial_j V_n, \partial_{j,n}\Omega)$ for each n and $j \geqslant 0$. By Lemma 2.11

$$s \geqslant k_\Omega(\partial_j V_n) \geqslant \log\left(1 + \frac{\ell_{Eucl}(\partial_j V_n)}{\varepsilon_{j,n}}\right).$$

Therefore, $\ell_{Eucl}(\partial_j V_n) \leqslant (e^s - 1)\, \varepsilon_{j,n}$, for $j \geqslant 0$.

We also have, for $j > 0$,

$$d_{Eucl}(\partial_j V_n, \partial_{j,n}\Omega) < \operatorname{diam}_{Eucl}(\partial_j V_n) < \frac{1}{2}\, \ell_{Eucl}(\partial_j V_n). \qquad \square$$

Lemma 3.2. *Let Ω be a plane domain, $E = \cup_n E_n$ an (r,s)-uniformly separated set in Ω and fix points $q_{j,n} \in \partial_{j,n}\Omega$ for $j > 0$. Then, for every $z \in V_n$,*

$$\frac{2}{e^s + 1}\, d_{Eucl}(z, \cup_{j>0}\, q_{j,n}) \leqslant d_{Eucl}(z, \cup_{j>0}\, \partial_{j,n}\Omega) \leqslant d_{Eucl}(z, \cup_{j>0}\, q_{j,n}).$$

Proof. The inequality $d_{Eucl}(z, \cup_{j>0}\, \partial_{j,n}\Omega) \leqslant d_{Eucl}(z, \cup_{j>0}\, q_{j,n})$ is immediate. Therefore, we just have to check the other one.

In order to do it, let w be a point in $\partial_{j,n}\Omega$ such that $d_{Eucl}(z, w) := d_{Eucl}(z, \partial_{j,n}\Omega)$. Applying the triangle inequality and Lemma 3.1,

$$\begin{aligned} d_{Eucl}(z, q_{j,n}) &\leqslant d_{Eucl}(z, w) + d_{Eucl}(w, q_{j,n}) \leqslant d_{Eucl}(z, w) + \operatorname{diam}_{Eucl}(\partial_j V_n) \\ &\leqslant d_{Eucl}(z, w) + \frac{1}{2}\, \ell_{Eucl}(\partial_j V_n) \leqslant d_{Eucl}(z, \partial_{j,n}\Omega) + \frac{e^s - 1}{2}\, d_{Eucl}(\partial_j V_n, \partial_{j,n}\Omega) \\ &\leqslant d_{Eucl}(z, \partial_{j,n}\Omega) + \frac{e^s - 1}{2}\, d_{Eucl}(z, \partial_{j,n}\Omega) = \frac{e^s + 1}{2}\, d_{Eucl}(z, \partial_{j,n}\Omega). \end{aligned}$$

Since this holds for every j, it holds also for the union of the points and sets concerned, which is the claim of the lemma. $\square$

Lemma 3.3. *Let Ω be a plane domain and $E = \cup_n E_n$ an (r,s)-uniformly separated set in Ω. Then, for every n and for every $z \in V_n$,*

$$\frac{2}{e^s - 1}\, d_{Eucl}(z, \partial\Omega \setminus \partial_{0,n}\Omega) \leqslant \delta_\Omega(z) \leqslant d_{Eucl}(z, \partial\Omega \setminus \partial_{0,n}\Omega).$$

Proof. The second inequality is trivial. On the other hand, applying Lemma 3.1 for $j = 0$, we obtain

$$\begin{aligned} d_{Eucl}(z, \partial_{0,n}\Omega) &\geqslant d_{Eucl}(\partial_0 V_n, \partial_{0,n}\Omega) \geqslant \frac{1}{e^s - 1}\, \ell_{Eucl}(\partial_0 V_n) \geqslant \frac{2}{e^s - 1}\, \operatorname{diam}_{Eucl}(\partial_0 V_n) \\ &\geqslant \frac{2}{e^s - 1}\, d_{Eucl}(z, \partial\Omega \setminus \partial_{0,n}\Omega), \end{aligned}$$

since $z \in \operatorname{int} V_n$ and $\partial\Omega \setminus \partial_{0,n}\Omega \subset \operatorname{int} \partial_0 V_n$.

By Remark 2.10(2), $s \geqslant 2\pi$, and thus $e^s - 1 \geqslant e^{2\pi} - 1 > 2$. This means that $2/(e^s - 1) < 1$, and, consequently,

$$\delta_\Omega(z) = \min\{d_{Eucl}(z, \partial_{0,n}\Omega), d_{Eucl}(z, \partial\Omega \setminus \partial_{0,n}\Omega)\} \geqslant \frac{2}{e^s - 1}\, d_{Eucl}(z, \partial\Omega \setminus \partial_{0,n}\Omega). \qquad \square$$

Corollary 3.4. *Let Ω be a plane domain and $E = \cup_n E_n$ an (r,s)-uniformly separated set in Ω. For each n and each $j > 0$, fix $q_{j,n} \in \partial_{j,n}\Omega$. If we consider the set $D_n := \mathbb{C} \setminus \cup_{j>0}\{q_{j,n}\}$, and define $\Omega^* := \Omega \setminus E$, $D_n^* := D_n \setminus E_n$ and $V_n^* := V_n \setminus E_n$, then:*

(1) $\dfrac{4}{e^{2s}-1}\, k_{\Omega|V_n} \leqslant k_{D_n|V_n} \leqslant k_{\Omega|V_n}$, *and*

(2) $\dfrac{4}{e^{2s}-1}\, k_{\Omega^*|V_n^*} \leqslant k_{D_n^*|V_n^*} \leqslant k_{\Omega^*|V_n^*}$.

Proof. Notice that $\Omega \subset D_n$ implies $\delta_{D_n}(z) \geqslant \delta_\Omega(z)$ for every $z \in \Omega$; hence the upper bound in (1) is clear. In order to prove the other inequality, let us consider $z \in V_n$; by Lemmas 3.2 and 3.3,

$$\delta_\Omega(z) \geqslant \frac{2}{e^s-1}\, d_{Eucl}(z, \partial\Omega \setminus \partial_{0,n}\Omega) \geqslant \frac{4}{e^{2s}-1}\, d_{Eucl}(z, \cup_{j>0}\, q_{j,n}) = \frac{4}{e^{2s}-1}\, \delta_{D_n}(z),$$

for every $z \in V_n$. Thus, we also have,

$$\frac{4}{e^{2s}-1}\, \delta_{D_n^*}(z) \leqslant \delta_{\Omega^*}(z) \leqslant \delta_{D_n^*}(z),$$

for every $z \in V_n^*$. □

Lemma 3.5. *Let us consider a plane domain Ω and a simply connected compact set $D \subset \Omega$. Then,* $\operatorname{diam}_{k_\Omega}(D) < 3\, k_\Omega(\partial D)$. *Consequently, D is $3\, k_\Omega(\partial D)$-hyperbolic.*

Proof. Without loss of generality we can assume that ∂D is a rectifiable curve, since otherwise $k_\Omega(\partial D) = \infty$, and the claim is clear. Let $x, y \in D$ be such that $k_\Omega(x,y) = \operatorname{diam}_{k_\Omega}(D)$. Let $a, b \in \partial D$ be nearest boundary points so that $\delta_D(x) = |x-a|$ and $\delta_D(y) = |y-b|$. We know that $a, b \in \Omega$, since D is a compact set contained in Ω. By the triangle inequality,

$$\operatorname{diam}_{k_\Omega}(D) = k_\Omega(x,y) \leqslant k_\Omega(x,a) + k_\Omega(a,b) + k_\Omega(b,y).$$

Clearly, $k_\Omega(a,b) \leqslant k_\Omega(\partial D)$. By symmetry, the claim will follow once we establish that $k_\Omega(x,a) < k_\Omega(\partial D)$. The claim is trivial if $a = x$, so we assume also that this is not the case.

We distinguish two cases. If $\delta_\Omega(a) \geqslant 2\,|a-x|$, we use that $B_{Eucl}(a, \delta_\Omega(a)) \subset \Omega$ and obtain

$$k_\Omega(x,a) \leqslant k_{B_{Eucl}(a,\delta_\Omega(a))}(x,a) = \log\left(\frac{\delta_\Omega(a)}{\delta_\Omega(a) - |a-x|}\right) \leqslant \log\left(1 + \frac{2\,|a-x|}{\delta_\Omega(a)}\right).$$

If $2|a-x| > \delta_\Omega(a)$, then we choose $b \in [a,x]$ such that $|b-a| = \frac{1}{2}\delta_\Omega(a)$ and obtain

$$k_\Omega(x,a) \leqslant k_{B_{Eucl}(a,\delta_\Omega(a))}(b,a) + k_{B_{Eucl}(x,|x-a|)}(x,b) = \log 2 + \log\frac{|x-a|}{|b-a|} = \log\frac{4\,|x-a|}{\delta_\Omega(a)}.$$

Thus we see that the inequality

$$k_\Omega(x,a) \leqslant \log\left(1 + \frac{4\,|a-x|}{\delta_\Omega(a)}\right)$$

holds in either case.

Since $B_{Eucl}(x,|a-x|) \subset D$, the isoperimetric inequality implies that $\ell_{\mathrm{Eucl}}(\partial D) \geqslant 2\pi\,|a-x|$. Therefore Lemma 2.11 yields the inequality

$$k_\Omega(\partial D) \geqslant \log\left(1+\frac{2\pi\,|a-x|}{\delta_\Omega(a)}\right).$$

Hence $k_\Omega(x,a) < k_\Omega(\partial D)$, which concludes the proof. □

Proposition 3.8 is interesting in itself and, furthermore, it is a main tool for proving the theorems in this paper. First, we need a definition.

Definition 3.6. Given an annulus $A = \{z \in \mathbb{C} : r_1 < |z-a| < r_2\}$, we define the *modulus* of A as $\operatorname{mod} A := \frac{r_2}{r_1} \in (1,\infty]$.

We say that an annulus A *separates the boundary* of a plane domain Ω if $A \subseteq \Omega$ and each connected component of $\overline{\mathbb{C}} \setminus A$ intersects $\partial\Omega$, where $\overline{\mathbb{C}}$ is the Riemann sphere (we consider $\partial\Omega \subset \overline{\mathbb{C}}$, in order to deal with the case $r_2 = \infty$).

We say that a plane domain Ω is *M-modulated* if $\operatorname{mod} A \leqslant M$ for every annulus A that separates the boundary of Ω.

We will also need the following well-known result.

Proposition 3.7 (Proposition 1, [22])**.** *If $T : \Omega_1 \to \Omega_2$ is a Möbius transformation with $\Omega_1, \Omega_2 \subset \mathbb{C}$, $\Omega_2 = T(\Omega_1)$ and Ω_1 is not M-modulated, then Ω_2 is not $f(M)$-modulated, where $f(M) := M + 8M^2$.*

Proposition 3.8. *Let us consider a plane domain $\Omega \subset \mathbb{C}$ such that $\partial\Omega \subset \overline{\mathbb{C}}$ consists of $N+1$ (with $N \geqslant 0$) connected components (N of them compact, and the remaining one containing ∞). If there exist simple closed curves $\gamma_1,\dots,\gamma_N$ each freely homotopic to a different connected component of $\partial\Omega$, with $\sum_{j=1}^N k_\Omega(\gamma_j) \leqslant c$, then Ω is δ-hyperbolic, with δ a constant which depends only on c.*

Remark 3.9. We allow some γ_j to be freely homotopic to the "exterior" connected component of $\partial\Omega$ (i.e., some γ_j might surround every compact connected component of $\overline{\mathbb{C}} \setminus \Omega$).

Proof. Since every non-trivial closed curve g satisfies $k_\Omega(g) \geqslant 2\pi$, we have $N \leqslant c/2\pi$.

We prove the proposition by induction on N. If $N = 0$, then Ω is a simply connected domain and it is well known that $k_\Omega/2 \leqslant h_\Omega \leqslant 2\,k_\Omega$, where h_Ω is the Poincaré metric. Since (Ω, h_Ω) is isometric to the unit disk $(\mathbb{D}, h_{\mathbb{D}})$, it is $\log\left(1+\sqrt{2}\,\right)$-thin (see, e.g. [4, p. 130]). Since (Ω, k_Ω) and (Ω, h_Ω) are quasi-isometric, Theorem 2.6 gives the result.

Assume now that the result holds for $0, 1, 2, \dots, N-1$.

Let us assume first that Ω is M-modulated, with $M := f^{-1}(3)$, where f is the function in Proposition 3.7. Then (Ω, k_Ω) is $(\alpha, 0)$-quasi-isometric to (Ω, h_Ω), with a constant α which just depends on M (see [7, Corollary 1]). Note that $\sum_{j=1}^N h_\Omega(\gamma_j) \leqslant 2\sum_{j=1}^N k_\Omega(\gamma_j) \leqslant 2c$, and therefore (Ω, h_Ω) is δ_1-hyperbolic, with δ_1 a constant which just depends on c (see [13, Theorem 7.10]). Thus Theorem 2.6 gives the result.

Assume now that Ω is not M-modulated. If Ω is a punctured plane $\Omega := \mathbb{C}\setminus\{z_0\}$, then it is isometric to an Euclidean cylinder with radius 1, which is π-thin. Hence, without loss of generality we can assume that Ω is not a punctured plane. By Proposition

3.7, applying a Möbius transformation if necessary, we can assume that Ω is not 3-modulated, the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$ separates the boundary of Ω, $3 \in \partial\Omega$, and if $N > 1$ then $B_{Eucl}(0,3)^c$ contains at least two connected components of $\overline{\mathbb{C}} \setminus \Omega$.

Define $Y := B_{Eucl}(0,2)^c$, $Z := \overline{B}_{Eucl}(0,2)$, $X_1 := \Omega \cap Z$, $X_2 := \Omega \cap Y$ and $\eta_{12} := X_1 \cap X_2 = \partial B_{Eucl}(0,2)$. Since

$$k_\Omega(\eta_{12}) = \int_{|z|=2} \frac{|dz|}{\delta_\Omega(z)} \leqslant \int_{|z|=2} |dz| = 4\pi,$$

the set $\{X_1, X_2\}$ is a $(4\pi, 0, 4\pi)$-tree decomposition of Ω. By Theorem 2.8, in order to finish the proof, we just need to show that $(X_1, k_{\Omega|X_1})$ and $(X_2, k_{\Omega|X_2})$ are δ_2-hyperbolic, with δ_2 a constant which only depends on c.

Let us consider the domains $\Omega_1 := X_1 \cup Y$ and $\Omega_2 := X_2 \cup Z$. Note that $\partial\Omega_1 = \partial X_1 \cup \{\infty\}$ whereas $\partial\Omega_2 = \partial X_2$, so the situation is not totally symmetric. If $N = 1$, then Ω_1 has two boundary components again, so we cannot use induction; we show below that the set is nevertheless hyperbolic. In the other cases the domains Ω_1 and Ω_2 have fewer boundary components than the original domain (here we use that $B_{Eucl}(0,3)^c$ contains at least two connected components of $\overline{\mathbb{C}} \setminus \Omega$).

It is easy to check that $k_{\Omega_1}(\eta_{12}), k_{\Omega_2}(\eta_{12}) \leqslant 4\pi$, and then X_1, Y is a $(4\pi, 0, 4\pi)$-tree decomposition of Ω_1, and X_2, Z is a $(4\pi, 0, 4\pi)$-tree decomposition of Ω_2. By the inductive assumption and Theorem 2.8 we know that $(X_1, k_{\Omega_1|X_1})$ and $(X_2, k_{\Omega_2|X_2})$ are δ_3-hyperbolic, with δ_3 a constant which only depends on c. Note that for every $z \in X_1$ we have that $\delta_\Omega(z)$ and $\delta_{\Omega_1}(z)$ are comparable with universal constants. In a similar way, for every $z \in X_2$ we obtain that $\delta_\Omega(z)$ and $\delta_{\Omega_2}(z)$ are comparable with universal constants. Since $(X_1, k_{\Omega|X_1})$ and $(X_1, k_{\Omega_1|X_1})$ (and $(X_2, k_{\Omega|X_2})$ and $(X_2, k_{\Omega_2|X_2})$) are quasi-isometric, Theorem 2.6 gives that $(X_1, k_{\Omega|X_1})$ and $(X_2, k_{\Omega|X_2})$ are δ_4-hyperbolic, with δ_4 a constant which only depends on c. Theorem 2.8 finishes this part of the proof.

It remains to consider the case $N = 1$ when the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$ separates the boundary of Ω, and $3 \in \partial\Omega$. We have $\Omega = \mathbb{C} \setminus (E \cup F)$, where E is a connected compact set and F is a closed set in $\overline{\mathbb{C}}$ with $3, \infty \in F$. We show that Ω is hyperbolic, with a universal constant.

Define $X_1 := \Omega \cap Z$, $X_2 := \Omega \cap Y$ and $\eta_{12} := X_1 \cap X_2$. Notice that $k_\Omega(\eta_{12}) \leqslant 4\pi$. Thus, X_1, X_2 is a $(4\pi, 0, 4\pi)$-tree decomposition of Ω. By Theorem 2.8, in order to prove that Ω is hyperbolic, we just need to show that $(X_1, k_{\Omega|X_1})$ and $(X_2, k_{\Omega|X_2})$ are hyperbolic, with a universal constant.

Since $h_{\Omega\setminus\overline{\mathbb{D}}}(\eta_{12}) \leqslant 2k_{\Omega\setminus\overline{\mathbb{D}}}(\eta_{12}) \leqslant 8\pi$, then $(\Omega \setminus \overline{\mathbb{D}}, h_{\Omega\setminus\overline{\mathbb{D}}})$ is δ_5-hyperbolic, with δ_5 a universal constant (see [31, Lemma 5.4] or [25, Theorem 5.6]). Furthermore, $\Omega \setminus \overline{\mathbb{D}}$ is uniformly perfect with a universal constant, and it follows that $(\Omega\setminus\overline{\mathbb{D}}, k_{\Omega\setminus\overline{\mathbb{D}}})$ is δ_6-hyperbolic, with δ_6 a universal constant, by Theorem 2.6. Note that, for every $z \in X_2$, $\delta_\Omega(z)$ and $\delta_{\Omega\setminus\overline{\mathbb{D}}}(z)$ are comparable with universal constants. Since $(X_2, k_{\Omega|X_2})$ and $(X_2, k_{\Omega\setminus\overline{\mathbb{D}}|X_2})$ are quasi-isometric, Theorems 2.6 and 2.8 give that $(X_2, k_{\Omega|X_2})$ is δ_7-hyperbolic, with δ_7 a universal constant.

We prove now that $(X_1, k_{\Omega|X_1})$ is hyperbolic. If $\operatorname{diam}_{Eucl} E \geqslant 1/2$, an argument similar to the one in the previous case shows that $(X_1, k_{\Omega|X_1})$ is hyperbolic. If $\operatorname{diam}_{Eucl} E < 1/2$, fix $w \in E$ and define $X_3 := \Omega \cap \overline{B}_{Eucl}(w, 2\operatorname{diam}_{Eucl} E)$ and $X_4 := \overline{B}_{Eucl}(0,2) \setminus B_{Eucl}(w, 2\operatorname{diam}_{Eucl} E)$. An argument similar to the one in the previous case shows that $(X_3, k_{\Omega|X_3})$ is hyperbolic. Note that, for every $z \in X_4$, $\delta_\Omega(z)$ and $\delta_{\Omega\setminus\{w\}}(z)$ are comparable with universal constants. Since $(X_4, k_{\Omega|X_4})$ and $(X_4, k_{\Omega\setminus\{w\}|X_4})$ are quasi-isometric, Theorems 2.6 and 2.8 give that $(X_4, k_{\Omega|X_4})$ is hyperbolic. Since X_3, X_4 is a tree decomposition of X_1, Theorem 2.8 gives that $(X_1, k_{\Omega|X_1})$ is hyperbolic. □

Proposition 3.10. *Let Ω be a plane domain and $E = \cup_n E_n$ an (r,s)-uniformly separated set in Ω. Then, $(V_n, k_{\Omega|V_n})$ is δ-hyperbolic for every n, with δ a constant which just depends on s.*

Proof. Let us fix n. By Corollary 3.4(1) and Theorem 2.6, it is sufficient to check that $(V_n, k_{D_n|V_n})$ are uniformly hyperbolic for every n, with D_n defined as in Corollary 3.4. Applying Proposition 3.8 to the domain D_n, where $\gamma_j = \partial_j V_n$, and taking into account that $\sum_{j>0} k_{D_n}(\partial_j V_n) \leqslant k_{D_n}(\partial V_n) \leqslant k_\Omega(\partial V_n) \leqslant s$, we conclude that D_n is δ_1-hyperbolic, with $\delta_1 = \delta_1(s)$. Let us define $X_0 := V_n$ and $X_1, \dots, X_N$ as the closure of each of the connected components of $D_n \setminus V_n$. Since $k_{D_n}(\partial V_n) \leqslant s$ as we have seen before, the family of sets $\{X_j\}_{j=0}^N$ is an $(s, 0, s)$-tree decomposition of $D_n \setminus V_n$. Therefore, by Theorem 2.8, every V_n is δ-hyperbolic, with $\delta = \delta(s)$. □

Lemma 3.11. *Let Ω be a plane domain and $E = \cup_n E_n$ an (r,s)-uniformly separated set in Ω. If we define $\Omega^* := \Omega \setminus E$ and $c_r := 2\Big(1 + \frac{1}{r(e^{r/2}-1)}\Big)$, then $\frac{1}{c_r} k_{\Omega^*}(\gamma) \leqslant k_\Omega(\gamma) \leqslant k_{\Omega^*}(\gamma)$ for every curve $\gamma \subset \Omega \setminus \cup_n \operatorname{int} V_n$ and hence*

$$\frac{1}{c_r}\, k_{\Omega^*|\Omega\setminus\cup_n \operatorname{int} V_n} \leqslant k_{\Omega|\Omega\setminus\cup_n \operatorname{int} V_n} \leqslant k_{\Omega^*|\Omega\setminus\cup_n \operatorname{int} V_n}.$$

Proof. Notice that $\delta_\Omega(z) \geqslant \delta_{\Omega^*}(z)$ for every $z \in \Omega^*$ and therefore $k_\Omega(\gamma) \leqslant k_{\Omega^*}(\gamma)$ for every curve $\gamma \subset \Omega^*$. In order to prove the other inequality, let us consider $z \notin \cup_n \operatorname{int} V_n$. There are two possible situations:

(1) $\delta_\Omega(z) \leqslant d_{Eucl}(z, E)$. In this case, we can conclude that $\delta_\Omega(z) = \delta_{\Omega^*}(z)$ and then $1/\delta_\Omega(z) = 1/\delta_{\Omega^*}(z)$.

(2) $\delta_\Omega(z) > d_{Eucl}(z, E) = \delta_{\Omega^*}(z)$. Let us consider a point $p \in E$ with $|z - p| = d_{Eucl}(z, E)$. Without loss of generality we can assume that $p = 0$ and $z > 0$; hence $z = d_{Eucl}(z, E) = \delta_{\Omega^*}(z) < \delta_\Omega(z)$.

Note that by the triangle inequality it is always true that $\delta_\Omega(z) \leqslant z + \delta_\Omega(0)$. And now let us consider the two following cases:

(2.1) If $\delta_\Omega(0) \leqslant z$, then $\delta_\Omega(z) \leqslant 2z = 2d_{Eucl}(z, E) = 2\delta_{\Omega^*}(z)$.

(2.2) If $\delta_\Omega(0) > z$, since E is an (r,s)-uniformly separated set in Ω, we have that $k_\Omega(\partial V_n, E_n) \geqslant r$. It implies that $k_\Omega(z, 0) \geqslant r$ because $z \notin \cup_n \operatorname{int} V_n$ and $0 \in E$. Note that in this case we do not have a straight upper bound for $\delta_\Omega(0)$ like in the previous situation, so our aim is trying to get it. In order to do it, we are going to compare the quasihyperbolic lengths of the geodesic joining z and 0 and the segment $[0, z]$.

For every $0 \leqslant t \leqslant e^{-r/2} z$, we have that $\delta_\Omega(t) \geqslant \delta_\Omega(0) - t > \delta_\Omega(0) - e^{-r/2}\delta_\Omega(0)$, and for every $t \in [0, z]$ we have $\delta_\Omega(t) > t$, since $\delta_\Omega(z) > z$, and consequently, $B_{Eucl}(z, z) \cap \partial\Omega = \varnothing$. Therefore,

$$r \leqslant k_\Omega(z, 0) \leqslant k_\Omega([0, z]) = \int_0^z \frac{dt}{\delta_\Omega(t)} \leqslant \int_0^{e^{-r/2}z} \frac{dt}{\delta_\Omega(t)} + \int_{e^{-r/2}z}^z \frac{dt}{t}$$
$$\leqslant \frac{r}{2} + \int_0^{e^{-r/2}z} \frac{dt}{\delta_\Omega(0)(1 - e^{-r/2})} = \frac{r}{2} + \frac{z}{\delta_\Omega(0)(e^{r/2} - 1)}.$$

Hence, $\delta_\Omega(0) \leqslant \dfrac{2z}{r(e^{r/2}-1)}$ and we have got the upper bound we were looking for. So,

$$\delta_\Omega(z) \leqslant z + \delta_\Omega(0) \leqslant \left(1 + \frac{2}{r(e^{r/2}-1)}\right) z = \left(1 + \frac{2}{r(e^{r/2}-1)}\right) \delta_{\Omega^*}(z).$$

Combining cases $(1), (2.1)$ and (2.2) gives us the inequality

$$\delta_\Omega(z) \leqslant \left(2 + \frac{2}{r(e^{r/2}-1)}\right)\delta_{\Omega^*}(z),$$

and we conclude that

$$\delta_{\Omega^*}(z) \leqslant \delta_\Omega(z) \leqslant 2\left(1 + \frac{1}{r(e^{r/2}-1)}\right)\delta_{\Omega^*}(z), \quad \text{for every } z \notin \cup_n \operatorname{int} V_n.$$

Therefore, $k_\Omega(\gamma) \leqslant k_{\Omega^*}(\gamma) \leqslant c_r k_\Omega(\gamma)$ for every curve $\gamma \subset \Omega \setminus \cup_n \operatorname{int} V_n$, as we wanted to prove. □

Proposition 3.12. *Let Ω be a plane domain, $E = \cup_n E_n$ an (r,s)-uniformly separated set in Ω, with E_n a connected set for every n, and $\Omega^* := \Omega \setminus E$. If $V_n^* := V_n \setminus E_n$, then $(V_n^*, k_{\Omega^*|V_n^*})$ is δ^*-hyperbolic, quantitatively.*

Proof. Let us fix n. By Corollary 3.4(2) and Theorem 2.6, it is sufficient to check that $(V_n^*, k_{D_n^*|V_n^*})$ are uniformly hyperbolic for every n, with D_n^* defined as in Corollary 3.4. Note that each $\partial_j V_n$ is freely homotopic to a point in $\partial D_n^* \subset \overline{\mathbb{C}}$, and none is freely homotopic to E_n.

Applying now Proposition 3.8 to the domain D_n^*, with $\gamma_j = \partial_j V_n$, we conclude that D_n^* is δ_1^*-hyperbolic, with $\delta_1^* = \delta_1^*(r,s)$, since, by Lemma 3.11, $\sum_{j>0} k_{D_n^*}(\partial_j V_n) = k_{D_n^*}(\partial V_n) \leqslant k_{\Omega^*}(\partial V_n) \leqslant c_r k_\Omega(\partial V_n) \leqslant c_r\, s$. Let us define $X_0 := V_n^*$ and $X_1, \dots, X_N$ as the closure of each of the connected components of $D_n^* \setminus V_n^*$. Since, $k_{D_n^*}(\partial V_n^*) = k_{D_n^*}(\partial V_n) \leqslant c_r\, s$, the family of sets $\{X_j\}_{j=0}^N$ is a $(c_r\, s, 0, c_r\, s)$-tree decomposition of $D_n^* \setminus V_n^*$. Therefore, by Theorem 2.8, every V_n^* is δ^*-hyperbolic with $\delta^* = \delta^*(r,s)$. □

4. The main results

Theorem 4.1. *Let Ω be a plane domain and $E = \cup_n E_n$ an (r,s)-uniformly separated set in Ω. Then $\Omega^* := \Omega \setminus E$ is δ^*-hyperbolic quantitatively if and only if Ω is δ-hyperbolic and $V_n^* := V_n \setminus E_n$ is uniformly hyperbolic for every n (with the metric $k_{\Omega^*|V_n^*}$).*

Proof. The normal neighborhoods $\{V_n\}_n$ of E_n are such that $V_n \setminus E_n$ is connected, $k_\Omega(\partial V_n, E_n) \geqslant r$ and $k_\Omega(\partial V_n) \leqslant s$ for every n; and $k_\Omega(V_n, V_m) \geqslant r$ for every $n \neq m$. In order to apply Theorem 2.8 in Ω^*, we need to check that $\{V_n^*\}_n \cup \{\Omega \setminus \cup_n \operatorname{int} V_n\}$ is a $(c, c/r, 0)$-tree decomposition, where c is a constant that just depends on r and s. Then, we have to prove that the two following conditions hold: on the one hand $k_{\Omega^*}(\partial V_n) \leqslant c$ and on the other hand $k_{\Omega^*}(\partial V_n, \partial V_m) \geqslant r$, since then $k_{\Omega^*}(\partial V_n) \leqslant (c/r) r \leqslant (c/r)\, k_{\Omega^*}(\partial V_n, \partial V_m)$. Note that once these two inequalities have been checked, the hyperbolicity of Ω^* is equivalent to the hyperbolicity of $\Omega \setminus \cup_n \operatorname{int} V_n$ with $k_{\Omega^*|\Omega \setminus \cup_n \operatorname{int} V_n}$, and the uniform hyperbolicity of $(V_n^*, k_{\Omega^*|V_n^*})$.

The first inequality is easily obtained applying Lemma 3.11 to each connected component of ∂V_n. To be more precise, $k_{\Omega^*}(\partial V_n) \leqslant c_r\, k_\Omega(\partial V_n) \leqslant c_r\, s =: c$. The second inequality is straightforward to obtain from the fact that $\delta_\Omega(z) \geqslant \delta_{\Omega^*}(z)$ for every $z \in \Omega^*$; then $k_{\Omega^*}(\partial V_n, \partial V_m) \geqslant k_\Omega(\partial V_n, \partial V_m) \geqslant r$.

Hence, by Theorem 2.8, Ω^* is hyperbolic if and only if $(\Omega \setminus \cup_n \operatorname{int} V_n, k_{\Omega^*|\Omega\setminus\cup_n \operatorname{int} V_n})$ is hyperbolic and $(V_n^*, k_{\Omega^*|V_n^*})$ are uniformly hyperbolic for every n. Since $k_\Omega \asymp k_{\Omega^*}$ in $\Omega \setminus \cup_n \operatorname{int} V_n$ by Lemma 3.11, we have that $(\Omega \setminus \cup_n \operatorname{int} V_n, k_{\Omega^*|\Omega\setminus\cup_n \operatorname{int} V_n})$ is hyperbolic if and only if $(\Omega \setminus \cup_n \operatorname{int} V_n, k_{\Omega|\Omega\setminus\cup_n \operatorname{int} V_n})$ is hyperbolic.

Note that $(V_n, k_{\Omega|V_n})$ are uniformly hyperbolic due to Proposition 3.10. Since $\{V_n\}_n \cup \{\Omega \setminus \cup_n \operatorname{int} V_n\}$ is an $(s, s/r, 0)$-tree decomposition of Ω, Theorem 2.8 gives that $(\Omega \setminus \cup_n \operatorname{int} V_n, k_{\Omega|\Omega\setminus\cup_n \operatorname{int} V_n})$ is hyperbolic if and only if (Ω, k_Ω) is hyperbolic. This finishes the proof. □

Theorem 4.2. *Let Ω be a plane domain and $E = \cup_n E_n$ an (r,s)-uniformly separated set in Ω, where every E_n is a connected set. Then $\Omega^* := \Omega \setminus E$ is δ^*-hyperbolic quantitatively if and only if Ω is δ-hyperbolic.*

Proof. The proof is straightforward from Theorem 4.1 and Proposition 3.12. □

Lemma 4.3. *Let Ω be a plane domain and $p \in \Omega$. For $t > 0$, let us define $D_t := B_{Eucl}(p, \delta_\Omega(p)\, e^{-t})$. Then,*

(1) $\log\left(\dfrac{2+e^t}{1+e^t}\right) \leqslant k_\Omega(p,z) \leqslant \log\dfrac{e^t}{e^t-1}$ *, for every $z \in \partial D_t$.*

(2) $k_\Omega(\partial D_t) \leqslant \dfrac{2\pi}{e^t-1}$.

Proof. Without loss of generality, applying a dilatation and a translation if necessary, we can assume that $p = 0$ and $\delta_\Omega(p) = 1$. In order to prove (1), let $z \in \partial D_t$. Then

$$k_\Omega(0,z) \leqslant k_{B_{Eucl}(0,1)}(0,z) = k_{B_{Eucl}(0,1)}(0,e^{-t}) = \int_0^{e^{-t}} \frac{dx}{1-x}$$
$$= -\log(1-e^{-t}) = \log\frac{e^t}{e^t-1}.$$

And, on the other hand, by Lemma 2.11,

$$k_\Omega(0,z) \geqslant \log\left(1 + \frac{e^{-t}}{1+e^{-t}}\right) = \log\left(\frac{2+e^t}{1+e^t}\right).$$

Next, let us prove (2),

$$k_\Omega(\partial D_t) \leqslant k_{B_{Eucl}(0,1)}(\partial D_t) = \int_0^{2\pi} \frac{e^{-t}}{1-e^{-t}}\, d\theta = \frac{2\pi e^{-t}}{1-e^{-t}} = \frac{2\pi}{e^t-1}.$$ □

Theorem 4.4. *Let Ω be a plane domain and E an r-uniformly separated set in Ω. Then $\Omega^* := \Omega \setminus E$ is δ^*-hyperbolic quantitatively if and only if Ω is δ-hyperbolic.*

Proof. Our goal is to prove that E is, as well, an (r', s')-uniformly separated set for some constants r' and s' that just depend on r. Notice that once this is proved, Theorem 4.2 gives the result.

Let us define $t := \log(e^{r/3}/(e^{r/3}-1))$. For every $p \in E$, let $V_p := B_{Eucl}(p, \delta_\Omega(p)\, e^{-t})$. First, notice that by Lemma 4.3(2),

$$k_\Omega(\partial V_p) \leqslant \frac{2\pi}{e^t-1} = 2\pi(e^{r/3}-1). \tag{4.5}$$

Next, applying Lemma 4.3(1), we deduce for every $z \in \partial V_p$

$$k_\Omega(p,z) \geqslant \log\left(\frac{2+e^t}{1+e^t}\right) = \log\left(\frac{3\,e^{r/3}-2}{2\,e^{r/3}-1}\right). \tag{4.6}$$

Finally, applying again Lemma 4.3(1), we obtain that $k_\Omega(p,z) \leqslant \log(e^t/(e^t-1)) = r/3$, for every $z \in \partial V_p$. This means that $V_p \subset B_{k_\Omega}(p, r/3)$, and therefore

$$k_\Omega(V_p, V_q) \geqslant \frac{r}{3}, \quad \text{for every } p,q \in E, \text{ with } p \neq q. \tag{4.7}$$

Now we define

$$r' := \min\left\{\frac{r}{3}, \log\left(\frac{3\,e^{r/3}-2}{2\,e^{r/3}-1}\right)\right\} = \log\left(\frac{3\,e^{r/3}-2}{2\,e^{r/3}-1}\right) \quad \text{and}$$
$$s' := 2\pi(e^{r/3}-1).$$

Then, by (4.5), (4.6) and (4.7), we know that E is an (r', s')-uniformly separated set, as we wanted to prove. □

If we consider $\Omega \setminus \{p_1, p_2\}$, where Ω is a plane domain and $p_1, p_2 \in \Omega$, several conformal invariants of $\Omega \setminus \{p_1, p_2\}$ degenerate when p_2 tends to p_1, e.g. the exponent of convergence and the isoperimetric constant with the Poincaré metric. We have the following surprising consequence of Theorem 4.4 about stability of hyperbolicity.

Corollary 4.8. *Let Ω be a δ-hyperbolic plane domain. Then, for each natural number n there exists a constant δ_n, which only depends on δ and n, such that $\Omega \setminus \{p_1, \dots, p_n\}$ is δ_n-hyperbolic, for any $p_1, \dots, p_n \in \Omega$.*

Proof. We prove the theorem by induction on n. Theorem 4.4 gives the result for $n = 1$ ($E = \{p_1\}$ is r-uniformly separated for any r). Let us assume that the result is true for $n-1$; then $\Omega^* := \Omega \setminus \{p_1, \dots, p_{n-1}\}$ is δ_{n-1}-hyperbolic, for any $p_1, \dots, p_{n-1} \in \Omega$. Theorem 4.4 gives that $\Omega^* \setminus \{p_n\}$ is δ_n-hyperbolic, where δ_n is a constant which only depends on δ_{n-1} and n ($E = \{p_n\}$ is r-uniformly separated in Ω^* for any r). □

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