CRITICAL VARIABLE EXPONENT FUNCTIONALS IN IMAGE RESTORATION

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ABSTRACT. We study variable exponent model for image restoration in the case that the exponent attains the critical value one. We prove existence and Γ-convergence. The results answer an open question by Li, Li and Pi [Variable exponent functionals in image restoration, *Appl. Math. Comput.* **216** (2010), no. 3, 870–882].

1. INTRODUCTION

Five years ago Chen, Levine & Rao [3, 9] (see also [1, 2]) proposed a variable exponent formulation for the problem of image restoration. The problem is the following: we are given an input signal f which equals the true signal u plus an additive, random noise (on a two dimensional rectangle, say). From f we must recover u. Since the noise is random, an obvious thing to do is to smooth the signal, which will lead the high frequency noise to cancel out. The problem with this approach is that it also looses critical information about object boundaries in the image. This problem can be overcome by smoothing only perpendicular to the direction of the gradient, so called total variation smoothing, a method proposed by Rudin, Osher & Fatemi [11]. The central problem with the second approach is that it too readily introduces boundaries, even when none exist in the true image, an effect which has been termed staircasing.

To understand the role of the variable exponent in the image restoration problem we look at the variational formulation of the previous two approaches. Isotropic smoothing corresponds to finding the minimum of the energy

$$\int_\Omega |\nabla u|^p + \lambda\,|u-f|^2\,dx, \tag{1.1}$$

with $p \equiv 2$, where $\lambda > 0$ is a parameter indicating the strength of the smoothing. Total variation smoothing, on the other hand, corresponds to minimizing the energy (1.1) with $p \equiv 1$.

The first minimization problem is naturally solved in the Sobolev space $W^{1,2}(\Omega)$, whereas the second is solved in the space $\mathrm{BV}(\Omega)$ of functions of bounded variation. Since we would like to combine the strengths of these two approaches, it is natural to formulate the minimization problem (1.1) for an exponent $p = p(x)$ varying in the interval $[1, 2]$. This is the essence of the model proposed in [3]:

$$\int_\Omega |\nabla u|^{p(x)} + \lambda\,|u-f|^2\,dx. \tag{1.2}$$

For an overview on such variational problems with variable exponent see [8]. Recently Li, Li and Pi studied this model in the case $p^- := \inf p > 1$ in [10]. In the end of their paper they ask whether it is possible to extend their results to the case $p^- = 1$. In this paper we

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propose a solution to this problem and show that the our energy operator is a natural limit of (1.2).

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipshitz boundary, $s \geqslant 1$ a fixed constant, and $p : \Omega \to [1, \infty)$ be a bounded lower semicontinuous exponent.

We denote $Y := \{x \in \Omega : p(x) = 1\}$,

$$\mathrm{BV}^{p(\cdot)}(\Omega) := \{\, u \in \mathrm{BV}(\Omega) \cap W^{1,p(\cdot)}(\Omega \setminus Y)\,\},$$

and

$$\varrho_{\mathrm{BV}^{p(\cdot)}(A)}(u) := \|\nabla u\|(Y \cap A) + \int_{A\setminus Y} |\nabla u|^{p(x)}\,dx$$

for every $A \subset \Omega$. Note that $\mathrm{BV}^{p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega \setminus Y)$ when $p^- > 1$; for more properties for $\mathrm{BV}^{p(\cdot)}$ see [7]. Our goal is to study the minimizing problem

$$\inf_u D_1(u) = \inf_u \Big(\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) + \lambda \int_\Omega |u - f|^s\,dx\Big)$$

in $\mathrm{BV}^{p(\cdot)}(\Omega) \cap L^s(\Omega)$, where λ is a fixed positive real number and $f \in L^s(\Omega)$ is the initial data. The following is our main result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be an open domain with Lipschitz boundary and let $p : \Omega \to [1, \infty)$ be lower semicontinuous. Then the minimizing problem*

$$\inf_{u\in \mathrm{BV}^{p(\cdot)}(\Omega)} \Big(\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) + \lambda \int_\Omega |u - f|^s\,dx\Big)$$

has a solution $u \in \mathrm{BV}^{p(\cdot)}(\Omega) \cap L^s(\Omega)$. Moreover, any minimizing sequence (u_i) has a subsequence such that $u_i \to u$ in $L^1(\Omega)$ and $\nabla u_i \rightharpoonup \nabla u$ weakly in $L^{p(\cdot)}(\Omega \setminus \overline{U})$ for every open $U \supset Y$.

In practice it is difficult to deal with the BV-part of the norm in $\mathrm{BV}^{p(\cdot)}(\Omega)$. Therefore we consider also approximating functionals which are defined as follows. For $\delta \geqslant 1$ we set $p_\delta := \max\{p, \delta\}$. We define energies

$$D_\delta(u) := \varrho_{\mathrm{BV}^{p_\delta(\cdot)}(\Omega)}(u) + \lambda \int_\Omega |u - f|^s\,dx$$

in $\mathrm{BV}^{p(\cdot)}(\Omega) \cap L^s(\Omega)$. We extend operators D_δ to $L^1(\Omega)$ by setting $D_\delta(u) := \infty$ for $u \in L^1(\Omega) \setminus \big(\mathrm{BV}^{p_\delta(\cdot)}(\Omega) \cap L^s(\Omega)\big)$. We prove Γ-convergence of our auxiliary functionals D_δ to D_1; thus we recall the following definition.

Definition 1.4. A sequence of functionals $F_\delta : L^1(\Omega) \to \overline{\mathbb{R}}$ is said to $\Gamma(L^1)$-converge to $F_1 : L^1(\Omega) \to \overline{\mathbb{R}}$ if the following hold for every sequence (δ_i) converging to one:

(1) for every $u \in L^1(\Omega)$ and for every sequence (u_i) in $L^1(\Omega)$ converging to u in $L^1(\Omega)$, we have

$$F_1(u) \leqslant \liminf_{i\to\infty} F_{\delta_i}(u_i);$$

(2) for every $u \in L^1(\Omega)$ there exists a $L^1(\Omega)$-sequence (u_i) (called a recovery sequence) such that $u_i \to u$ in $L^1(\Omega)$ and

$$F_1(u) \geqslant \limsup_{i\to\infty} F_{\delta_i}(u_i).$$

For Γ-convergence we require a stronger assumption on the exponent, namely so-called strong $\log$-Hölder continuity:

$$|p(x) - p(y)| \leqslant \frac{C}{\log(e + 1/|x-y|)}$$

for all $x, y \in \Omega$ and

$$\lim_{x\to y} \big|p(x) - 1\big| \log \frac{1}{|x-y|} = 0$$

for every $y \in Y$.

Theorem 1.5. *Let Ω be an open rectangle and let p be strongly $\log$-Hölder continuous. Then D_δ $\Gamma(L^1)$-converges to D_1.*

It can be seen from the proof that in fact we also have $\Gamma(\text{w-}L^1)$ convergence with respect to weak-L^1.

2. Existence and lower semicontinuity

We first give a lower semicontinuity result which will be used both for existence and Γ-convergence.

Theorem 2.1. *If $u_i \rightharpoonup u$ in $L^1(\Omega)$ and $\delta_i \to 1^+$, then*

$$D_1(u) \leqslant \liminf D_{\delta_i}(u_i).$$

In particular, if the limit inferior is finite, then $u \in \mathrm{BV}^{p(\cdot)}(\Omega) \cap L^s(\Omega)$

Proof. Let (u_i) be a sequence in $L^1(\Omega)$ converging weakly to u in $L^1(\Omega)$. By picking a subsequence, if necessary, we may assume that (u_i) gives the limit inferior (and thus so does its every subsequence). Denote $p_i := p_{\delta_i}$. To estimate the derivatives we are free to assume that $\alpha := \liminf_{i\to\infty} D_{\delta_i}(u_i) < \infty$ and $D_{\delta_i}(u_i) < \infty$ for every i.

Then $u_i - f$ is bounded in $L^s(\Omega)$, so it converges weakly to some function (by reflexivity if $s > 1$, and by assumption when $s = 1$); uniqueness of the limit implies that this function is $u - f$. Hence weak lower semicontinuity of the integral yields

$$\lambda \int_\Omega |u - f|^s \, dx \leqslant \liminf_{i\to\infty} \lambda \int_\Omega |u_i - f|^s \, dx. \tag{2.2}$$

Denote $\Omega_k := \{p > 1 + \frac{1}{k}\}$; since p is lower semicontinuous, Ω_k is open. Then

$$\int_{\Omega_k} |\nabla u_i|^{p(x)} \, dx \leqslant \int_{\Omega_k} |\nabla u_i|^{p_i(x)} + 1 \, dx \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(\nabla u_i) + |\Omega| \leqslant 2\alpha + |\Omega|,$$

when i is large enough. Hence (∇u_i) is a bounded sequence in the reflexive space $L^{p(\cdot)}(\Omega_k)$. By reflexivity, $\nabla u_i \rightharpoonup g$ (up to a subsequence) in $L^{p(\cdot)}(\Omega_k)$ for some function g. Since $u_i \rightharpoonup u$ in $L^1(\Omega)$, we see that if $\varphi \in C_0^\infty(\Omega_k)$, then

$$\int_{\Omega_k} g \cdot \varphi \, dx = \lim \int_{\Omega_k} \nabla u_i \cdot \varphi \, dx = -\lim \int_{\Omega_k} u_i \operatorname{div} \varphi \, dx = \int_{\Omega_k} u \operatorname{div} \varphi \, dx,$$

so actually $g = \nabla u$ in Ω_k.

By Young's inequality, $a^{p(\cdot)} \leqslant a^{p_i(\cdot)} + (p_i - p) \leqslant a^{p_i(\cdot)} + \delta_i - 1$. Hence by the weak lower semicontinuity of the modular, we have

$$\begin{aligned}\int_{\Omega_k} |\nabla u|^{p(x)} \, dx &\leqslant \liminf_{i\to\infty} \int_{\Omega_k} |\nabla u_i|^{p(x)} \, dx = \liminf_{i\to\infty} \int_{\Omega_k} |\nabla u_i|^{p_i(x)} + (\delta_i - 1) \, dx \\ &= \liminf_{i\to\infty} \int_{\Omega_k} |\nabla u_i|^{p_i(x)} \, dx \leqslant \liminf_{i\to\infty} \int_{\Omega\setminus Y} |\nabla u_i|^{p_i(x)} \, dx\end{aligned}$$

for every k. Letting $k \to \infty$ we obtain by the monotone convergence that $|\nabla u| \in L^{p(\cdot)}(\Omega \setminus Y)$.

To finish the proof, we choose for every $\varepsilon > 0$ an open neighborhood $U \subset \Omega$ of Y such that $|U \setminus Y| < \varepsilon$, $\int_{U\setminus Y} |\nabla u|^{p(x)}\,dx < \varepsilon$ and $|\partial U| = 0$. Since $u_i \rightharpoonup u$ in $L^1(\Omega)$ we obtain by [6, Theorem 1, p.172] that $u \in \mathrm{BV}(\Omega)$ and $\|\nabla u\|(U) \leqslant \liminf_{i\to\infty} \|\nabla u_i\|(U)$. By the argument in the beginning of this proof,

$$\int_{\Omega\setminus\overline{U}} |\nabla u|^{p(x)}\,dx \leqslant \liminf_{i\to\infty} \int_{\Omega\setminus\overline{U}} |\nabla u_i|^{p_i(x)}\,dx.$$

Hence by the pointwise inequality $|t|^{p(x)} \leqslant |t|^{p_i(t)} + \delta_i - 1$, we conclude that

$$\begin{aligned}\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) &\leqslant \|\nabla u\|(U) + \int_{\Omega\setminus\overline{U}} |\nabla u|^{p(x)}\,dx + \int_{U\setminus Y} |\nabla u|^{p(x)}\,dx \\ &\leqslant \liminf_{i\to\infty} \|\nabla u_i\|(U) + \liminf_{i\to\infty} \int_{\Omega\setminus\overline{U}} |\nabla u_i|^{p_i(x)}\,dx + \varepsilon\end{aligned}$$

We consider the subsequences with $\delta_i = 1$ and $\delta_i > 1$ separately. In the former case, since $|t| \leqslant |t|^{p_i(t)} + 1$, we find that

$$\|\nabla u_i\|(U) = \|\nabla u_i\|(Y) + \int_{U\setminus Y} |\nabla u_i|\,dx \leqslant \|\nabla u_i\|(Y) + \int_{U\setminus Y} |\nabla u_i|^{p_i(x)}\,dx + |U \setminus Y|,$$

while in the latter case, since $\nabla u_i \in L^{p_i(\cdot)}(\Omega)$,

$$\|\nabla u_i\|(U) = \int_U |\nabla u_i|\,dx \leqslant \int_U |\nabla u_i|^{p_i(x)}\,dx + |U \setminus Y| + (\delta_i - 1)|Y|.$$

In both cases we thus have

$$\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) \leqslant \liminf_{i\to\infty} \varrho_{\mathrm{BV}^{p_i(\cdot)}(\Omega)}(u_i) + 2\epsilon.$$

As $\varepsilon \to 0$, the claim follows from this and (2.2). □

We can then prove the existence of minimizers.

Proof of Theorem 1.3. Let

$$E := \inf_{u\in\mathrm{BV}^{p(\cdot)}(\Omega)} \Big(\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) + \int_\Omega \lambda|u - f|^s\,dx\Big);$$

since 0 is an admissible test function, $E \leqslant \lambda\,\varrho_{L^s(\Omega)}(f) < \infty$. Let (u_i) be a minimizing sequence. Then $\|u_i\|_{BV(\Omega)} \leqslant c < \infty$ so by [6, Theorem 4, p.176] we may choose a subsequence which converges in $L^1(\Omega)$. By Theorem 2.1 with $\delta_i = 1$, $D_1(\lim u_i) \leqslant \liminf D_1(u_i) = E$. Hence $\lim u_i$ is the desired minimizer. The other claims were proved in the proof of the said theorem. □

3. The recovery sequence

We construct the recovery sequence using a suitable convolution. For this we need that p is strongly log-Hölder continuous, as defined in the introduction.

We denote

$$\Omega_{a,b} := \{x \in \Omega \,:\, a < \mathrm{dist}(x, \partial\Omega) < b\},$$

and set $\Omega_a := \Omega_{a,\infty}$. For brevity, we also write

$$\varrho_{LBV^{p(\cdot)}(E)}(u) := \varrho_{L^{p(\cdot)}(E)}(u) + \varrho_{BV^{p(\cdot)}(E)}(u)$$

for $u \in \mathrm{BV}^{p(\cdot)}(\Omega)$ and

$$\varrho_{W^{1,p(\cdot)}(E)}(u) := \varrho_{L^{p(\cdot)}(E)}(u) + \varrho_{L^{p(\cdot)}(E)}(|\nabla u|)$$

for a Sobolev function $u \in W^{1,p(\cdot)}(\Omega)$ whenever $E \subset \Omega$ is measurable. Clearly

$$\varrho_{LBV^{p(\cdot)}(E)}(u) = \varrho_{W^{1,p(\cdot)}(E)}(u) \quad \text{if } u \in W^{1,p(\cdot)}(\Omega).$$

We need the following extension result:

Theorem 3.1. *Let Q be an open rectangle, and let $a > 0$ be less than half the length of its shorter side. If $u \in W^{1,p(\cdot)}(Q_a)$, then there exists an extension $Eu \in W^{1,p(\cdot)}(Q)$ such that*

$$\varrho_{W^{1,p(\cdot)}(Q_{0,a})}(Eu) \leqslant c\, \varrho_{W^{1,p(\cdot)}(Q_{a,2a})}(u).$$

Proof. Existence of an extension is proved e.g. in [4, Theorem 8.5.12]. The inequality can be seen by analyzing the proof. An easier proof for this case is in [5]. □

Let u_δ be the standard mollification of u.

Proposition 3.2 (Theorem 4.6, [7]). *Let p be strongly $\log$-Hölder continuous with $p^+ < \infty$. Assume that $u \in \mathrm{BV}^{p(\cdot)}(\Omega)$ and $F \subset \Omega$ is closed. Then $u_\delta \to u$ in $L^{p(\cdot)}(\Omega)$ as $\delta \to 0^+$ and*

$$\limsup_{\delta \to 0^+} \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u_\delta) \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u).$$

Lemma 3.3. *Let p be strongly $\log$-Hölder continuous with $p^+ < \infty$. Let Q be a rectangle. Then for every $u \in \mathrm{BV}^{p(\cdot)}(Q)$ and $\epsilon > 0$ there exists $\lambda_0 > 1$ and $\tilde{u} \in W^{1,p_\lambda(\cdot)}(Q)$ such that*

$$\varrho_{W^{1,p_\lambda(\cdot)}(Q)}(\tilde{u}) < \varrho_{LBV^{p(\cdot)}(Q)}(u) + \epsilon \quad \textit{and} \quad \varrho_{L^{p(\cdot)}(Q)}(u - \tilde{u}) < \epsilon$$

for every $\lambda \in (1, \lambda_0)$.

Proof. Fix $u \in \mathrm{BV}^{p(\cdot)}(Q)$ and $\epsilon \in (0,1)$. Let us choose first $a > 0$ such that $\varrho_{LBV^{p(\cdot)}(Q_{0,3a})}(u) < \epsilon$. Using Proposition 3.2, we then choose $\delta \in (0, \frac{a}{2})$ such that

$$\varrho_{LBV^{p(\cdot)}(Q_{a,2a})}(u_\delta) < \epsilon, \quad \varrho_{L^{p(\cdot)}(Q_a)}(u - u_\delta) < \epsilon,$$

and

$$\varrho_{LBV^{p(\cdot)}(\overline{Q_a})}(u_\delta) < \varrho_{LBV^{p(\cdot)}(\overline{Q_a})}(u) + \epsilon.$$

Since $u_\delta \in C^\infty(Q)$, the dominated convergence theorem implies that

$$\lim_{\lambda \to 1^+} \varrho_{LBV^{p_\lambda(\cdot)}(E)}(u_\delta) = \varrho_{LBV^{p(\cdot)}(E)}(u_\delta)$$

for any measurable set E with $\overline{E} \subset Q$. Hence we can choose $\lambda_0 > 1$ such that

$$\varrho_{LBV^{p_\lambda(\cdot)}(Q_{a,2a})}(u_\delta) < \epsilon \quad \text{and} \quad \varrho_{LBV^{p_\lambda(\cdot)}(\overline{Q_a})}(u_\delta) < \varrho_{LBV^{p(\cdot)}(\overline{Q_a})}(u) + \epsilon$$

for every $\lambda \in (1, \lambda_0)$.

Suppose that $\lambda \in (1, \lambda_0)$. By Theorem 3.1, we extend $u_\delta|_{Q_a}$ to a function $\tilde{u} \in W^{1,p_\lambda(\cdot)}(Q)$. Then

$$\begin{aligned}
\varrho_{LBV^{p_\lambda(\cdot)}(Q)}(\tilde{u}) &= \varrho_{LBV^{p_\lambda(\cdot)}(\overline{Q_a})}(u_\delta) + \varrho_{LBV^{p_\lambda(\cdot)}(Q_{0,a})}(\tilde{u}) \\
&\leqslant \varrho_{LBV^{p(\cdot)}(\overline{Q_a})}(u) + \epsilon + c\varrho_{LBV^{p_\lambda(\cdot)}(Q_{a,2a})}(u_\delta) \\
&\leqslant \varrho_{LBV^{p(\cdot)}(Q)}(u) + c\epsilon.
\end{aligned}$$

Note that the extension is independent of λ, i.e. the same function $\tilde{u}$ can be used for every $\lambda \in (1, \lambda_0)$. Hence the first inequality is proved.

To prove the $L^{p(\cdot)}$-inequality, we estimate

$$\begin{aligned}\varrho_{L^{p(\cdot)}(Q)}(u-\tilde u) &\leqslant \varrho_{L^{p(\cdot)}(Q_a)}(u-u_\delta) + 2^{p^+}\Big(\varrho_{L^{p(\cdot)}(Q_{0,a})}(u) + \varrho_{L^{p(\cdot)}(Q_{0,a})}(\tilde u)\Big)\\ &\leqslant \epsilon + 2^{p^+}(\epsilon + c\epsilon).\end{aligned}$$

Here again the latter inequality follows from Theorem 3.1 and the choice of a. □

We are now ready to prove the Γ-convergence.

Proof of Theorem 1.5. Condition (1) of Γ-convergence was established in Theorem 2.1. Here we prove Condition (2). So let $\delta_i \to 1^+$ and $u \in L^1(Q)$. If $\varrho_{BV^{p(\cdot)}(Q)}(u) = \infty$, there is nothing to prove. So we assume that $u \in BV^{p(\cdot)}(Q)$.

Let $\tilde u_j$ be the function $\tilde u$ from Lemma 3.3 corresponding to $\epsilon = \frac{1}{j}$ and let $\lambda_j > 1$ be less than the corresponding λ_0. We are free to assume that (λ_j) decreases to 1. Fix $\epsilon > 0$. For each i, let $j(i)$ be the largest index j for which $\lambda_j > \delta_i$. Since $\delta_i \to 1^+$, we have $j(i) \to \infty$. Now we choose $u_i = \tilde u_{j(i)}$ as the sequence in Condition (2). By Lemma 3.3,

$$\limsup_{i\to\infty} \varrho_{W^{1,p_{\delta_i}(\cdot)}(Q)}(u_i) \leqslant \varrho_{LBV^{p(\cdot)}(Q)}(u) \quad \text{and} \quad \lim_{i\to\infty} \varrho_{L^{p(\cdot)}(Q)}(u_i) = \varrho_{L^{p(\cdot)}(Q)}(u).$$

Therefore

$$\limsup_{i\to\infty} \varrho_{BV^{p_{\delta_i}(\cdot)}(Q)}(u_i) \leqslant \varrho_{BV^{p(\cdot)}(Q)}(u).$$

and $u_i \to u$ in $L^1(Q)$ by Hölder's inequality. Hence Condition (2) holds. □

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