ISOMETRIES OF THE HALF-APOLLONIAN METRIC

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Abstract. The half-apollonian metric is a generalization of the hyperbolic metric, similar to the Apollonian metric. It can be defined in arbitrary domains in euclidean space and has the advantages of being easy to calculate and estimate. We show that the half-apollonian metric has quite many geodesics and use this fact to show that in most domains all the isometries of the metric are similarity mappings.

In this paper we introduce a new metric which we call the half-apollonian metric. It is bilipschitz equivalent to the Apollonian metric which A. Beardon [1] introduced as a generalization of the hyperbolic metric. The new metric is simpler than the Apollonian metric, for instance we show here that the isometry question can be completely solved. The most interesting features of the Apollonian metric are that it is simple to define, calculate and estimate, that it is Möbius invariant and that it equals the hyperbolic metric in balls and half-spaces. The main inconvenience of this metric is that it is not geodesic. The half-apollonian metric considered in this paper goes some way towards remedying this short-coming in that it has lots of geodesics. Specifically, through every point there is at least one geodesic ray. Also the half-apollonian metric is easy to define, calculate and estimate, however, it is only Möbius quasi-invariant.

The question of the isometries of the Apollonian metric was posed by A. Beardon [1], and turned out to be quite difficult. F. Gehring and K. Hag [2], Z. Ibragimov [6],[7] and P. Hästö [5] have made some progress in deriving a solution, but the problem remains open in most cases. In this paper we show that the question of isometries is much more tractable in the case of the half-apollonian metric, mainly on account of the geodesics. The main result, Theorem 3.10, shows that in most domains every homeomorphic half-apollonian isometry is a similarity mapping.

Given this, and since the Apollonian and half-apollonian metric are bilipschitz equivalent (Theorem 2.1), it seems reasonable to ask, whether the half-apollonian metric would be a more useful tool than the Apollonian metric in higher dimensional geometric function theory.

The paper is organized as follows: In the next section we give the definitions of the half-apollonian, the Apollonian and the j_G metrics. In Section 2 we give some simple comparison results between the metrics, in particular, we show that

2000 *Mathematics Subject Classification.* Primary 51M15; Secondary 30F45.

Key words and phrases. Half-apollonian metric, Apollonian metric, geodesic ray, regular boundary point, half-apollonian isometry.

PH was supported in part by the Academy of Finland through the Graduate School of Analysis and Logic. HL was supported in part by the Finnish Academy of Science and Letters through the Vilho, Yrjö and Kalle Väisälä fund.

the Apollonian and half-apollonian metrics are bilipschitz equivalent. This implies that comparison results of the Apollonian metric, like those derived by A. Rhodes [8], P. Seittenranta [9] and P. Hästö [4], immediately give results for the half-apollonian metric, as well. In Section 3 we study geodesics of the half-apollonian metric and prove the main result.

1. The metrics

Throughout this paper we consider domains G, i.e. open and connected subsets of $\mathbb{R}^n$ and metrics defined in them. We denote by $B^n(x,r)$ the open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ and put $\mathbf{B}^n = B^n(0,1)$. By $S^{n-1}(x,r)$ we denote the sphere $\partial B^n(x,r)$ and by $\mathbf{H}^n$ the upper half-space.

Our main interest lies in the metric which we will refer to as the half-apollonian metric η_G (for ἥμισυς hemisys, gr. half). Note that a related metric h_G was introduced already in [11, 2.38].

Definition 1.1. Let G be a proper open subset of $\mathbb{R}^n$. For $x, y \in G$ we define the *half-apollonian metric* η_G by

$$\eta_G(x,y) = \sup_{z \in \partial G} \left| \log \frac{|x-z|}{|y-z|} \right|.$$

The following result contains some basic properties of η_G.

Theorem 1.2. *The half-apollonian distance η_G is a pseudometric in G, and a proper metric whenever $\mathbb{R}^n \setminus G$ is not a subset of a hyperplane. Furthermore, η_G is invariant under similarity mappings.*

Proof. We first prove that η_G is a pseudometric irrespective of ∂G. Symmetry and positivity are clear, so we need only prove the triangle inequality. For $x, y \in G$ denote by $w_{x,y}$ a point in ∂G for which $\eta_G(x,y) = |\log(|x-w_{x,y}|/|y-w_{x,y}|)|$. For $x, y, z \in G$ we have

$$\begin{aligned} \eta_G(x,y) + \eta_G(y,z) &= \left| \log \frac{|x-w_{x,y}|}{|y-w_{x,y}|} \right| + \left| \log \frac{|y-w_{y,z}|}{|z-w_{y,z}|} \right| \\ &\geq \left| \log \frac{|x-w_{x,z}|}{|y-w_{x,z}|} \right| + \left| \log \frac{|y-w_{x,z}|}{|z-w_{x,z}|} \right| \\ &\geq \left| \log \frac{|x-w_{x,z}|}{|y-w_{x,z}|} \frac{|y-w_{x,z}|}{|z-w_{x,z}|} \right| = \left| \log \frac{|x-w_{x,z}|}{|z-w_{x,z}|} \right| \\ &= \eta_G(x,z), \end{aligned}$$

the triangle inequality.

Now let $x, y \in G$ be arbitrary. Clearly

$$\sup_{z \in \partial G} \left| \log \frac{|x-z|}{|y-z|} \right| = 0$$

if and only if $|x-z|/|z-y| = 1$ for all $z \in \partial G$. Assuming $x \neq y$ the condition $|x-z|/|z-y| = 1$ holds only for points z located on the mid-point normal hyperplane of the line segment $[x, y]$. Thus we see that $\eta_G(x, y) = 0$ is possible for $x \neq y$ if and only if ∂G is a subset of a hyperplane.

Since a similarity is defined as a mapping which leaves the ratio of euclidean distances invariant, it is clear that η_G is similarity invariant. □

The question of whether there exist mappings leaving η_G invariant that are not similarities will be further addressed in Section 3.

Before that we want to compare the half-apollonian metric with some other metrics. In the Möbius space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, define the Apollonian metric α_G, intimately connected to η_G, by

$$\alpha_G(x, y) = \sup_{z,w \in \partial G} \log |z, x, y, w| = \sup_{z \in \partial G} \log \frac{|z - y|}{|z - x|} + \sup_{w \in \partial G} \log \frac{|w - x|}{|w - y|},$$

where the cross ratio is defined by

$$|a, b, c, d| = \frac{|a - c|}{|a - b|} \frac{|b - d|}{|c - d|} \quad \text{with} \quad \frac{|\infty - c|}{|\infty - b|} = 1 \text{ etc.}$$

The reason η_G is called the half-apollonian metric is that it is defined using only one of the two parts of the Apollonian metric. We will see later that this means that half-apollonian metric is about half as large as the Apollonian metric.

We will also need the *j_G-metric*, which is a modification from [10] of a metric introduced in [3]. It is defined for $x, y \in G \subsetneq \mathbb{R}^n$ by

$$j_G(x, y) = \log \Big(1 + \frac{|x - y|}{\min\{d(x), d(y)\}} \Big),$$

where $d(z) = d(z, \partial G)$.

2. Basic properties of the half-apollonian metric

Let us first show that the Apollonian and half-apollonian metrics are always bilipschitz equivalent.

Theorem 2.1. *Let $G \subsetneq \mathbb{R}^n$ be a domain. Then the double inequality*

$$\frac{1}{2} \alpha_G(x, y) \le \eta_G(x, y) \le \alpha_G(x, y)$$

holds for all $x, y \in G$. Both inequalities are sharp in the sense that there exists domains such that equality holds for some points x and y.

Proof. Since G does not contain the point at infinity we have

$$\sup_{w \in \partial G} \log \frac{|w - y|}{|w - x|} \ge 0$$

for all $x, y \in G$. Assume without loss of generality that x and y are ordered so that $\eta_G(x, y) = \sup_{w \in \partial G} \log(|x - w|/|y - w|)$. Then

$$\begin{aligned} \eta_G(x, y) &= \sup_{z \in \partial G} \log \frac{|z - x|}{|z - y|} \\ &\le \sup_{z \in \partial G} \log \frac{|z - x|}{|z - y|} + \sup_{w \in \partial G} \log \frac{|w - y|}{|w - x|} = \alpha_G(x, y). \end{aligned}$$

Thus $\eta_G(x,y) \le \alpha_G(x,y)$. For the inequality in the other direction we have

$$\begin{aligned}\alpha_G(x,y) &= \sup_{z\in\partial G}\log\frac{|z-x|}{|z-y|}+\sup_{w\in\partial G}\log\frac{|w-y|}{|w-x|}\\ &\le 2\max\Big\{\sup_{z\in\partial G}\log\frac{|z-x|}{|z-y|},\sup_{w\in\partial G}\log\frac{|w-y|}{|w-x|}\Big\}=2\eta_G(x,y).\end{aligned}$$

It remains to prove that the inequalities are sharp. For the lower bound choose $G=\mathbf{B}^n$. Then $\alpha_{\mathbf{B}^n}(-e_1/2,e_1/2)=\log 9$ and $\eta_{\mathbf{B}^n}(-e_1/2,e_1/2)=\log 3$. For the upper bound choose $G=\mathbf{H}^n$. Then $\alpha_{\mathbf{H}^n}(e_n,2e_n)=\eta_{\mathbf{H}^n}(e_n,2e_n)=\log 2$. □

Corollary 2.2. *Let $f\colon G\to\mathbb{R}^n$ be a Möbius mapping. Then*

$$\frac{1}{2}\eta_G(x,y)\le\eta_{f(G)}(f(x),f(y))\le 2\eta_G(x,y)$$

for all $x,y\in G$. Both inequalities are sharp in the sense of Theorem 2.1.

Proof. Since the Apollonian metric is Möbius invariant [1] we have

$$\eta_{f(G)}(f(x),f(y))\le\alpha_{f(G)}(f(x),f(y))=\alpha_G(x,y)\le 2\eta_G(x,y).$$

The lower bound follows similarly.

For optimality consider the domain $\mathbf{B}^n$ and the inversion in $S^{n-1}(-e_n,\sqrt{2})$ denoted by f. Then we have $\eta_{\mathbf{B}^n}(re_n,-re_n)=\log\frac{1+r}{1-r}$ for $0<r<1$ and

$$\eta_{f(\mathbf{B}^n)}(f(re_n),f(-re_n))=\eta_{\mathbf{H}^n}\Big(\frac{1-r}{1+r}e_n,\frac{1+r}{1-r}e_n\Big)=2\log\frac{1+r}{1-r},$$

so the constant 2 is attained. The same inversion from $\mathbf{H}^n$ to $\mathbf{B}^n$ shows that the constant 1/2 is also optimal. □

In [1] it was shown that $\alpha_G\le 2j_G$. We will see next that the half-apollonian metric again does credit to its name as the constant 2 can be removed from in front of the j_G-metric, when comparing with η_G.

Theorem 2.3. *For every domain $G\subsetneq\mathbb{R}^n$ the inequality $\eta_G(x,y)\le j_G(x,y)$ holds for all $x,y\in G$. The inequality is sharp in the sense of Theorem 2.1.*

Proof. Let $z\in\partial G$ be arbitrary, and denote $d(x)=d(x,\partial G)$. Now, by the euclidean triangle inequality we have that

$$\begin{aligned}\log\frac{|x-z|}{|y-z|} &\le \log\frac{|y-z|+|x-y|}{|y-z|}=\log\Big(1+\frac{|x-y|}{|y-z|}\Big)\\ &\le \log\Big(1+\frac{|x-y|}{\min\{d(x),d(y)\}}\Big)=j_G(x,y).\end{aligned}$$

Since this is true for all $z\in\partial G$, in fact

$$\eta_G(x,y)=\sup_{z\in\partial G}\left|\log\frac{|x-z|}{|y-z|}\right|\le j_G(x,y).$$

For sharpness use $j_{\mathbf{H}^n}(e_n,2e_n)=\eta_{\mathbf{H}^n}(e_1,2e_1)=\log 2$. □

Since the half-apollonian metric is only a pseudo-metric in some domains, it is clear that it can not be estimated from below by a constant times the j_G-metric. We do, however, have the following result, which resembles [9, Theorem 3.11].

Theorem 2.4. *For every domain $G \subsetneq \mathbb{R}^n$ the inequality*

$$j_G(x,y) - \log 3 \le \eta_G(x,y)$$

holds for all $x, y \in G$. The inequality is sharp in the sense of Theorem 2.1.

Proof. Assume that $d(x) \le d(y)$ and that $z \in \partial G$ is such that $|x-z| = d(x)$. This implies $|x-z| \le |y-z|$ and $|x-y| \le 2|y-z|$. Thus

$$\begin{aligned} j_G(x,y) &= \log\Big(1+\frac{|x-y|}{d(x)}\Big) = \log\Big(1+\frac{|x-y|}{|x-z|}\Big) \\ &\le \log\Big(1+\frac{2|y-z|}{|x-z|}\Big) \le \sup_{w\in\partial G}\log\Big(1+2\frac{|y-w|}{|x-w|}\Big) \\ &\le \sup_{w\in\partial G}\log\Big(\frac{|y-w|}{|x-w|}+2\frac{|y-w|}{|x-w|}\Big) \le \sup_{w\in\partial G}\log\Big(3\frac{|y-w|}{|x-w|}\Big) \\ &= \log 3 + \eta_G(x,y). \end{aligned}$$

Equality holds in the inequality if $G = \mathbb{R}^n \setminus \{0\}$ and $y = -x$. □

If we want a lower bound for the half-apollonian metric in terms of a multiple of j_G then we have to restrict our attention to a certain class of domains, which P. Hästö [4] called interior double ball domains. Indeed, because of Theorem 2.1 it follows directly from [4, Theorem 5.13] that $j_G/K \le \eta_G$ for some constant $K > 0$ in exactly the same class of domains. A similar trick works for all properties that only require metrics to be approximately equal. Therefore the most important differences between the Apollonian and the half-apollonian metric come into play only when we study finer properties, such as the geometry of the metric, to which we now turn.

3. Geodesics and isometries

In this section we study geodesics of the half-apollonian metric in order to get a grip of its isometries. Recall that every similarity is an η_G-isometry. Our main objective in this section is to prove that in many cases these are all the isometries. We first need some definitions and auxiliary results.

The following lemma parallels [2, Lemma 3.18].

Lemma 3.1. *Let $x, y, z \in G$ be points such that $\eta_G(x,y) = \eta_G(x,z) + \eta_G(z,y)$. Then there exists a point $w \in \partial G$ such that*

$$\eta_G(x,y) = \left|\log\frac{|x-w|}{|y-w|}\right|, \qquad \eta_G(x,z) = \left|\log\frac{|x-w|}{|z-w|}\right|$$

$$\textit{and} \quad \eta_G(z,y) = \left|\log\frac{|z-w|}{|y-w|}\right|.$$

Proof. Swap x and y if necessary and assume that $\eta_G(x,y) = \log \frac{|x-w|}{|y-w|}$. Then

$$\begin{aligned}\eta_G(x,y) &= \log\frac{|x-w|}{|z-w|} + \log\frac{|z-w|}{|y-w|} \le \left|\log\frac{|x-w|}{|z-w|}\right| + \left|\log\frac{|z-w|}{|y-w|}\right| \\ &\le \eta_G(x,z) + \eta_G(z,y).\end{aligned}$$

It follows by assumption that we have equality throughout. Therefore

$$\log\frac{|x-w|}{|z-w|} = \left|\log\frac{|x-w|}{|z-w|}\right|, \quad \log\frac{|z-w|}{|y-w|} = \left|\log\frac{|z-w|}{|y-w|}\right|,$$

$$\eta_G(x,z) = \left|\log\frac{|x-w|}{|z-w|}\right| \quad \text{and} \quad \eta_G(z,y) = \left|\log\frac{|z-w|}{|y-w|}\right|.$$

□

Definition 3.2. Let $G \subset \mathbb{R}^n$ be a domain. We say that a euclidean path $\gamma \subset G$ is a geodesic if for every $x, y, z \in \gamma$ ordered appropriately we have $\eta_G(x,y) = \eta_G(x,z) + \eta_G(z,y)$.

Corollary 3.3. *Let $\gamma \subset G$ be a geodesic. Then there exists a point $w \in \partial G$ such that for every $x, y \in \gamma$ we have*

$$\eta_G(x,y) = \left|\log\frac{|x-w|}{|y-w|}\right|.$$

By a *geodesic ray* we mean a geodesic whose (euclidean) closure intersects the domain boundary. We easily see that this intersection point (for there can be only one) is the point w in the previous corollary. We call this point the *base-point* of γ and denote it by γ^0. For a geodesic ray and $t > 0$ we denote by γ^t the point on the geodesic with $|\gamma^t - \gamma^0| = t$. (Note that γ^t is well-defined only when η_G is a metric, i.e. when $\mathbb{R}^n \setminus G$ is not contained in a hyperplane.)

Example 3.4. In $\mathbf{H}^n$ the geodesic rays emanating from 0 are half of the half-circles which are orthogonal to $\partial\mathbf{H}^n$ at 0 and the positive e_n-axis, see Figure 1. Since every geodesic is associated with a boundary point this means that every geodesic in $\mathbf{H}^n$ is a segment, a ray or an arc of a circle.

Let G be an arbitrary domain and suppose that $x \in G$. If $z \in \partial G$ is such that $d(x) = |x - z|$ then the half-open segment $(z, x]$ is a geodesic ray. Hence through every $x \in G$ there is at least one geodesic ray.

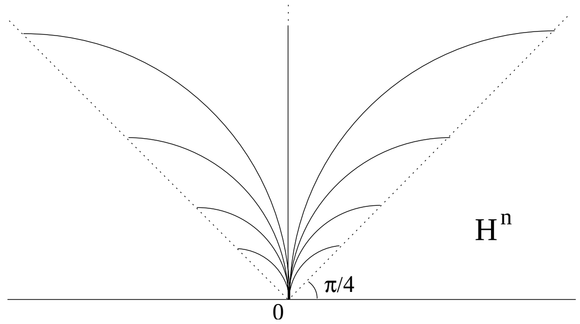

FIGURE 1. Geodesic rays emanating from 0 in $\mathbf{H}^n$.

Lemma 3.5. *Let $G \subsetneq \mathbb{R}^n$ and $f\colon G \to \mathbb{R}^n$ be a homeomorphic η_G-isometry. For every geodesic ray γ there exists a number $K_\gamma > 0$ such that*

$$|f(x) - f(\gamma)^0| = K_\gamma |x - \gamma^0|,$$

for every $x \in \gamma$.

Proof. Let $\gamma \subset G$ be a geodesic ray. Then $f(\gamma)$ is also a geodesic ray. Fix $y \in \gamma$ and let also $x \in \gamma$. By Corollary 3.3 we have

$$\log \frac{|x - \gamma^0|}{|y - \gamma^0|} = \eta_G(x, y) = \eta_{f(G)}(f(x), f(y)) = \log \frac{|f(x) - f(\gamma)^0|}{|f(y) - f(\gamma)^0|}.$$

In other words,

$$|f(x) - f(\gamma)^0| = \frac{|f(y) - f(\gamma)^0|}{|y - \gamma^0|} |x - \gamma^0|.$$

$\square$

Definition 3.6. Let $G \subset \mathbb{R}^n$ and $x \in \partial G$. We say that x is *circularly accessible* if there exists a ball $B \subset G$ such that $x \in \partial B$. If x is circularly accessible by two distinct balls whose surfaces intersect at more than one point then we say that x is a *corner point*. If x is circularly accessible, but not a corner point, we say that x is a *regular point*.

Example 3.7. Let $G_n = \mathbf{B}^n \setminus [0, e_1)$. In the case $n = 2$ every boundary point of G_2 except 0 and e_1, is regular. The origin is a corner point. For $n \geq 3$, all the points of $[0, e_1)$ are corner points of G_n.

If w is a regular boundary point, then it is the base-point of some geodesic ray, namely, if $B^n(x, r) \subset G$ is such that $w \in S^{n-1}(x, r)$, then $(w, x]$ is a geodesic ray.

Lemma 3.8. *Let $G \subsetneq \mathbb{R}^n$ and let $w \in \partial G$ be a regular boundary point. Then through every point $x \in G$ there exists at most one geodesic ray with w as base-point. If it exists, the geodesic is a circular arc, a segment or a ray.*

Proof. Let γ be a geodesic ray and let $x, y \in \gamma$ be distinct such that $|x - \gamma^0| > |y - \gamma^0|$. By Corollary 3.3 we have

$$\left| \log \frac{|x - z|}{|y - z|} \right| \leq \log \frac{|x - \gamma^0|}{|y - \gamma^0|},$$

for all $z \in \partial G$. This means that the balls

$$\left\{ z \in \mathbb{R}^n \colon \frac{|x - z|}{|y - z|} \geq \frac{|x - \gamma^0|}{|y - \gamma^0|} \right\} \quad \text{and} \quad \left\{ z \in \mathbb{R}^n \colon \frac{|y - z|}{|x - z|} \geq \frac{|x - \gamma^0|}{|y - \gamma^0|} \right\}$$

(the so-called *Apollonian balls*, see [1]) are contained in G. Therefore γ^0 is circularly accessible.

In the above ball-argument we can fix x and let $y \to x$. Then the boundary of the balls, which is squashed between x and y, approaches x. Thus if γ^0 is a regular boundary point, there is a unique ball in G whose boundary contains γ^0 and x. It follows that $\gamma\colon [0, 1] \to G$ is differentiable at $\gamma^{-1}(x)$, and this sphere determines the derivative of γ at x. The same holds for all points on γ, and so the geodesic is uniquely determined by the base-point and x. Let H be the half-space which is tangential to the boundary at w. Then the geodesics in G are subsets of geodesics in H, hence, by Example 3.4, circular arcs, segments or rays. $\square$

Corollary 3.9. *Let $G \subsetneq \mathbb{R}^n$ and $f\colon G \to fG \subset \mathbb{R}^n$ be a homeomorphic η_G-isometry. Let γ be a geodesic ray. Then $f(\gamma)^0$ is a regular or corner point according as γ^0 is.*

Proof. As in the proof of the previous lemma we see that $f(\gamma)^0$ is circularly accessible. It is easy to see that that if γ^0 is a corner point, then there exists $x \in G$ and two distinct geodesic rays through x with base-point γ^0. This property is clearly preserved by a homeomorphic η_G-isometry. Therefore corner points correspond to corner points and regular boundary points to regular boundary points. $\square$

Theorem 3.10. *Let $G \subsetneq \mathbb{R}^n$ be a domain which has at least n regular boundary points which span a hyperplane. Then $f\colon G \to fG \subset \mathbb{R}^n$ is a homeomorphic η_G-isometry if and only if it is a similarity.*

Proof. Denote the n regular boundary points which span a hyperplane by $w_1,\dots,$ w_n. To each w_i there corresponds a geodesic ray which is a euclidean half-open segment; we denote it by γ_i.

By Lemma 3.5 we assume, without loss of generality, that

$$|f(x) - f(\gamma_1)^0| = |x - \gamma_1^0| \tag{3.11}$$

for $x \in \gamma_1$. For the other boundary points we have

$$|f(x) - f(\gamma_i)^0| = K_i|x - \gamma_i^0|$$

for some $K_i > 0$ and every $x \in \gamma_i$.

Let us show that in fact $K_2 = 1$. Let $y \in \gamma_1$ and fix $x \in \gamma_2$. We have

$$\log\frac{|x-\gamma_1^0|}{|y-\gamma_1^0|} \le \eta_G(x,y) = \eta_{f(G)}(f(x),f(y)) \le j_{f(G)}(f(x),f(y)).$$

Since w_1 is a regular boundary point it follows from Corollary 3.9 that so is $f(\gamma_1)^0$. Furthermore, since the geodesic rays are circular arcs orthogonal to the boundary, we have $|z - f(\gamma_1)^0|/d(z,\partial fG) \to 1$ as $d(z,\partial fG) \to 0$ along a geodesic ray. Then we see that

$$\begin{aligned} j_{f(G)}(f(x),f(y)) \quad &\le \quad \log\Big(1+\frac{|f(x)-f(y)|}{d(f(y),\partial fG)}\Big) \\ &\xrightarrow{y\to\gamma_1^0} \quad \log\frac{|f(x)-f(y)|+|f(y)-f(\gamma_1)^0|}{|f(y)-f(\gamma_1)^0|}, \end{aligned}$$

i.e. the previous expression converges asymptotically to the latter one as $y \to \gamma_1^0$. By (3.11) this implies as $|y-\gamma_1^0| \to 0$ that

$$|x-\gamma_1^0| \le |f(x)-f(y)| = |f(x)-f(\gamma_1)^0|.$$

Similarly we derive a lower bound, so that in fact $|x-\gamma_1^0| = |f(x)-f(\gamma_1)^0|$. Now letting $x \to \gamma_2^0$ gives $|\gamma_2^0-\gamma_1^0| = |f(\gamma_1)^0 - f(\gamma_2)^0|$.

We can use the same argument by first letting $x \to \gamma_2^0$ for fixed $y \in \gamma_1^0$ and then letting $y \to \gamma_1^0$. This gives $|\gamma_1^0-\gamma_2^0| = |f(\gamma_1)^0 - f(\gamma_2)^0|/K_2$. Since $|\gamma_1^0-\gamma_2^0| =$ $|f(\gamma_1)^0 - f(\gamma_2)^0|$ and the boundary points are distinct, we conclude that $K_2 = 1$.

The same argument applies to K_i for $i = 3, \dots n$, as well, so they all equal 1 and, moreover, the argument showed that f acts as an isometry on $\{\gamma_1^0, \dots, \gamma_n^0\}$. Hence we may assume that $f(\gamma_i)^0 = \gamma_i^0$ for all $i = 1, \dots n$.

Let then $x \in G$ be arbitrary and fix $i = 1 \dots n$. Let $y \in \gamma_i$ be close to the base-point. As before we find that

$$\begin{aligned} \log \frac{|x - \gamma_i^0|}{|y - \gamma_i^0|} &\leq \eta_G(x, y) = \eta_{f(G)}(f(x), f(y)) \\ &\leq j_{f(G)}(f(x), f(y)) \xrightarrow{y \to \gamma_i^0} \log \frac{|f(x) - f(y)| + |f(y) - \gamma_i^0|}{|f(y) - \gamma_i^0|}. \end{aligned}$$

As $y \to \gamma_i^0$ we thus get $|x - \gamma_i^0| \leq |f(x) - \gamma_i^0|$. Similarly we derive the bound from below. Thus $|x - \gamma_i^0| = |f(x) - \gamma_i^0|$. Since this holds for n distinct points that span a hyperplane, we see that $f(x)$ can be in only one of two positions, either $f(x) = x$ or $f(x)$ is the reflection of x in the hyperplane spanned by the w_i. Since f is continuous we conclude that either it is the identity for all the points or it is a reflection for all the points. In either case f is a similarity mapping, which was to be shown. □

In some special cases we do not need to make any assumptions on the regularity of the boundary.

Theorem 3.12. *Let $K \subset \mathbb{R}^n$ be compact and not contained in an $n-2$-plane. Define $G = \mathbb{R}^n \setminus K$. Then every homeomorphic η_G-isometry $f\colon G \to fG \subset \mathbb{R}^n$ is a similarity.*

Proof. Choose n half-spaces H_i such that $G \subset H_i$ and $x_i \in \partial G \cap \partial H_i$ are distinct. Let γ_i be the ray emanating from x_i which is normal to H_i and does not intersect G.

Choose x and y on γ_1 and γ_2 with $d(x) = d(y) = 1$. Suppose that $d'(f(x)) < d'(f(y))$. Since f is a euclidean similarity on the geodesic rays by Lemma 3.5, we see that γ_1^s and γ_2^s are mapped to points on the geodesics such that $d'(f(\gamma_1^s)) = d'(f(x))s$ and $d'(f(\gamma_2^s)) = d'(f(y))s$, for $s > 0$. Using the base point of γ_1 as a test point, we find that

$$\eta_{f(G)}(f(\gamma_1^s), f(\gamma_2^s)) \geq \log\left(\frac{d'(f(\gamma_2^s))}{d'(f(\gamma_1^s))}\right) = \log\left(\frac{d'(f(y))}{d'(f(x))}\right) > 0.$$

But obviously $\eta_G(\gamma_1^s, \gamma_2^s) \to 0$ as $s \to \infty$, a contradiction. Therefore $d'(f(x)) = d'(f(y))$, and we may assume, using an auxiliary scaling, that f is a euclidean isometry on every geodesic ray.

After this we can proceed as in the proof of Theorem 3.10 to show that f is the identity or a reflection. □

Theorem 3.13. *Let $K \subset \mathbb{R}^n$ be a discrete set of points not contained in an $(n-2)$-plane. Define $G = \mathbb{R}^n \setminus K$. Then every homeomorphic η_G-isometry $f\colon G \to \mathbb{R}^n$ is a similarity.*

Proof. Choose n boundary points which span a hyperplane. With the notation as in the proof of Theorem 3.10 we have

$$\begin{aligned}\log\frac{|x-\gamma_1^0|}{|y-\gamma_1^0|} &\le \eta_G(x,y)=\eta_{f(G)}(f(x),f(y))\\ &\le j_{f(G)}(f(x),f(y))=\log\frac{|f(x)-f(y)|+|f(y)-f(\gamma_1)^0|}{|f(y)-f(\gamma_1)^0|},\end{aligned}$$

since $f(\gamma_1)^0$ is the only boundary point near $f(y)$ for small $|y-\gamma_1^0|$. Using this we obtain as in the proof of Theorem 3.10 that f is an isometry on the n boundary points that we have chosen, hence the identity or a reflection. ☐

Let us conclude with two examples. The first one indicates why we study *homeomorphic* η_G-isometries.

Example 3.14. Let $s\colon \overline{\mathbf{H}^2}\to\{1,-1\}$ be an arbitrary mapping. Define $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ by $f(x,y)=(x,s(xe_1+|y|e_2)y)$. Let $K\subset\mathbb{R}$ and $G=\mathbb{R}^2\setminus(K\times\{0\})$. Then f is an η_G-isometry, since $|x-z|=|f(x)-z|$ for every $z\in\partial G$. If s is non-constant on $\mathbf{H}^2$ then f is not continuous and hence not a homeomorphism.

It is not clear whether the phenomenon of the previous example can occur when the boundary is not contained in a hyperplane. However, if we reduce the boundary further, then we even get homeomorphic η_G-isometries which are not similarities:

Example 3.15. Let $G=\mathbb{R}^2\setminus\{0\}$. Let $\theta\colon[0,\infty)\to[0,2\pi)$ be a continuous function. Then $f\colon\mathbb{R}\times[0,2\pi)\to\mathbb{R}\times[0,2\pi)$ defined in polar coordinates by $f(r,\phi)=(r,\phi+\theta(r))$ is a homeomorphic η_G-isometry. If θ is non-constant on $(0,\infty)$ then f is not a similarity. Clearly the same argument applies also in higher dimensions n as long as the dimension of the boundary is less than or equal to $n-2$.

Acknowledgements. The authors would like to thank Matti Vuorinen for suggesting studying the half-apollonian metric and the referee for comments.

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