SOLUTION OF TWO CONJECTURES OF KLÉN, LINDÉN, VUORINEN AND WANG

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ABSTRACT. This note contains a solution to two conjectures of Klén, Lindén, Vuorinen and Wang in a recent paper: the visual angular metric v_G can be estimate from above by $c\, j_G$ when G is a ball or a half-space, where the constant $c \approx 1.432404247$.

In a recent paper, Klén, Lindén, Vuorinen and Wang [1] studied the visual angular metric v_G and proposed in Conjectures 3.13 and 3.28 that it can be estimated from above by $c\, j_G$, where the constant c belongs to $(1.431, 1.432)$. Their conjecture is proven in this note (although the constant fits this range only taking round-off error into account).

The definitions of the metrics in question are the following:

$$v_G(x,y) := \sup_{z \in \partial G} \measuredangle xzy$$

and

$$j_G(x,y) := \log\Big(1 + \frac{|x-y|}{\min\{\delta(x), \delta(y)\}}\Big).$$

where $x, y \in G \subset \mathbb{R}^n$ and $\delta(x)$ is the distance of x to the boundary ∂G.

Theorem. *Let G be a ball or a half-space. Then $v_G(x,y) \leqslant c j_G(x,y)$, where*

$$c = \frac{1+2\alpha}{1+\alpha^2} \approx 1.432404247036$$

and α is the positive solution of the equation

$$2(1+\alpha^2)\arctan\alpha = (1+2\alpha)\log(1+2\alpha),$$

Proof. We consider first the case when G is a ball. Since both metrics are invariant under similarities, it suffices to consider the case when G is the unit ball. Let $x \in G$ and $r > 0$. Let y vary over the set $S(x,r) \cap B(0, 1-\delta(x))$ (wide grey curve in Figure 1): for all these point we have $j_G(x,y) = \log(1 + r/\delta(x))$. On the other hand, $v_G(x,y)$ is maximized for such y that $\delta(y) = \delta(x)$: indeed, for a given segment, the points from which the segment subtends a given angle form to partial arcs of a circle (drawn with thin black line in the figure). If y is moved away from the extremal position (marked z in the figure), then one needs larger half-circles, i.e. a smaller angle.

Hence $v_G(x,y)$ is maximized when $\delta(y) = \delta(x)$. In this case [1, (3.4)] gives us

$$v_G(x,y) = 2\arctan\frac{|x|\sin\theta}{1-|x|\cos\theta},$$

where θ is the half of the angle $\measuredangle x0y$. Also, we have $|x-y| = 2|x|\sin\theta$ so the claim becomes

$$2\arctan\frac{|x|\sin\theta}{1-|x|\cos\theta} \leqslant c\log\Big(1 + \frac{2|x|\sin\theta}{1-|x|}\Big).$$

Date: October 24, 2015.

2010 *Mathematics Subject Classification.* 30F45(51M10).

Key words and phrases. Visual angle metric, j-metric.

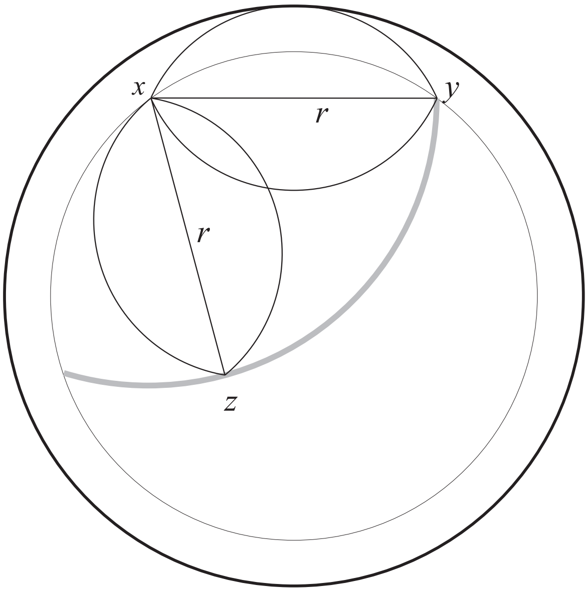

FIGURE 1. Finding the extremal configuration of points

Denote $s := |x|\sin\theta$ and $t := |x|$ so that the inequality is

$$2\arctan\frac{s}{1-\sqrt{t^2-s^2}} \leqslant c\log\Big(1+\frac{2s}{1-t}\Big), \quad 0<s\leqslant t<1.$$

Let us fix $\frac{s}{1-t}=\alpha\in(0,\infty)$ and maximize the left-hand side under this constraint. The reciprocal of the argument of the arc-tangent equals

$$\frac{1-\sqrt{t^2-s^2}}{s}=\frac{1}{s}-\sqrt{(\tfrac{1}{s}-\tfrac{1}{\alpha})^2-1}.$$

Let us denote $u := (\frac{1}{s}-\frac{1}{\alpha})^2\geqslant 1$. Then

$$\frac{1-\sqrt{t^2-s^2}}{s}=\sqrt{u}+\frac{1}{\alpha}-\sqrt{u-1}=\frac{1}{\alpha}+\frac{1}{\sqrt{u}+\sqrt{u-1}},$$

which is decreasing in u, so that

$$\min_{\frac{s}{1-t}=\alpha}\frac{1-\sqrt{t^2-s^2}}{s}=\min_{u\geqslant 1}\sqrt{u}+\tfrac{1}{\alpha}-\sqrt{u-1}=\tfrac{1}{\alpha}.$$

We have shown that

$$2\sup_{\frac{s}{1-t}=\alpha}\arctan\frac{s}{1-\sqrt{t^2-s^2}}=2\arctan\alpha.$$

Thus the claim is equivalent to the inequality

$$2\arctan\alpha\leqslant c\log(1+2\alpha).$$

Let $f(\alpha) := c\log(1+2\alpha)-2\arctan\alpha$. Then $f'(\alpha)=0$ if and only if $c=\frac{1+2\alpha}{1+\alpha^2}$. Furthermore, f' has profile either $+$ or $+|-|+$. The critical value of c is that for which $f(\alpha_2)=0$ at the larger zero of the derivative, which is the unique positive solution of the equation

$$2\arctan\alpha=\frac{1+2\alpha}{1+\alpha^2}\log(1+2\alpha). \tag{1}$$

With Wolfram Alpha we approximate said solution as $\alpha \approx 1.128833543610632$, in which case $c \approx 1.432404247036$.

It remains to consider the case when B is a half-space. As in the case of a ball, it suffices to consider the upper half-space and for fixed x and $j_G(x,y)$ the value $v_G(x,y)$ is maximized for such y that $\delta(x) = \delta(y)$. Now we may further normalize the situation by a similarity and a scaling and assume without loss of generality that $x = e_n - te_1$ and $y = e_n + te_1$ for some $t > 0$. In this case

$$j_G(x,y) = \log\Big(1 + \frac{|x-y|}{\delta(x)}\Big) = \log(1+2t)$$

and by [1, (3.20)]

$$v_G(x,y) = \arccos\frac{4\delta(x)^2 - |x-y|^2}{4\delta(x)^2 + |x-y|^2} = \arccos\frac{1-t^2}{1+t^2} = 2\arctan t.$$

Thus the same inequality as before holds in this case also. □

In the newer work [3], Vuorinen and Wang derive an inequality between v_G and j_G in convex domains. This inequality is quite complex, and it remains an open question whether the technique of this note can be combined with their estimate to yield the optimal constant c for which $v_G \leqslant c j_G$ in all convex domains. See also [2] for further results.

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