

Iterated maximal functions in variable exponent Lebesgue spaces

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Abstract. In this paper we study the iterated Hardy–Littlewood maximal operator in variable exponent Lebesgue spaces with exponent allowed to reach the value 1. We use modulars where the $L^{p(\cdot)}$ -modular is perturbed by a logarithmic-type function, and the results hold also in the more general context of such Musielak–Orlicz spaces.

Key words. Variable exponent Lebesgue spaces; Generalized Lebesgue spaces; Musielak–Orlicz spaces; Maximal functions; Non-uniformly convex norms; Riesz potentials; Sobolev inequalities

1. Introduction

Variable exponent spaces have been the subject of much interest recently, as surveyed in [15, 18, 26, 35, 44]. These investigations have dealt both with the spaces themselves, e.g. [10, 19, 21, 25, 30], with related differential equations [4, 6, 7, 12, 20, 46], and with applications [1–3, 8, 43]. A critical step in the development of the theory was establishing the boundedness of the Hardy–Littlewood maximal operator [13], and many subsequent papers have extended this result. That is the purpose also of the present paper. To describe our results, we use the by now standard notation of variable exponent spaces. The reader may consult Section 2 if necessary.

Based on Diening [13], Cruz-Uribe, Fiorenza and Neugebauer [11] and Nekvinda [40] established, independently, that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ provided p is log-Hölder continuous and satisfies $1 < p^- \leq p^+ < \infty$; moreover, Pick and Růžička [41] showed that the continuity condition is essentially optimal, but see also Lerner [33]. In a recent paper, with Diening, we showed that the condition $1 < p^- \leq p^+ < \infty$ is not necessary [14].

To consider the boundedness of the maximal operator when $p^- = 1$, it is necessary to move beyond Lebesgue spaces with variable exponent, to the framework of Musielak–Orlicz spaces. Recall that the Musielak–Orlicz space with modular $\Phi(x, t)$ is defined by the Luxemburg-type norm

$$\|f\|_{L^\Phi(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi \left(x, \frac{f(x)}{\lambda} \right) dx \leq 1 \right\}$$

[38]. The following function plays an important role in this paper:

$$\psi_p(t) := \begin{cases} \log(e + |t|), & \text{for } |t| < e^{p'} - e \\ 2p' - \frac{e^{p'}}{e+|t|} p', & \text{for } |t| \geq e^{p'} - e. \end{cases}$$

Note that $t \mapsto t^p \psi_p(t)$ is convex on $[0, \infty)$ and that

$$\psi_p(t) \approx \min \{ p', \log(e + |t|) \}.$$

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The norm $\|f\|_{L^{p(\cdot)}\psi_{p(\cdot)}(L)}$ is given by the modular

$$\Phi(x, t) = |t|^{p(x)}\psi_{p(x)}(t).$$

We see that $\|f\|_{L^{p(\cdot)}\psi_{p(\cdot)}(L)} \approx \|f\|_{L^{p(\cdot)}}$ if $p^- > 1$, but the constant of proportionality blows up as $p^- \rightarrow 1$. With this notation, the main result from [14] reads:

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain and let $1/p: \Omega \rightarrow \mathbb{R}$ be log-Hölder continuous with $1 \leq p^- \leq p^+ \leq \infty$. Then*

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \lesssim \|f\|_{L^{p(\cdot)}\psi_{p(\cdot)}(L)(\Omega)}.$$

Moreover, the embedding is sharp in the sense that $Mf \in L^{p(\cdot)}(B)$ if and only if $f \in L^{p(\cdot)}\psi_{p(\cdot)}(L)(B)$ in the case of a ball B .

In some settings, e.g. extrapolation theorems following Rubio de Francia [10,42], it is necessary to iterate the maximal operator. If the operator maps the space to itself, iteration is easy. To deal with the case $p = 1$ in the classical setting we need to know that perturbing the L^p scale by further logarithms does not affect the behavior of the maximal operator. Specifically,

$$Mf \in L^p(\log L)^q \text{ iff } f \in L^p(\log L)^q \ (p > 1) \quad \text{and} \quad Mf \in L(\log L)^q \text{ iff } f \in L(\log L)^{q+1}$$

for $q \geq 0$ [45, 5.2(d)]. Iterating, we obtain

$$M^k f \in L^p(\log L)^q \text{ iff } f \in L^p(\log L)^q \ (p > 1) \quad \text{and} \quad M^k f \in L(\log L)^q \text{ iff } f \in L(\log L)^{q+k}$$

for $k \in \mathbb{Z}^+$.

In this paper we are interested in similar results for the variable exponent context. In view of Theorem 1, we expect that

$$M^k f \in L^{p(\cdot)} \text{ iff } f \in L^{p(\cdot)}\psi_{p(\cdot)}^k(L).$$

If $k \geq 1$, then $L^{p(\cdot)}\psi_{p(\cdot)}^k(L)$ is not a variable exponent Lebesgue space. Therefore a straightforward iteration of Theorem 1 is not possible. Two alternative approaches might be tried: we can either generalize the theorem to space $L^{p(\cdot)}\Psi(\cdot, L)$ where Ψ is a function which can be chosen to equal $\psi_{p(\cdot)}^k$, or we can think of M^k as a single operator and work directly with it. In this paper we use the former.

To describe the result, we introduce the abbreviation $\|f\|_{L^{p(\cdot)}\Psi(\cdot, L)\psi_{p(\cdot)}^k(L)}$ for the Musielak–Orlicz space with modular

$$\Phi(x, t) = |t|^{p(x)}\Psi(x, |t|)\psi_{p(x)}^k(t).$$

We assume that Ψ is a function of logarithmic type such that Φ is convex. More specifically:

- (Ψ 1) there exists $c_1 > 0$ such that $\Psi(x, t^2) \leq c_1\Psi(x, t)$ for $x \in \Omega$ and $t \geq 1$;
- (Ψ 2) there exists $c_2 > 0$ such that $\Psi(x, t) \leq c_2\Psi(y, t)$ when $x \in \Omega$, $y \in B(x, r) \cap \Omega$ and $t \in [1, r^{-n}]$;
- (Ψ 3) $\inf_x \Psi(x, 0) > 0$ and $\sup_x \Psi(x, t) < \infty$ for every $t \geq 0$; and
- (Ψ 4) $t \mapsto \Psi(x, t)$ is increasing on $[0, \infty)$.

Although the four conditions might appear complicated, we think it is better to prove directly a general result than piece-meal generalizations. An important fact is that the conditions are general enough to cover all the cases mentioned above, i.e. powers of logarithms and powers of $\psi_{p(\cdot)}$. For instance

$$\Psi(x, t) := (\log(e + t))^{q(x)}$$

satisfies these conditions if $q \in L^\infty$ is non-negative and log log-Hölder continuous, i.e.,

$$|q(x) - q(y)| \leq \frac{C}{\log \log(e^2 + 1/|x - y|)}.$$

Of course we may also add further logarithms, e.g.

$$\Psi(x, t) := (\log(e + t))^{q(x)} (\log \log(e^2 + t))^{r(x)},$$

etc. We can now state our main results:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain, let $p: \Omega \rightarrow \mathbb{R}$ be log-Hölder continuous with $1 \leq p^- \leq p^+ < \infty$, and suppose that Ψ satisfies conditions $(\Psi 1)$ – $(\Psi 4)$. Then*

$$\|M^k f\|_{L^{p(\cdot)\Psi(\cdot,L)}(\Omega)} \lesssim \|f\|_{L^{p(\cdot)\Psi(\cdot,L)\psi_{p(\cdot)}^k(L)}(\Omega)}.$$

Note that this result is new in the variable exponent context even for the case $p^- > 1$, and generalizes [37]. In addition, it holds in the case $p^+ = \infty$ when $\Psi \equiv 1$; this is in fact trivial, since we have the equivalence $\|f\|_{L^{p(\cdot)\psi_{p(\cdot)}^k(L)}(\Omega)} \approx \|f\|_{L^{p(\cdot)}(\Omega)}$ in the part of the domain where p is large. The next theorem shows that the embedding space of the previous result is optimal.

Theorem 3. *Let $B \subset \mathbb{R}^n$ be a ball, let $p: B \rightarrow \mathbb{R}$ be log-Hölder continuous with $1 \leq p^- \leq p^+ < \infty$. Then*

$$M^k f \in L^{p(\cdot)}(B) \quad \text{if and only if} \quad f \in L^{p(\cdot)\psi_{p(\cdot)}^k(L)}(B).$$

The structure of the rest of this paper is as follows. In Section 2 we recall the definitions and conventions used throughout the paper. In Section 3 we prove Theorem 2. In Section 4 we prove the converse inequality, thus establishing Theorem 3. The paper is concluded by two short sections with complementary results. In Section 5 we consider modular versions of the claims, and show that they only hold when the exponents are independent of x . In the final section, we study the Riesz potential and obtain a Sobolev inequality in the space $L^{p(\cdot)\Psi(\cdot,L)}$.

Some possible future generalizations include the weighted case and the case of metric measure spaces. Special classes of weights have been studied by Kokilashvili, N. Samko and S. Samko (e.g., [29,30]) and general $A_{p(\cdot)}$ -weights were recently introduced by Diening and Hästö [17]. The metric measure space case was studied by the authors and collaborators (in various combinations) in [22,25,27,36] as well as by Samko et al. [5,31].

Remark 1. If $p^- > 1$, then we can consider more general functions Ψ which include also negative powers of logarithms. In this case condition $(\Psi 4)$ is not needed, because $t \mapsto t^{p_0-1}\Psi(x,t)$ is increasing uniformly in $x \in \Omega$ for $1 < p_0 < p^-$.

2. Notation and conventions

In this section we present several basic results, which can be found for instance in [15].

By C we denote a generic constant, i.e. a constant whose value may change from appearance to appearance. We write $f \lesssim g$ if there exists a constant C such that $f \leq Cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. We always assume that $\Omega \subset \mathbb{R}^n$ is an open bounded set. For $f \in L^1_{\text{loc}}(\Omega)$ and $A \subset \mathbb{R}^n$ with positive finite measure we write

$$f_A := \int_A f(y) dy := |A|^{-1} \int_{A \cap \Omega} f(y) dy.$$

By M we denote the centered *Hardy–Littlewood maximal operator*, that is $Mf(x) = \sup_{r>0} |f|_{B(x,r)}$.

Let $p: \Omega \rightarrow [1, \infty]$ be a measurable function, called a variable exponent. For $A \subset \mathbb{R}^n$ we write $p_A^+ = \text{ess sup}_{x \in A \cap \Omega} p(x)$ and $p_A^- = \text{ess inf}_{x \in A \cap \Omega} p(x)$, and abbreviate $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$. The variable exponent *modular* is defined for measurable functions by

$$\varrho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx.$$

Here we use the convention $t^\infty = \infty \chi_{(1,\infty)}(t)$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of those measurable functions $f: \Omega \rightarrow \mathbb{R}$ with $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1 \}.$$

Here Ω could of course be replaced by some subset as in $\|f\|_{L^{p(\cdot)}(A)}$; the abbreviation $\|f\|_{p(\cdot)}$ is used for the norm $\|f\|_{L^{p(\cdot)}(\Omega)}$ over all of Ω .

The following continuity condition has emerged as central in the theory of variable exponent spaces.

Definition 1. We say that $g \in C(\Omega)$ is log-Hölder continuous if there exists a constant $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\log(e + 1/|x - y|)}$$

for all $x, y \in \Omega$.

Note that the previous definition suffices for our purposes since we only deal with bounded domains. It follows easily from the definition that

$$|B|^{-|g(x)-g(y)|} \leq (2e/\Omega_n)^{n c_{\log}}$$

for all balls $B \subset \mathbb{R}^n$ containing x and y when g is log-Hölder continuous; here Ω_n is the volume of the unit ball.

Lemma 1. If p is log-Hölder continuous, then $\psi_{p(x)}(t)$ satisfies conditions $(\Psi 1) - (\Psi 4)$.

Proof. Properties $(\Psi 1)$, $(\Psi 3)$ and $(\Psi 4)$ are clear. To show that $\psi_{p(x)}(t)$ satisfies condition $(\Psi 2)$, it suffices to note that if $y \in B(x, r)$ and $t \in [1, r^{-n}]$, then

$$\left| \frac{1}{p'(x)} - \frac{1}{p'(y)} \right| = \left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \frac{C}{\log(e + t)},$$

so that

$$\frac{1}{p'(y)} \leq \frac{1}{p'(x)} + \frac{C}{\log(e + t)} \leq \frac{1 + C}{\psi_{p(x)}(t)},$$

which gives

$$\psi_{p(x)}(t) \leq (1 + C)p'(y).$$

This is the claim when $p'(y) < \log(e + |t|)$. The case $p'(y) \geq \log(e + |t|)$ is trivial, since $\psi_{p(x)}(t) \leq \log(e + |t|)$.

Assume that Ψ satisfies conditions $(\Psi 1) - (\Psi 4)$. From these assumptions one easily derives the following properties:

$(\Psi 5)$ there exist constants $A_1, A_2 > 0$ such that

$$\Psi(x, t) \geq A_1 \quad \text{and} \quad \Psi(x, 2t) \leq A_2 \Psi(x, t)$$

for all $x \in \Omega$ and $t \geq 0$;

$(\Psi 6)$ if $r \geq 1$, then there exist constants $A, \alpha > 0$ such that

$$\Psi(x, t^r) \leq A r^\alpha \Psi(x, t)$$

for all $x \in \Omega$ and $t \geq 0$;

$(\Psi 2')$ if $a_1, a_2 > 0$, then there exists a constant $A(a_1, a_2) > 0$ such that $\Psi(x, t) \leq A(a_1, a_2)\Psi(y, t)$ when $x \in \Omega, y \in B(x, r) \cap \Omega$ and $t \in [a_1, a_2 r^{-n}]$.

3. The embedding theorem

In the proof of Theorem 1 we used a theorem of Nekvinda [39] to combine a sum of norms to a norm of the sum. This was possible since the exponent tended to 1 (clearly $\|\sum f_i\|_1 = \sum \|f_i\|_1$ for non-negative functions). This result has thereafter been generalized by Hästö [28] to unbounded domains and exponents with other limits. However, these results pertain only to variable exponent Lebesgue spaces, not to more general Musielak–Orlicz spaces.

Surprisingly, the sum and the $L^1\Psi(\cdot, L)$ -norm “commute” provided the functions satisfy a polynomial growth condition (with respect to the index):

Lemma 2. Let Ω be a bounded open set, partitioned into measurable sets Ω_i , and let Ψ satisfy $(\Psi 1)$ and $(\Psi 4)$. Then

$$\sum_i i \|f\|_{L^1\Psi(\cdot, L)(\Omega_i)} \lesssim 1 + \sum_i i \varrho_{L^1\Psi(\cdot, L)(\Omega_i)}(f)$$

if there exists $\varepsilon > 0$ such that $f|_{\Omega_i} \geq i^\varepsilon$.

Proof. Fix $\varepsilon > 0$ such that $f|_{\Omega_i} \geq i^\varepsilon$. We prove the claim by exhibiting a sequence λ_i with the properties $\|f\|_{L^1\Psi(\cdot, L)(\Omega_i)} \leq \lambda_i$ and

$$\sum_i i\lambda_i \lesssim 1 + \sum_i i\varrho_{L^1\Psi(\cdot, L)(\Omega_i)}(f).$$

We define $\lambda_i = i^{-3}$ if $\|f\|_{L^1\Psi(\cdot, L)(\Omega_i)} \leq i^{-3}$, and denote the set of indices for which this is the case by I . Set $d = 3\varepsilon^{-1} + 1$ and

$$\lambda_i = \max \left\{ i^{-3}, \beta d^\alpha \varrho_{L^1\Psi(\cdot, L)(\Omega_i)}(f) \right\}$$

for $i \notin I$, where α is from assumption $(\Psi 1)$ and $\beta \geq 1$ will be fixed later. Then

$$\sum_{i=1}^{\infty} i\lambda_i = \sum_{i \in I} i\lambda_i + \sum_{i \notin I} i\lambda_i \leq \sum_{i=1}^{\infty} i^{-2} + \beta d^\alpha \sum_{i \notin I} i\varrho_{L^1\Psi(\cdot, L)(\Omega_i)}(f).$$

So the lemma will be proved once we show that our choice of variables λ_i satisfy also the other condition, $\|f/\lambda_i\|_{L^1\Psi(\cdot, L)(\Omega_i)} \leq 1$. For $i \in I$ this is clear by definition.

Assume then that $i \notin I$. If $c_i \geq i^\varepsilon$, then $i^3 c_i \leq c_i^d$. Hence by $(\Psi 4)$ and $(\Psi 6)$ we have

$$\Psi(x, i^3 c_i) \lesssim \Psi(x, c_i^d) \lesssim d^\alpha \Psi(x, c_i).$$

We apply this estimate to the function f in the domain Ω_i . Hence

$$\begin{aligned} \varrho_{L^1\Psi(\cdot, L)(\Omega_i)}(f/\lambda_i) &= \frac{1}{\lambda_i} \int_{\Omega_i} f(x) \Psi(x, f(x)/\lambda_i) dx \\ &\lesssim \frac{1}{\lambda_i} \int_{\Omega_i} f(x) \Psi(x, i^3 f(x)) dx \\ &\lesssim \frac{d^\alpha}{\lambda_i} \int_{\Omega_i} f(x) \Psi(x, f(x)) dx = \frac{1}{\beta}. \end{aligned}$$

We choose β so large as to take care of the implicit constants, and then conclude that the modular is at most one, as required. \square

The proof of Theorem 1 was based on a generalization of Dening's trick from [13] which says that we can pull a variable exponent inside the maximal operator, like an application of Hölder's inequality in the constant exponent case. We need a similar statement for includes Ψ .

Lemma 3. *Suppose that p is log-Hölder continuous, Ψ satisfies $(\Psi 1)$ – $(\Psi 4)$, and let $\varepsilon \in (0, 1]$. Then*

$$Mf(x)^{p(x)} \Psi(x, Mf(x)) \lesssim M[f^{p(\cdot)} \Psi(\cdot, f)](x) + M\chi_{\{0 < f < \varepsilon\}}(x)$$

for $x \in \Omega$, whenever $f \in L^1_{loc}(\Omega)$ is nonnegative on the open bounded set Ω with $\rho_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}(f) \leq 1$.

Proof. Let f be a nonnegative measurable function on Ω with $\|f\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)} \leq 1$; and extend it to \mathbb{R}^n by zero. Fix $x \in \Omega$, $r > 0$, set $B := B(x, r)$ and define

$$F := \int_B f(y)^{p(y)} \Psi(y, f(y)) dy.$$

Since the modular of f is at most one, we obtain

$$F \leq \frac{1}{|B|} \int_\Omega f(y)^{p(y)} \Psi(y, f(y)) dy \leq Cr^{-n}.$$

Suppose first that $F \geq 1$ and set $\eta := F^{1/p(x)} \Psi(x, F)^{-1/p(x)}$. Then

$$\frac{F}{\eta^{p(x)} \Psi(x, \eta)} = \frac{\Psi(x, F)}{\Psi(x, \eta)} \lesssim 1$$

by $(\Psi 6)$ and $(\Psi 5)$. With this and $(\Psi 5)$, the inequality $F \geq 1$ implies that $\eta \geq c$. By $(\Psi 3)$, we find that $\eta \lesssim F \lesssim r^{-n}$. Then since $\eta \in [\frac{1}{C}, Cr^{-n}]$, the log-Hölder continuity of p and condition $(\Psi 2')$ imply that

$$\eta^{p(y)} \Psi(y, \eta) \approx \eta^{p(x)} \Psi(x, \eta)$$

for $y \in B$. The function $t \mapsto \Psi(x, t)$ is increasing for every x , hence $\Psi(y, f(y))/\Psi(y, \eta) > c > 0$ when $f(y) \geq \eta$. For the next estimate we first split the integral into the sets where f is smaller or greater than η , use the increasing property in the latter, and finally move the η -terms out of the integral by the estimate on the previous formula line. Thus we obtain

$$\begin{aligned} \int_B f(y) dy &\leq \eta + \frac{C}{|B|} \int_B f(y) \left(\frac{f(y)}{\eta} \right)^{p(y)-1} \frac{\Psi(y, f(y))}{\Psi(y, \eta)} dy \\ &\lesssim \eta + \frac{F}{\eta^{p(x)-1} \Psi(x, \eta)} \lesssim \eta. \end{aligned}$$

From this estimate and the definitions of η it follows that $f_B^{p(x)} \lesssim F \Psi(x, F)^{-1}$. Since $F \geq 1$, the inequality

$$f_B \leq \int_B \varepsilon + \varepsilon^{1-p^+} f(y)^{p(y)} dy \lesssim \varepsilon + F \quad (1)$$

implies that $F \gtrsim f_B$. Hence, by $(\Psi 4)$, $\Psi(x, f_B) \lesssim \Psi(x, F)$, and it follows that

$$f_B^{p(x)} \Psi(x, f_B) \lesssim \frac{F}{\Psi(x, F)} \cdot \Psi(x, F) = F = \int_B f(y)^{p(y)} \Psi(y, f(y)) dy$$

by the previous estimate and the definition of η and F .

Consider next the case $F \leq 1$. Now (1) gives $f_B \leq c_0$. Hence

$$\begin{aligned} f_B^{p(x)} \Psi(x, f_B) &\lesssim c_0^{p^+-1} \sup_x \Psi(x, c_0) f_B \\ &\lesssim \int_B \chi_{\{0 < f < \varepsilon\}}(y) + f(y)^{p(y)} \Psi(y, f(y)) dy \\ &\leq M \chi_{\{0 < f < \varepsilon\}}(x) + M[f^{p(\cdot)} \Psi(\cdot, f)](x). \end{aligned}$$

Thus irrespective of the value of F we have established that

$$f_B^{p(x)} \Psi(x, f_B) \lesssim M \chi_{\{0 < f < \varepsilon\}}(x) + M[f^{p(\cdot)} \Psi(\cdot, f)](x).$$

The claim follows from this when we take the supremum over B . \square

Corollary 1. *Under the assumptions of Lemma 3,*

$$Mf(x)^{p(x)} \Psi(x, Mf(x)) \lesssim \left[M \left(f^{\frac{p(\cdot)}{r}} \Psi(\cdot, f)^{\frac{1}{r}} \right) (x) + M \chi_{\{0 < f < 1\}}(x) \right]^r$$

for $r \leq p^-$.

Proof. We apply Lemma 3 with the exponent $\frac{p(\cdot)}{r}$ and perturbation $\Psi^{1/r}$ to the function f/C . The constant C is chosen so that $\varrho_{L^{\frac{p(\cdot)}{r}} \Psi(\cdot, L)^{\frac{1}{r}}(\Omega)}(f/C) \leq |\Omega|^{1-1/r} \varrho_{L^{p(\cdot)} \Psi(\cdot, L)(\Omega)}(f/C) \leq 1$, where the first inequality follows from Hölder's inequality. Note that such a C can be chosen independent of f . Then Lemma 3 is applicable, and it implies that

$$Mf(x)^{\frac{p(x)}{r}} \Psi(x, Mf(x))^{\frac{1}{r}} \lesssim M \left[f^{\frac{p(\cdot)}{r}} \Psi(\cdot, f)^{\frac{1}{r}} \right] (x) + M \chi_{\{0 < f < C\varepsilon\}}(x).$$

Choosing $\varepsilon = \frac{1}{C}$ and raising this inequality to the power r , we obtain the result. \square

The proof of the following proposition is somewhat similar to that of Proposition 4.2 in [14], although more technical.

Proposition 1. *Let p and Ψ be as in Lemma 3 and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then*

$$\|Mf\|_{L^{p(\cdot)} \Psi(\cdot, L)(\Omega)} \lesssim \|f\|_{L^{p(\cdot)} \Psi(\cdot, L) \psi_{p(\cdot)}(L)(\Omega)}$$

for every $f \in L^{p(\cdot)} \Psi(\cdot, L) \psi_{p(\cdot)}(L)(\Omega)$, where the constant depends only on p , Ω and the dimension n .

Proof. By the homogeneity of the claim, it suffices to consider such non-negative functions f that

$$\int_{\Omega} f(x)^{p(x)} \Psi(x, f(x)) \psi_{p(x)}(f(x)) dx \leq 1 \quad (2)$$

and prove the uniform boundedness of the $L^{p(\cdot)}\Psi(\cdot, L)$ -modular of Mf . We split f into small and large parts as follows:

$$f_s := f \chi_{\{y \in \Omega: f(y) \leq \exp(p'(y)) - e\}} \quad \text{and} \quad f_l := f - f_s.$$

By Lemma 3 with $\varepsilon = 1$,

$$Mf_s(x)^{p(x)} \Psi(x, Mf_s(x)) \lesssim M[f_s^{p(\cdot)} \Psi(\cdot, f_s)](x) + 1$$

for $x \in \Omega$. This and the embedding $M: L \log L \hookrightarrow L^1$ imply that

$$\begin{aligned} & \int_{\Omega} Mf_s(x)^{p(x)} \Psi(x, Mf_s(x)) dx \\ & \lesssim \int_{\Omega} f_s(y)^{p(y)} \Psi(y, f_s(y)) \log(e + f_s(y)^{p(y)} \Psi(y, f_s(y))) dy + |\Omega|. \end{aligned} \quad (3)$$

By $(\Psi 5)$ and $(\Psi 3)$, $\Psi(y, f_s(y)) \leq A_2^m \Psi(y, 1) \lesssim A_2^m$, where $m \in \mathbb{N}$ is the smallest integer with $f_s(y) \leq 2^m$. Hence $f_s(y)^{p(y)} \Psi(y, f_s(y)) \lesssim 1 + f_s(y)^q$ where q is a constant depending also on p^+ . Since $f_s(y) \leq \exp(p'(y)) - e$, we conclude from this that

$$\log(e + f_s(y)^{p(y)} \Psi(y, f_s(y))) \lesssim \log(e + f_s(y)) \leq \psi_{p(\cdot)}(f(y)).$$

Thus (3) implies that $\rho_{L^{p(\cdot)}\Psi(\cdot, L)\psi_{p(\cdot)}(L)}(Mf_s) \lesssim 1 + |\Omega|$.

Next we treat f_l . Let us define $r_0 := \infty$, and $r_i := 1 + 1/i$ and

$$\Omega_i := \{x \in \Omega: p(x) \in [r_i, r_{i-1}]\}$$

for all $i \geq 1$, so that $\Omega = \cup \Omega_i$. The sequences $(1)_{i=1}^{\infty}$ and $(r_i)_{i=0}^{\infty}$ satisfy the criterion of Nekvinda [39, Theorem 4.1] so we conclude that $l^1 \cong l^{(r_i)}$. Thus we can fix $K > 0$ such that

$$\sum_i x_i < K \quad \text{whenever} \quad \sum_i x_i^{r_i} < c, \quad (4)$$

where c is some constant depending only on the dimension.

Denote $p_i(x) := \max\{p(x), r_i\}$, $f_i = f_l \chi_{\Omega_i}$ and $g_i := (r_i - 1)f_i / \|f_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}$. By (2) we obtain $\|f_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)} \leq 1$ and by definition $f_i(x) \geq e^{p'(x)} - 1 \geq i$ in Ω_i , and hence $g_i(x) \notin (0, 1)$. Thus there is no ‘‘error term’’ in Corollary 1 and we conclude that

$$Mg_i(x)^{p_i(x)} \Psi(x, Mg_i(x)) \lesssim [M[g_i^{p_i(\cdot)/r_i} \Psi(\cdot, g_i)^{1/r_i}]](x)^{r_i}$$

for $x \in \Omega$. Integrating this inequality over $x \in \Omega$, we find by the usual constant exponent r_i -maximal inequality that

$$\begin{aligned} & \int_{\Omega} Mg_i(x)^{p_i(x)} \Psi(x, Mg_i(x)) dx \\ & \lesssim \|M[g_i^{p_i(\cdot)/r_i} \Psi(\cdot, g_i)^{1/r_i}]\|_{L^{r_i}(\Omega)}^{r_i} \\ & \lesssim (r_i - 1)^{-r_i} \|g_i^{p_i(\cdot)/r_i} \Psi(\cdot, g_i)^{1/r_i}\|_{L^{r_i}(\Omega)}^{r_i} \\ & \lesssim \underbrace{\left\| \left(\frac{f_i}{\|f_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}} \right)^{p_i(\cdot)} \Psi\left(\cdot, \frac{f_i}{\|f_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}}\right) \right\|_{L^1(\Omega)}}_{=1}. \end{aligned}$$

It follows that $\|Mg_i\|_{L^{p_i(\cdot)}\Psi(\cdot, L)(\Omega)} \lesssim 1$ from which we obtain

$$\|Mf_i\|_{L^{p_i(\cdot)}\Psi(\cdot, L)(\Omega)} \lesssim i \|f_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}$$

by the definition of g_i and the homogeneity of the maximal operator and the norm.

By the subadditivity of the maximal operator and the triangle inequality we conclude that

$$\begin{aligned} \|Mf_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)} &\leq \sum_i \|Mf_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)} \lesssim \sum_i \|Mf_i\|_{L^{p_i(\cdot)}\Psi(\cdot, L)(\Omega)} \\ &\lesssim \sum_i i \|f_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}. \end{aligned}$$

Then, in view of (4) and $i^{r_i-1} \approx i$, it suffices to show that

$$\sum_i i \|f_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}^{r_i-1} \leq C.$$

We show below that

$$\|f_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}^{r_i-1} \lesssim \|f_i^{p(\cdot)}\|_{L^1\Psi(\cdot, L)(\Omega_i)}. \quad (5)$$

From this, Lemma 2 and $i \leq p'(x) \leq \psi_{p(x)}(f(x))$ for $x \in \Omega_i$, it follows that

$$\begin{aligned} \sum_i i \|f_i\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}^{r_i-1} &\lesssim \sum_i i \|f_i^{p(\cdot)}\|_{L^1\Psi(\cdot, L)(\Omega_i)} \\ &\lesssim 1 + \sum_i i \varrho_{L^1\Psi(\cdot, L)(\Omega_i)}(f_i^{p(\cdot)}) \\ &\lesssim 1 + \varrho_{L^{p(\cdot)}\Psi(\cdot, L)\psi_{p(\cdot)}(L)(\Omega)}(f) \leq C. \end{aligned}$$

So it remains only to prove (5), i.e. that

$$\left\| \frac{f_i}{\|f_i^{p(\cdot)}\|_{L^1\Psi(\cdot, L)(\Omega)}^{1/r_i-1}} \right\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}^{r_i-1} \leq C.$$

Let us denote $h_i = f_i^{p(\cdot)}$ and $c_i = \|h_i\|_{L^1\Psi(\cdot, L)(\Omega)}$. Since $p_{\Omega_i}^+ = r_{i-1}$ and $c_i \lesssim 1$ (by (2)) we see that $c_i^{-p(x)/r_{i-1}} \lesssim c_i^{-1}$. The norm is bounded if and only if the modular is bounded, so it suffices to bound

$$\rho_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)}\left(\frac{f_i}{c_i^{1/r_{i-1}}}\right) \lesssim \int_{\Omega} \frac{h_i(x)}{c_i} \Psi\left(x, \left(\frac{h_i(x)}{c_i}\right)^{\frac{1}{p(x)}}\right) dx \lesssim \int_{\Omega} \frac{h_i(x)}{c_i} \Psi\left(x, \frac{h_i(x)}{c_i}\right) dx$$

where we used that Ψ is of log-type and increasing. But the latter expression is bounded by one, since c_i is the $L^1\Psi(\cdot, L)(\Omega)$ -norm of h_i , so we are done. \square

Proof (of Theorem 2). By Proposition 1 we obtain

$$\|M^{k-j}f\|_{L^{p(\cdot)}\Psi(\cdot, L)(\Omega)\psi_{p(\cdot)}^j(L)} \lesssim \|M^{k-j-1}f\|_{L^{p(\cdot)}\Psi(\cdot, L)\psi_{p(\cdot)}^{j+1}(L)(\Omega)}$$

for $j = 0, \dots, k-1$. The claim follows by combining these inequalities. \square

4. Optimality of the embedding results

We next show that the space we constructed is optimal for the boundedness of the maximal operator. First we have a constant exponent lemma.

Lemma 4. *Let $p \in (1, 2]$ and fix an open bounded set $\Omega \subset \mathbb{R}^n$. Assume that $f \in L^1(\Omega)$ is a non-negative function with $\int_{\Omega} f(x) dx \leq 1$ which does not take values in the range $(0, e^{Kp'})$ for some $K > 1$. Then there exists a constant c_n depending only on the dimension such that*

$$(p')^k \int_{\Omega} f(x)^p dx \lesssim \int_{\tilde{\Omega}} Mf(x)^p \psi_p^{k-1}(Mf(x)) dx,$$

where $\tilde{\Omega} := \{x \in \mathbb{R}^n : d(x, \Omega) < c_n e^{-(K-1)p'/n}\}$ is a dilatation of Ω .

Proof. Let $a := e^{(K-1)p'}$. We first note that

$$\begin{aligned} (p')^k \int_{\Omega} f(x)^p dx &\leq \frac{e}{e-1} (p')^{k-1} \int_{\Omega} f(x) \int_a^{f(x)} p \lambda^{p-2} d\lambda dx \\ &\lesssim \int_{\Omega} f(x) \int_a^{f(x)} p \lambda^{p-2} \psi_p^{k-1}(\lambda) d\lambda dx \\ &= \int_a^{\infty} p \lambda^{p-2} \psi_p^{k-1}(\lambda) \int_{\{x \in \Omega : f(x) > \lambda\}} f(x) dx d\lambda; \end{aligned} \quad (6)$$

here we used in the second step that $\psi_{p(x)}(t) \approx p'$ when $t \in (a, f(x))$, since $f(x) \notin (0, e^{Kp'})$.

Let $c_n \in (0, 1)$ be the constant from [45, Chapter 1, 5.2(b)]. Since the support of f lies in Ω and $\int_{\Omega} f(x) dx \leq 1$, we see that $Mf(x) < c_n a$ if $d(x, \Omega) > (\Omega_n c_n a)^{-1/n}$. Since $Mf < c_n \lambda$ in the set $\mathbb{R}^n \setminus \tilde{\Omega}$ for $\lambda \geq a$, we obtain

$$|\{x \in \tilde{\Omega} : Mf(x) > c_n \lambda\}| = |\{x \in \mathbb{R}^n : Mf(x) > c_n \lambda\}| \geq \frac{1}{2^n \lambda} \int_{\{f > \lambda\}} f(x) dx$$

for every non-negative $f \in L^1(\Omega)$ and $\lambda \geq a$, where the inequality is [45, Chapter 1, 5.2(b)]. Continuing the estimate (6), we thus find that

$$\begin{aligned} (p')^k \int_{\Omega} f(x)^p dx &\lesssim \int_a^{\infty} p \lambda^{p-1} \psi_p^{k-1}(\lambda) |\{x \in \tilde{\Omega} : Mf(x) > c_n \lambda\}| d\lambda \\ &= \int_{\tilde{\Omega}} \int_a^{\infty} p \lambda^{p-1} \psi_p^{k-1}(\lambda) \chi_{\{Mf(x) > c_n \lambda\}} d\lambda dx \\ &\lesssim \int_{\tilde{\Omega}} M(f/c_n)(x)^p \psi_p^{k-1}(M(f/c_n)(x)) dx, \end{aligned}$$

which yields the claim. \square

With the help of the previous lemma, we can prove the following result which generalizes [14, Theorem 4.6] to the case $k > 1$. This result together with Theorem 2 implies Theorem 3.

Lemma 5. *Let p be bounded and log-Hölder continuous in a ball $B \subset \mathbb{R}^n$. If $Mf \in L^{p(\cdot)} \psi_{p(\cdot)}^{k-1}(L)(B)$, then $f \in L^{p(\cdot)} \psi_{p(\cdot)}^k(L)(B)$.*

Proof. We extend p by reflection in ∂B to the dilated ball $\frac{3}{2}B$, and then to all of \mathbb{R}^n without changing the log-Hölder constant by [16, Proposition 3.6]. The extension can be done so that p is bounded away from 1 in the complement of $2B$. Let $f \in L^1(B)$ be a function with $\int_B Mf(x)^{p(x)} \psi_{p(x)}^{k-1}(Mf(x)) dx < \infty$. As in the proof of [23, Theorem 2.7] we conclude by reflection in ∂B that Mf is in fact bounded in $L^{p(\cdot)} \psi_{p(\cdot)}^{k-1}(L)(\frac{3}{2}B)$. Since f is supported in B , we have $Mf(x) \lesssim |x|^{-n}$ in $\mathbb{R}^n \setminus \frac{3}{2}B$. Thus we conclude, that $Mf \in L^{p(\cdot)} \psi_{p(\cdot)}^{k-1}(L)(\mathbb{R}^n)$.

We may assume that f is non-negative since otherwise we may estimate function $|f|$. The claim obviously holds in the set $\{p \geq 2\}$ since $Mf \geq f$ a.e. Thus we assume in the rest of the proof that $p < 2$. Since we may scale the function f by a constant without affecting the claim, we assume that $\varrho_{L^{p(\cdot)} \psi_{p(\cdot)}^{k-1}(L)(\mathbb{R}^n)}(Mf) \leq 1$ and $\|f\|_1 \leq 1$.

We split f into small and large parts as before:

$$f_s := f \chi_{\{f \leq e^{p'(\cdot)-e}\}} \quad \text{and} \quad f_l := f \chi_{\{f > e^{p'(\cdot)-e}\}}.$$

Then we find that

$$\int_B f(x)^{p(x)} \psi_{p(x)}^k(f(x)) dx \leq \int_B f_s(x)^{p(x)} \log^k(e + f_s(x)) dx + \int_B p'(x)^k f_l(x)^{p(x)} dx. \quad (7)$$

Thus it suffices to bound the terms on the right hand side. Since $f_s(x)^{p(x)-1} \leq e^{p(x)} \leq e^2$ we see that

$$\int_B f_s(x)^{p(x)} \log^k(e + f_s(x)) dx \leq e^2 \int_B f_s(x) \log^k(e + f_s(x)) dx.$$

Since

$$\int_B Mf_s(x) \log^{k-1}(e + Mf_s(x)) dx \leq C \int_B Mf(x)^{p(x)} \psi_{p(x)}^{k-1}(Mf(x)) dx,$$

it follows by [45, Chapter 1, 5.2(b)] that the first integral on the right hand side of (7) is finite.

So it remains to estimate

$$\int_B p'(x)^k f_l(x)^{p(x)} dx.$$

We divide B into the sets $B_i := \{r_{i+1} < p < r_{i-1}\}$, $r_i := 1 + 2^{-i}$, and write $f_i := f_l \chi_{B_i}$. Let \tilde{B}_i be the dilation of B_i , as in Lemma 4, corresponding to exponent r_i . Since $r'_i = 2^i + 1$, we get

$$d(\partial\tilde{B}_i, \partial B_i) \leq C_n e^{-(K-1)(2^i+1)/n}.$$

On the other hand,

$$d(\partial B_{i-j}, \partial B_i) \geq \exp\left(-c \log \frac{r_{i-j+1} r_{i-1}}{r_{i-j+1} - r_{i-1}}\right) = e^{-c \log 2^{i+3}/|2^j-4|}$$

for $j > 2$ by the log-Hölder continuity of the exponent. A similar estimate holds for $j < -1$. Now we have $\tilde{B}_i \subset \bigcup_{j \in J} B_{i+j}$ for every $i \in \mathbb{N}$, where $J \subset \mathbb{Z}$ is finite and depends on n . We can then use Lemma 4 in the sets B_i as follows

$$\begin{aligned} \int_B p'(x)^k f_l(x)^{p(x)} dx &\lesssim \sum_i \int_{B_i} (r'_{i-1})^k \left(f_i(x)^{p(x)/r_{i-1}}\right)^{r_{i-1}} dx \\ &\lesssim \sum_i \int_{\tilde{B}_i} M[f_i^{p(\cdot)/r_{i-1}}](x)^{r_{i-1}} \psi_{r_{i-1}}^{k-1}(M[f_i^{p(\cdot)/r_{i-1}}](x)) dx. \end{aligned}$$

By Corollary 1,

$$M(f_i^{p(\cdot)/r_{i-1}})(x) \lesssim (Mf_i(x) + 1)^{p(x)/r_{i-1}}.$$

Using this and the finite overlap of \tilde{B}_i , we obtain that

$$\begin{aligned} \int_B p'(x)^k f_l(x)^{p(x)} dx &\lesssim \sum_i \int_{\tilde{B}_i} Mf_i(x)^{p(x)} \psi_{p(x)}^{k-1}(Mf_i(x)) dx + C \\ &\lesssim \sum_i \int_{\tilde{B}_i} Mf(x)^{p(x)} \psi_{p(x)}^{k-1}(Mf(x)) dx + C \\ &\lesssim |J| \sum_i \int_{B_i} Mf(x)^{p(x)} \psi_{p(x)}^{k-1}(Mf(x)) dx + C \\ &\lesssim \int_{\mathbb{R}^n} Mf(x)^{p(x)} \psi_{p(x)}^{k-1}(Mf(x)) dx + C. \end{aligned}$$

Thus also the integral over the large part of the function is bounded, which concludes the proof. \square

Proof (of Theorem 3). If $f \in L^{p(\cdot)} \psi_{p(\cdot)}^k(L)(B)$, then $M^k f \in L^{p(\cdot)}(B)$ by Theorem 2. If conversely $M^k f \in L^{p(\cdot)}(B)$, then $M^{k-1} f \in L^{p(\cdot)} \psi_{p(\cdot)}(L)(B)$, by Lemma 5 applied to the function $M^{k-1} f$; hence $M^{k-2} f \in L^{p(\cdot)} \psi_{p(\cdot)}^2(L)(B)$ by the same lemma; and so on, until we conclude that $f \in L^{p(\cdot)} \psi_{p(\cdot)}^k(L)(B)$.

5. Modular inequalities

Lerner [34] has shown that the maximal inequality essentially never holds in a modular form in the variable exponent Lebesgue space. In this section we show that this claim also holds with logarithmic perturbations of the scale.

Lemma 6 (Constant exponent case). *Let $p > 1$ and $q \in \mathbb{R}$, or $p = 1$ and $q \geq 0$. If $f: \mathbb{R}^n \rightarrow [0, \infty]$ is locally integrable and*

$$J := \int_{B(x,r)} f(y)^p (\log(e + f(y)))^q dy, \quad \text{then} \quad f_{B(x,r)} \lesssim J^{1/p} (\log(e + J))^{-q/p}.$$

Proof. For $\eta > 0$ we have

$$f_{B(x,r)} \leq \eta + C \int_{B(x,r)} f(y) \left(\frac{f(y)}{\eta} \right)^{p-1} \left(\frac{\log(e+f(y))}{\log(e+\eta)} \right)^q dy = \eta + \frac{CJ}{\eta^{p-1}(\log(e+\eta))^q}.$$

With the choice $\eta = J^{1/p}(\log(e+J))^{-q/p}$ we obtain the claim. \square

Theorem 4. *The inequality*

$$\int_{\Omega} Mf(x)^{p(x)} (\log(e+Mf(x)))^{q(x)} dx \lesssim \int_{\Omega} |f(x)|^{p(x)} (\log(e+|f(x)|))^{q(x)} dx \quad (8)$$

holds for all f on Ω if and only if p and q are constants and $p > 1$.

Proof. Suppose first that p and q are constants and $p > 1$. Then (8) follows from Lemma 6 by standard arguments (we extend f to \mathbb{R}^n by zero outside Ω), cf. the proof of Corollary 1.

Suppose next that (8) holds and that p is not constant. Then we can find numbers p_1 and p_2 such that $1 \leq p_1 < p_2 < \infty$, and both $E_1 := \{x \in \Omega : p(x) \leq p_1\}$ and $E_2 := \{x \in \Omega : p(x) \geq p_2\}$ have positive measure. If $f = k\chi_{E_1}$ with $k > 1$, then

$$Mf(x) \gtrsim k \quad \text{for } x \in E_2$$

so that

$$\int_{\Omega} Mf(x)^{p(x)} (\log(e+Mf(x)))^{q(x)} dx \gtrsim k^{p_2} (\log(e+k))^{q^-}.$$

On the other hand

$$\int_{\Omega} |f(x)|^{p(x)} (\log(e+|f(x)|))^{q(x)} dx \lesssim k^{p_1} (\log(e+k))^{q^+}.$$

In view of (8) we have

$$k^{p_2} (\log(e+k))^{q^-} \lesssim k^{p_1} (\log(e+k))^{q^+},$$

which gives a contradiction as $k \rightarrow \infty$. Hence p is constant. Once we have established that p is constant, the same argument can be used to prove that q is constant. \square

6. The Riesz potential and Sobolev's inequality

For a locally integrable function we define the Riesz potential by

$$I_{\alpha}f(x) := \int_{\Omega} |x-y|^{\alpha-n} f(y) dy.$$

Lemma 7. *Suppose that p is log-Hölder continuous with $p^+ < \frac{n}{\alpha}$, and Ψ satisfies $(\Psi 1)$ – $(\Psi 4)$. Then*

$$\int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \lesssim \delta^{\alpha-n/p(x)} \Psi(x,\delta)^{-1/p(x)}$$

for all $x \in \Omega$, $\delta \in (0, \text{diam}\Omega)$ and $f \geq 0$ satisfying $\|f\|_{L^{p(\cdot)\Psi(\cdot,L)}(\Omega)} \leq 1$.

Proof. Since the proof is similar to previous ones, we only sketch it. We denote $g(y) := f(y)^{p(y)} \Psi(y, f(y))$ and $k_x(y) := |x-y|^{-n/p(x)} \Psi(x, |x-y|^{-1})^{-1/p(x)}$. Then

$$\begin{aligned} \int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy &\leq \int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} k_x(y) dy \\ &\quad + \int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) \left(\frac{f(y)}{k_x(y)} \right)^{p(y)-1} \frac{\Psi(y, f(y))}{\Psi(y, k_x(y))} dy \\ &\lesssim \delta^{\alpha-n/p(x)} \Psi(x, \delta^{-1})^{-1/p(x)} \\ &\quad + \int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} g(y) k_x(y)^{-p(y)+1} \Psi(y, k_x(y))^{-1} dy \\ &\lesssim \delta^{\alpha-n/p(x)} \Psi(x, \delta^{-1})^{-1/p(x)} \\ &\quad + \int_{\Omega \setminus B(x,\delta)} g(y) |x-y|^{\alpha-n/p(x)} \Psi(x, |x-y|^{-1})^{-1/p(x)} dy \\ &\lesssim \delta^{\alpha-n/p(x)} \Psi(x, \delta^{-1})^{-1/p(x)}. \quad \square \end{aligned}$$

Theorem 5. *Suppose that p is log-Hölder continuous with $p^+ < n/\alpha$, and Ψ satisfies $(\Psi 1)$ – $(\Psi 4)$. Let $1/p^*(x) = 1/p(x) - \alpha/n$. Then*

$$\|I_\alpha f\|_{L^{p^*(\cdot)}\Psi(\cdot, L)^{p^*(\cdot)/p(\cdot)}(\Omega)} \lesssim \|f\|_{L^{p(\cdot)}\Psi(\cdot, L)^{\psi_{p(\cdot)}(L)}(\Omega)}.$$

Proof. By a scaling argument, it suffices to show that there exists a constant $C > 0$ such that

$$\int_{\Omega} \{I_\alpha f(x)\Psi(x, I_\alpha f(x))^{1/p(x)}\}^{p^*(x)} dx \leq C$$

for all $f \geq 0$ with $\|f\|_{L^{p(\cdot)}\Psi(\cdot, L)^{\psi_{p(\cdot)}(L)}(\Omega)} \leq 1$. For $\delta > 0$, we have

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x, \delta)} |x - y|^{\alpha-n} f(y) dy + \int_{\Omega \setminus B(x, \delta)} |x - y|^{\alpha-n} f(y) dy \\ &\lesssim \delta^\alpha Mf(x) + \delta^{\alpha-n/p(x)} \Psi(x, \delta^{-1})^{-1/p(x)} \end{aligned}$$

by Lemma 7. Now, letting $\delta = \min\{\text{diam}\Omega, Mf(x)^{-p(x)/n} \Psi(x, Mf(x))^{-1/n}\}$, we obtain

$$I_\alpha f(x) \lesssim 1 + Mf(x)^{p(x)/p^*(x)} \Psi(x, Mf(x))^{-\alpha/n},$$

so that

$$\{I_\alpha f(x)\Psi(x, I_\alpha f(x))^{1/p(x)}\}^{p^*(x)} \lesssim 1 + Mf(x)^{p(x)} \Psi(x, Mf(x)).$$

Integrating this inequality gives the desired conclusion. \square

We immediately obtain the Sobolev inequality for smooth functions with compact support, $C_0^\infty(\Omega)$. Of course, the same result holds in the appropriate Sobolev space by the density of smooth functions, but we do not here delve into the definition of such spaces.

Corollary 2. *Suppose that p is log-Hölder continuous with $p^+ < n/\alpha$, and Ψ satisfies $(\Psi 1)$ – $(\Psi 4)$. Let $1/p^*(x) = 1/p(x) - \alpha/n$. Then*

$$\|u\|_{L^{p^*(\cdot)}\Psi(\cdot, L)^{p^*(\cdot)/p(\cdot)}(\Omega)} \lesssim \|\nabla u\|_{L^{p(\cdot)}\Psi(\cdot, L)^{\psi_{p(\cdot)}(L)}(\Omega)}$$

for $u \in C_0^\infty(\Omega)$.

Proof. It is shown in [24, Lemma 7.14, p. 154] that $|u| \leq I_1|\nabla u|$ for $u \in C_0^\infty(\Omega)$. Therefore

$$\|u\|_{L^{p^*(\cdot)}\Psi(\cdot, L)^{p^*(\cdot)/p(\cdot)}(\Omega)} \lesssim \|I_1|\nabla u|\|_{L^{p^*(\cdot)}\Psi(\cdot, L)^{p^*(\cdot)/p(\cdot)}(\Omega)} \lesssim \|\nabla u\|_{L^{p(\cdot)}\Psi(\cdot, L)^{\psi_{p(\cdot)}(L)}(\Omega)}$$

by the previous theorem. \square

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