THE MAXIMAL OPERATOR ON WEIGHTED VARIABLE LEBESGUE SPACES

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Abstract

We study the boundedness of the maximal operator on the weighted variable exponent Lebesgue spaces $L^{p(\cdot)}_\omega(\Omega)$. For a given log-Hölder continuous exponent p with $1 < \inf p \leqslant \sup p < \infty$ we present a necessary and sufficient condition on the weight ω for the boundedness of M. This condition is a generalization of the classical Muckenhoupt condition.

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1. Introduction

The purpose of this paper is to give a condition for the maximal operator to be bounded on the weighted variable exponent Lebesgue space $L^{p(\cdot)}_\omega(\Omega)$. Below we will recall the precise definition of this space; briefly, we can think of it as the Banach function space consisting of all measurable functions f such that

$$\int_\Omega |f(x)\omega(x)|^{p(x)}\,dx < \infty.$$

These are generalizations of the variable exponent Lebesgue spaces which have been the subject of considerable attention for nearly two decades: see the survey article by Samko [39] or the monograph [11] for further details and references.

By the maximal operator we mean the Hardy–Littlewood maximal operator: given a locally integrable function f, define the maximal function

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of f by

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes (with sides parallel to the coordinate axes) that contain x and $\fint dy$ denotes the mean value integral $|Q|^{-1} \int_Q dy$. If we replace cubes by balls we get an operator that is pointwise equivalent with a constant depending only on the dimension.

When $\omega \equiv 1$, the question of the boundedness of the maximal operator on $L^{p(\cdot)}(\Omega)$ has been studied extensively, and we refer to [5, 11] for details and further references.

In the weighted case the problem has been studied by a number of authors from two different perspectives. Besides the definition given above, we can also define the weighted variable exponent space $L^{p(\cdot)}(\Omega, \sigma)$ consisting (intuitively) of measurable functions f such that

$$\int_\Omega |f(x)|^{p(x)} \sigma(x)\, dx < \infty.$$

Clearly these definitions are interchangeable, since we can take $\sigma(x) = w(x)^{p(x)}$. However, while equivalent, these approaches are usually treated separately. We will refer to the approach taken here as treating the weight as a multiplier; the alternative approach we will refer to as treating the weight as a measure.

Stefan Samko and his collaborators were the first to investigate variable exponent Sobolev spaces with weights. They started from weighted inequalities considered in the power weight case: $\omega(x) = |x|^a$, $a \in \mathbb{R}$, or more generally, weights of the form

$$\omega(x) = \prod_{k=1}^{m} |x - c_k|^{a_k},$$

or variable power weights of the form $\omega(x) = |x|^{a(x)}$ or $\omega(x) = (1 + |x|)^{a(x)}$. (Variable power weights arise naturally when considering the Hardy operator and its variants; see, for instance, [13].) The first results of this kind we proved by Kokilashvili, Samko and their collaborators [25, 40, 41, 42, 43]; these results were more recently extended to classes of weights who oscillate between powers: see [1, 21, 22, 23, 24, 26, 27, 37, 38]. Other results in this direction have been proved by a number of authors; see, for example, [4, 3, 15, 20, 32, 33]. More general metric measure spaces have been studied for instance in [16, 18, 19, 28, 34]; the discrete weighted case was studied in [36]. In the majority of these papers the weights are treated as multipliers. Our main result is in terms of weights as multipliers; however, to put our work in context we will first review some results with weights as measures.

For the Hardy–Littlewood maximal operator a significant question has been to extend the Muckenhoupt A_p condition from the theory of weighted norm inequalities to the variable exponent setting. When p is constant, then it is a classical result (see, for example [14]) that the maximal operator is bounded on $L^p_w(\mathbb{R}^n)$, $1 < p < \infty$, if and only if ω satisfies the A_p condition: for every cube Q,

$$\|\omega\chi_Q\|_p\|\omega^{-1}\chi_Q\|_{p'} \leqslant C|Q|.$$

We can rewrite this condition as

$$\|\chi_Q\|_{L^p_\omega}\|\chi_Q\|_{(L^p_\omega)'} \leqslant C|Q|, \tag{1.1}$$

where $(L^p_\omega)'$ denotes the associate space (the closed subspace of the dual generated by measurable functions, see [11, Definition 2.7.1]) of L^p_ω, which can be identified with $L^{p'}_{\omega^{-1}}$.

In the constant exponent case, a necessary condition for the maximal operator to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is that p satisfies (see Diening [10] and Kopaliani [29]):

$$\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \leqslant C|Q|. \tag{1.2}$$

This condition is similar to (**1.1**), since $(L^{p(\cdot)})' = L^{p'(\cdot)}$ (see [11, Theorem 3.1.13]).

It has been shown by Kopaliani [29] that condition (**1.2**) is also sufficient if p is constant outside of a large ball. However, if p is not constant outside a large ball, then the condition is not sufficient: counter-examples are given in [30] and [11, Theorem 5.3.4]. In this case the condition has to be replaced by a condition on families of disjoint cubes (see class $\mathcal{A}$ below) in order to get a sufficient condition (see [9] and [11, Chapter 5]).

An A_p type condition on weights, treating ω as a measure, was first considered in [12] by the latter two authors. Given $p \in \mathcal{P}(\mathbb{R}^n)$, they defined the class $A^\dagger_{p(\cdot)}$ to consist of all weights ω such that for every cube Q,

$$\|\omega\chi_Q\|_1\|\omega^{-1}\chi_Q\|_{p'(\cdot)/p(\cdot)} < C|Q|^{p_Q},$$

where p_Q is the harmonic mean of p on Q ($p_Q^{-1} = \fint_Q p(x)^{-1}\,dx$), and $\|\cdot\|_{p'(\cdot)/p(\cdot)}$ is defined using the definition of the norm even when $p'(\cdot)/p(\cdot)$ is not greater than or equal to 1. With this definition they proved the following (see Section **2** for the definitions of the other notation).

THEOREM **1.1**. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. Then the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n, \omega)$ if and only if $\omega \in A^\dagger_{p(\cdot)}$.*

An alternative class $A^\sharp_{p(\cdot)}$ is the following: given p, the weight $\omega \in A^\sharp_{p(\cdot)}$ if

$$\|\omega^{1/p(\cdot)}\chi_Q\|_{p(\cdot)}\|\omega^{-1/p(\cdot)}\chi_Q\|_{p'(\cdot)} \leqslant C|Q|.$$

The conditions (**1.1**) and (**1.2**) can be restated as $\omega \in A^\sharp_{p(\cdot)}$ and $1 \in A^\sharp_{p(\cdot)}$, respectively; in [12] it was shown that if $\omega \in A^\dagger_{p(\cdot)}$ and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then $\omega \in A^\sharp_{p(\cdot)}$. A comparison of Theorem **1.1** and Theorem **1.3** below shows that $A^\sharp_{p(\cdot)} = A^\dagger_{p(\cdot)}$ for all $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$.

Another approach in terms of weights as measures was developed independently in [6]. Given $p \in \mathcal{P}(\mathbb{R}^n)$, such that $1 < p^- \leqslant p^+ < \infty$ and $p(x) \to p_\infty$ as $|x| \to \infty$, a weight $\omega \in A^*_{p(\cdot)}$ if:

(a) For every $x_0 \in \mathbb{R}^n$ there exists $\varepsilon = \varepsilon(x_0) > 0$ such that ω satisfies the $A_{p(x_0)}$ condition on $Q_\varepsilon(x_0)$, the cube centered at x_0 of side-length ε. That is, for every cube $Q \subset Q_\varepsilon(x_0)$,

$$\fint_Q \omega(x)\,dx \left(\fint_Q \omega(x)^{1-p'(x_0)}\,dx\right)^{p(x_0)-1} \leqslant K < \infty.$$

(b) There exists $N > 0$ such that ω satisfies the A_{p_∞} condition on $\mathbb{R}^n \setminus Q_N(0)$.

Intuitively, we can think of the $A^*_{p(\cdot)}$ condition as a modular version of the $A^\sharp_{p(\cdot)}$ condition.

THEOREM **1.2**. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. If $\omega \in A^*_{p(\cdot)}$, then the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n, w)$.*

An advantage of this approach is that it makes it much easier to apply the theory of Muckenhoupt A_p weights, since there is an immediate connection between this class and $A^*_{p(\cdot)}$. A drawback is that the $A^*_{p(\cdot)}$ condition is not necessary except if ω is assumed to behave locally like a power weight. (See [6] for a precise statement.)

REMARK **1.1**. The three conditions discussed above—$A^\dagger_{p(\cdot)}$, $A^\sharp_{p(\cdot)}$, and $A^*_{p(\cdot)}$—are all called $A_{p(\cdot)}$ in the literature. We have introduced this notation for clarity. Note that $A^*_{p(\cdot)} \subsetneqq A^\dagger_{p(\cdot)} = A^\sharp_{p(\cdot)}$ for $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$.

We turn now to our main result, which is a weighted norm inequality for the maximal operator in which we treat the weight as a multiplier. Given $p \in \mathcal{P}(\mathbb{R}^n)$, we say that $\omega \in \mathcal{A}_{p(\cdot)}$ if for every cube Q,

$$\|\omega \chi_Q\|_{p(\cdot)} \|\omega^{-1}\chi_Q\|_{p'(\cdot)} \leqslant C|Q|. \tag{1.3}$$

The smallest constant is called the $\mathcal{A}_{p(\cdot)}$-constant of ω. Note that $\omega \in \mathcal{A}_{p(\cdot)}$ is equivalent to $\omega^{\frac{1}{p(\cdot)}} \in A^{\sharp}_{p(\cdot)}$. The opposite inequality of (**1.3**) follows by Hölder's inequality, so we have $\|\omega\chi_Q\|_{p(\cdot)}\|\omega^{-1}\chi_Q\|_{p'(\cdot)} \approx |Q|$.

We prove the following result.

THEOREM **1.3**. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. Then the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}_{\omega}(\mathbb{R}^n)$ if and only if $\omega \in \mathcal{A}_{p(\cdot)}$.*

The validity of Theorem **1.3** was conjectured in [12, Remark 3.10]. In the special when p is constant outside of a large cube Theorem **1.3** was proved in [6]. Our proof below adapts to the variable exponent case techniques used to prove two-weight norm inequalities for a variety of operators; see [7] for details and further references.

2. Preliminaries

By c we denote a generic constant, whose value may change between appearances even within a single line. By $f \equiv g$ we mean that there exists c such that $\frac{1}{c} f \leqslant g \leqslant c\, f$.

For a measurable set $\Omega \subset \mathbb{R}^n$ we define $\mathcal{P}(\Omega)$ to consist of all measurable functions (called variable exponents) $p : \Omega \to [1,\infty]$. For $p \in \mathcal{P}(\Omega)$ we define $p^- := \operatorname{ess\,inf}_\Omega p$ and $p^+ := \operatorname{ess\,sup}_\Omega p$.

We say that a function $\alpha\colon \Omega \to \mathbb{R}$ is log-*Hölder continuous* on Ω if there exists $c_{\log}(p) \geqslant 0$ and $\alpha_\infty \in \mathbb{R}$ such that

$$|\alpha(x) - \alpha(y)| \leqslant \frac{c_{\log}(p)}{\log(e + 1/|x-y|)} \quad \text{and} \quad |\alpha(x) - \alpha_\infty| \leqslant \frac{c_{\log}(p)}{\log(e+|x|)}$$

for all $x, y \in \Omega$. We can drop the decay condition if Ω is bounded. We define $\mathcal{P}^{\log}(\Omega)$ to consist of those exponents $p \in \mathcal{P}(\Omega)$ for which $\frac{1}{p} : \Omega \to [0,1]$ is log-Hölder continuous on Ω (with the convention $\frac{1}{\infty} = 0$). If $p \in \mathcal{P}(\Omega)$ is bounded, then $p \in \mathcal{P}^{\log}(\Omega)$ is equivalent to the log-Hölder continuity of p.

We use the calligraphic letter $\mathcal{A}$ to indicate that we treat the weight as a multiplier.

A measurable function $\omega : \mathbb{R}^n \to (0,\infty)$ is called a *weight*. For $p \in \mathcal{P}(\mathbb{R}^n)$ we define

$$\varphi_{p(\cdot),\omega}(x,t) := \varphi_{p(\cdot)}\big(x, t\omega(x)\big) = \varphi_{p(x)}\big(t\omega(x)\big),$$

where $\varphi_{p(\cdot)}(x,t)$ is one of the following two choices:

$$\widetilde{\varphi}_{p(\cdot)}(x,t) := \frac{1}{p(x)} t^{p(x)} \qquad \text{or} \qquad \bar{\varphi}_{p(\cdot)}(x,t) := t^{p(x)},$$

with the convention that t^∞ equals ∞ for $t > 1$ and equals 0 for $t \in [0,1]$. We define the corresponding *Musielak–Orlicz* space [35] by

$$L^{p(\cdot)}_\omega(\Omega) := L^{\varphi_{p(\cdot),\omega}}(\Omega).$$

We refer to $L^{p(\cdot)}_\omega$ as a weighted variable exponent space or as a weighted variable Lebesgue space. Both choices $\widetilde{\varphi}_{p(\cdot)}$ and $\bar{\varphi}_{p(\cdot)}$ induce the same space with equivalent norms. The norm $\|\cdot\|_{p(\cdot),\omega}$ of $L^{p(\cdot)}_\omega(\Omega)$ satisfies

$$\|f\|_{p(\cdot),\omega} = \|f\omega\|_{p(\cdot)}. \tag{2.1}$$

It is for this reason, as we noted above, that we refer to the weight ω as a multiplier.

The conjugate function of $\varphi_{p(\cdot),\omega}(t)$ is given by

$$(\varphi_{p(\cdot),\omega})^*(x,t) := \sup_{s \geqslant 0} \big(st - \varphi_{p(\cdot),\omega}(x,s)\big).$$

We have $(\widetilde{\varphi}_{p(\cdot),\omega})^* = \widetilde{\varphi}_{p'(\cdot),1/\omega}$. Unfortunately, for general weights the spaces $L^{p(\cdot)}_\omega$ is not a Banach function space, since simple functions need not be contained in $L^{p(\cdot)}_\omega$. However, if $\omega \in L^{p(\cdot)}_{\mathrm{loc}}$, then all characteristic functions of cubes are contained in $L^{p(\cdot)}_\omega$. As a consequence the restriction $L^{p(\cdot)}_\omega(Q)$ is a Banach function space for every cube Q. Moreover, the associate space of $L^{p(\cdot)}_\omega(Q)$ is given by $L^{p'(\cdot)}_{\omega^{-1}}(Q)$ (see [11, Theorem 2.7.4]). Hence, the norm conjugate formula holds for $L^{p(\cdot)}_\omega(Q)$ (see [11, Corollary 2.7.5]): more precisely,

$$\frac{1}{2}\|f\|_{L^{p(\cdot)}_\omega(Q)} \leqslant \sup_{\|g\|_{L^{p'(\cdot)}_{\omega^{-1}}(Q)} \leqslant 1} \int_Q |f(x)|\,|g(x)|\,dx \leqslant 2\,\|f\|_{L^{p(\cdot)}_\omega(Q)}. \tag{2.2}$$

Therefore, by monotone convergence the norm conjugate formula also holds for $L^{p(\cdot)}_\omega(\mathbb{R}^n)$ if $\omega \in L^{p(\cdot)}_{\mathrm{loc}}$, even when $L^{p(\cdot)}_\omega(\mathbb{R}^n)$ is not a Banach function space. By the same argument $L^{p'(\cdot)}_{\omega^{-1}}(\mathbb{R}^n)$ is the associate space of $L^{p(\cdot)}_\omega(\mathbb{R}^n)$.

Given $p \in \mathcal{P}(\mathbb{R}^n)$ and a weight ω, we have that $\omega \in \mathcal{A}_{p(\cdot)}$ is equivalent to

$$\Big\| \chi_Q \fint_Q |f|\,dx \Big\|_{p(\cdot),\omega} \leqslant \|f\|_{p(\cdot),\omega} \tag{2.3}$$

for all cubes $Q \subset \mathbb{R}^n$ and all $f \in L^{p(\cdot)}_\omega(\mathbb{R}^n)$, where χ_Q is the characteristic function of Q. (This follows from the conjugate norm formula (**2.2**) and Hölder's inequality; see [11, Theorem 4.5.7, Remark 4.5.8] for details.)

We say that $\varphi_{p(\cdot),\omega} \in \mathcal{A}$ if

$$\Big\| \sum_{Q\in\mathcal{Q}} \chi_Q \fint_Q |f|\,dx \Big\|_{p(\cdot),\omega} \leqslant \|f\|_{p(\cdot),\omega} \tag{2.4}$$

for all families $\mathcal{Q}$ of disjoint cubes and all $f \in L^{p(\cdot)}_\omega(\mathbb{R}^n)$. Clearly $\varphi_{p(\cdot),\omega} \in \mathcal{A}$ implies $\omega \in \mathcal{A}_{p(\cdot)}$. It was shown in [11, Section 4.4] that a necessary condition for the maximal operator to be bounded on $L^{p(\cdot)}_\omega$ is that $\varphi_{p(\cdot),\omega} \in \mathcal{A}$, and this condition is sufficient for the maximal operator to be of weak type $\varphi_{p(\cdot),\omega}$, i.e. $\sup_{\lambda>0} \|\lambda \chi_{Mf>\lambda}\|_{p(\cdot)} \leqslant c\|f\|_{p(\cdot)}$ for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

For $p \in \mathcal{P}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$, the condition $\varphi_{p(\cdot)} \in \mathcal{A}$ is equivalent to the boundedness of M on $L^{p(\cdot)}$, see [8] or [11, Theorem 5.7.2]. If $p \in \mathcal{P}(\mathbb{R}^n)$ is constant outside a large ball and satisfies $1 \in \mathcal{A}_{p(\cdot)}$, then $\varphi_{p(\cdot)} \in \mathcal{A}$ as was shown in [29] (see also [31]). Based on arguments as in [11, Section 7.3] this extra requirement can be relaxed to the decay condition of the log-Hölder continuity. However, it was shown in [11, Theorem 5.3.4] that there exists $p \in \mathcal{P}(\mathbb{R}^n)$ with $\frac{3}{2} \leqslant p^- \leqslant p^+ \leqslant 3$ and $1 \in \mathcal{A}_{p(\cdot)}$ but $\varphi_{p(\cdot)} \notin \mathcal{A}$, so M fails to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. This shows the importance of $\mathcal{A}$, although the condition $\omega \in \mathcal{A}_{p(\cdot)}$ is more elegant and easier to verify. The following concept lets us reduce for certain exponents p the verification that $\varphi_{p(\cdot),\omega} \in \mathcal{A}$ to showing that $\omega \in \mathcal{A}_{p(\cdot)}$.

Definition **2.1**. Let $p \in \mathcal{P}(\mathbb{R}^n)$ and let ω be a weight. We say that $\varphi_{p(\cdot),\omega} \in \mathcal{G}$ if

$$\sum_{Q\in\mathcal{Q}} \|\chi_Q f\|_{p(\cdot),\omega} \|\chi_Q g\|_{p'(\cdot),\omega^{-1}} \leqslant K\, \|f\|_{p(\cdot),\omega} \|g\|_{p'(\cdot),\omega^{-1}}$$

for all $f \in L^{p(\cdot)}_\omega(\mathbb{R}^n)$, $g \in L^{p'(\cdot)}_{\omega^{-1}}(\mathbb{R}^n)$, and disjoint families $\mathcal{Q}$ of cubes. The smallest constant K is called the *$\mathcal{G}$-constant of* $\varphi_{p(\cdot),\omega}$.

The name "$\mathcal{G}$" is derived from the works of Berezhnoi [2] for ideal Banach spaces. In the notation of Berezhnoi, the fact that $\varphi_{p(\cdot),\omega} \in \mathcal{G}$ is denoted by $(L^{p(\cdot)}_\omega, (L^{p(\cdot)}_\omega)') \in \mathbf{G}(\mathcal{X}^n)$, where $\mathcal{X}^n$ is the set of all cubes in $\mathbb{R}^n$.

LEMMA **2.1**. *Let $\varphi_{p(\cdot)} \in \mathcal{G}$ and let ω be a weight. Then $\varphi_{p(\cdot),\omega} \in \mathcal{G}$. Furthermore, if $\omega \in \mathcal{A}_{p(\cdot)}$, then $\varphi_{p(\cdot),\omega} \in \mathcal{A}$.*

P r o o f. The property $\varphi_{p(\cdot),\omega} \in \mathcal{G}$ follows immediately from the definition of $\mathcal{G}$ and (**2.1**). Now let $\omega \in \mathcal{A}_{p(\cdot)}$. Then [11, Corollary 7.3.7] implies $\varphi_{p(\cdot),\omega} \in \mathcal{A}$. $\square$

DEFINITION **2.2**. Let $p \in \mathcal{P}(\mathbb{R}^n)$. Then for every cube $Q \subset \mathbb{R}^n$ we define

$$M_{p(\cdot),Q} f := \frac{\|\chi_Q f\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}}.$$

Moreover, we define the maximal function $M_{p(\cdot)}$ by

$$M_{p(\cdot)} f(x) := \sup_{Q \ni x} M_{p(\cdot),Q} f.$$

LEMMA **2.2**. *Let $1 \in \mathcal{A}_{p(\cdot)}$ and let ω be a weight. Then $\omega \in \mathcal{A}_{p(\cdot)}$ if and only if*

$$M_{p(\cdot),Q}(\omega) M_{p'(\cdot),Q}(\omega^{-1}) \approx 1$$

uniformly for all cubes $Q \subset \mathbb{R}^n$.

P r o o f. Since $1 \in \mathcal{A}_{p(\cdot)}$ we have $\|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{p'(\cdot)} \approx |Q|$. On the other hand, by $\omega \in \mathcal{A}_{p(\cdot)}$ we have $\|\chi_Q \omega\|_{p(\cdot)} \|\chi_Q \omega^{-1}\|_{p'(\cdot)} \approx |Q|$. If we take the quotient of these equivalences, we prove the claim. $\square$

If $1 \in \mathcal{A}_{p(\cdot)}$, then the equivalence $\|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{p'(\cdot)} \approx |Q|$ and Hölder's inequality applied to $⨍_Q fg\,dx$ yields that

$$M_Q(fg) = ⨍_Q |fg|\,dx \leqslant c\, M_{p(\cdot),Q}(f)\, M_{p'(\cdot),Q}(g) \qquad (\mathbf{2.5})$$

for all $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$. The constant only depends on the $\mathcal{A}_{p(\cdot)}$-constant of 1.

Finally, we will need the following result, which is proved in [11, Theorem 7.3.27].

THEOREM **2.1**. *Let $p, q, s \in \mathcal{P}^{\log}(\mathbb{R}^n)$ such that $p(x) = q(x)s(x)$ and $s^- > 1$. Then $M_{q(\cdot)}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. The operator norm of $M_{q(\cdot)}$ depends only on $c_{\log}(p)$, $c_{\log}(q)$, $c_{\log}(s)$, and s^-.*

3. The main result

For the proof of Theorem **1.3** we need a version of the classical Calderón–Zygmund decomposition. This result is implicit in [17]; for an explicit proof, see [7].

LEMMA **3.1**. *Let f be a measurable function such that $\fint_Q |f(x)|dx \to 0$ as $|Q| \to \infty$. Fix $b \geqslant 2^{n+1}$ and let $D_k := \{b^{k+1} \geqslant Mf > b^k\}$ for $k \in \mathbb{Z}$. Then $\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}} D_k$ up to a set of measure zero and there exists a family $\{Q_i^k\}_{k\in\mathbb{Z}, i\in\mathbb{N}}$ of cubes such that the following holds.*

(a) *$D_k \subset \bigcup_{i\in\mathbb{N}} 3Q_i^k$ for all $k \in \mathbb{Z}$.*
(b) *for all k, if $i \neq j$, $Q_i^k \cap Q_j^k = \emptyset$.*
(c) *For all k,*

$$b^{k+1} \geqslant \fint_{Q_i^k} |f|\, dx > b^k.$$

(d) *Let $\Omega_k = \bigcup_j Q_j^k$; then $\Omega_{k+1} \subset \Omega_k$, and if $F_i^k = Q_i^k \setminus \Omega_{k+1}$, then the family $\{F_i^k\}_{i,k}$ is pairwise disjoint and $|Q_i^k| \leqslant 2|F_i^k|$.*

If $p^+ < \infty$ and $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, we can apply this lemma to functions $f \in L^{p(\cdot)}(\mathbb{R}^n)$: by Hölder's inequality and [11, Corollary 4.5.9],

$$\fint_Q |f(x)|\, dx \leqslant |Q|^{-1} \|f\|_{p(\cdot)} \|Q\|_{p'(\cdot)} \leqslant c|Q|^{-1/p_Q},$$

and the right-hand term tends to zero as $|Q| \to \infty$.

Proof of Theorem **1.3**. As was mentioned earlier, it was shown in [11, Section 4.4] that $\varphi_{p(\cdot),\omega} \in \mathcal{A}$ (which implies $\omega \in A_{p(\cdot)}$) is a necessary condition for the maximal operator to be bounded on $L^{p(\cdot)}_\omega$. Therefore, we need to prove that the $A_{p(\cdot)}$ condition is sufficient.

Since $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, we have $p \in \mathcal{G} \cap \mathcal{A}$ by [11, Theorems 4.4.8 and 7.3.22]. Let $\omega \in \mathcal{A}_{p(\cdot)}$; then $\varphi_{p(\cdot),\omega} \in \mathcal{A}$ by Lemma **2.1** and $\varphi_{p'(\cdot),1/\omega} \in \mathcal{A}$ by conjugation. Since the class $\mathcal{A}$ is left open (see [11, Theorem 5.4.15]), there exists $s \in (\min\{1/p^-, 1/(p')^-\}, 1)$ such that $\varphi_{sp(\cdot),\omega^{1/s}}, \varphi_{sp'(\cdot),\omega^{-1/s}} \in \mathcal{A}$; here we have used that $\bar{\varphi}_{sp(\cdot),\omega^{1/s}}(t) = \bar{\varphi}_{p(\cdot),\omega}(t^s)$ and $\bar{\varphi}_{sp'(\cdot),\omega^{-1/s}}(t) = \bar{\varphi}_{p'(\cdot),\omega}(t^s)$.

Define $u, v \in \mathcal{P}^{\log}(\mathbb{R}^n)$ by

$$\frac{1}{u'(x)} = s - \frac{1}{p(x)} \qquad \text{and} \qquad \frac{1}{v(x)} = s - \frac{1}{p'(x)}$$

for all $x \in \mathbb{R}^n$. Since $s \in (\min\{1/p^-, 1/(p')^-\}, 1)$, the exponents u and v are well defined. Moreover, $u' = \frac{1}{s}(sp)'$ and $v := \frac{1}{s}(sp')'$ and

$$\begin{aligned}\frac{p'(x)}{v'(x)} &= p'(x)(1-s) + 1 \geqslant (p')^-(1-s) + 1,\\ \frac{p(x)}{u(x)} &= p(x)(1-s) + 1 \geqslant p^-(1-s) + 1.\end{aligned}$$

Thus, by Theorem **2.1**, $M_{u(\cdot)}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and $M_{v'(\cdot)}$ is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$.

Let $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and b, D_k, Q_j^k and F_j^k be as in Lemma **3.1**. Further, let $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ and abbreviate $\hat{Q}_j^k := 3Q_j^k$. Then by (a) and (c) of the lemma it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} |Mf||g|\,dx &\leqslant \sum_k \int_{D_k} b^{k+1}|g|\,dx\\ &\leqslant \sum_{j,k} b^{k+1} \int_{\hat{Q}_j^k} |g|\,dx\\ &\leqslant b \sum_{j,k} \fint_{Q_j^k} |f|\,dx \int_{\hat{Q}_j^k} |g|\,dx\\ &\leqslant b\,3^n \sum_{j,k} |\hat{Q}_j^k|\, M_{\hat{Q}_j^k} f\, M_{\hat{Q}_j^k} g\end{aligned}$$

We use **(2.5)** with exponents u and v to get

$$\begin{aligned}&\int_{\mathbb{R}^n} |Mf||g|\,dx\\ &\quad\leqslant c \sum_{j,k} |\hat{Q}_j^k|\, M_{u(\cdot),\hat{Q}_j^k}(f\omega) M_{u'(\cdot),\hat{Q}_j^k}(\omega^{-1}) M_{v'(\cdot),\hat{Q}_j^k}(g\omega^{-1}) M_{v(\cdot),\hat{Q}_j^k}(\omega).\end{aligned}$$

We now claim that

$$M_{u'(\cdot),\hat{Q}_j^k}(\omega^{-1}) M_{v(\cdot),\hat{Q}_j^k}(\omega) \approx 1. \tag{3.1}$$

This estimate is analogous to the reverse Hölder estimate of (classical) Muckenhoupt weights. By Lemma **2.1**, $\omega^{1/s} \in \mathcal{A}_{sp(\cdot)}$ and $\omega^{-1/s} \in \mathcal{A}_{sp'(\cdot)}$. Then Lemma **2.2** applied to the exponents $sp(\cdot)$ and $sp'(\cdot)$ and weights $\omega^{1/s}$

and $\omega^{-1/s}$, respectively, implies that

$$M_{sp(\cdot),\hat{Q}_j^k}(\omega^{1/s})M_{(sp(\cdot))',\hat{Q}_j^k}(\omega^{-1/s}) \approx 1,$$
$$M_{sp'(\cdot),\hat{Q}_j^k}(\omega^{-1/s})M_{(sp'(\cdot))',\hat{Q}_j^k}(\omega^{1/s}) \approx 1.$$

It follows at once from the definition of the norm that $\|h\|_{sq(\cdot)}^s \approx \||h|^s\|_{q(\cdot)}$ for any $h \in L^{sq(\cdot)}(\mathbb{R}^n)$. Hence, we can rewrite this as

$$M_{p(\cdot),\hat{Q}_j^k}(\omega)M_{u'(\cdot),\hat{Q}_j^k}(\omega^{-1}) \approx 1,$$
$$M_{p'(\cdot),\hat{Q}_j^k}(\omega^{-1})M_{v(\cdot),\hat{Q}_j^k}(\omega) \approx 1.$$

This combined with $M_{p(\cdot),\hat{Q}_j^k}(\omega)M_{p'(\cdot),\hat{Q}_j^k}(\omega^{-1}) \approx 1$ (Lemma **2.2**) implies (**3.1**).

Therefore,

$$\int_{\mathbb{R}^n} |Mf||g|\,dx \leqslant c \sum_{j,k} |\hat{Q}_j^k|\, M_{u(\cdot),\hat{Q}_j^k}(f\omega)\, M_{v'(\cdot),\hat{Q}_j^k}(g\omega^{-1}).$$

Since $|\hat{Q}_j^k| = 3^n|Q_j^k| \leqslant 2\cdot 3^n|F_j^k|$ and the family $\{F_j^k\}_{j,k}$ is pairwise disjoint, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |Mf||g|\,dx &\leqslant c \int_{\mathbb{R}^n} \sum_{j,k} \chi_{F_j^k} M_{u(\cdot)}(f\omega)\, M_{v'(\cdot)}(g\omega^{-1})\,dx \\ &\leqslant c \int_{\mathbb{R}^n} M_{u(\cdot)}(f\omega)\, M_{v'(\cdot)}(g\omega^{-1})\,dx.\end{aligned}$$

Then by Hölder's inequality with p and p', the boundedness of $M_{u(\cdot)}$ on $L^{p(\cdot)}(\mathbb{R}^n)$ and the boundedness of $M_{v'(\cdot)}$ on $L^{p'(\cdot)}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |Mf||g|\,dx \leqslant c\,\|f\omega\|_{p(\cdot)}\|g\omega^{-1}\|_{p'(\cdot)} = c\,\|f\|_{p(\cdot),\omega}\|g\|_{p'(\cdot),\omega^{-1}}.$$

The desired inequality now follows from the norm conjugate formula for $L^{p(\cdot)}_\omega$. □

REMARK **3.1**. Note that in Theorem **1.3** we do not require explicitly $\varphi_{p(\cdot),\omega} \in \mathcal{A}$ but only $\omega \in \mathcal{A}_{p(\cdot)}$ (although the latter follows automatically by the theorem). The implication $\omega \in \mathcal{A}_{p(\cdot)} \Rightarrow \varphi_{p(\cdot),\omega} \in \mathcal{A}$ is a consequence of the log-Hölder continuity of p in Theorem **1.3**. Based on arguments as in [11, Section 7.3] it can be shown that the decay condition of the log-Hölder continuity suffices for this implication.

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