BESOV SPACES WITH VARIABLE SMOOTHNESS AND INTEGRABILITY

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Abstract. In this article we introduce Besov spaces with variable smoothness and integrability indices. We prove independence of the choice of basis functions, as well as several other basic properties. We also give Sobolev-type embeddings, and show that our scale contains variable order Hölder–Zygmund spaces as special cases. We provide an alternative characterization of the Besov space using approximations by analytic functions.

1. Introduction

Spaces of variable integrability, also known as variable exponent function spaces, can be traced back to 1931 and W. Orlicz [29], but the modern development started with the paper [24] of Kováčik and Rákosník in 1991. Corresponding PDE with non-standard growth have been studied since the same time. For an overview we refer to the surveys [14, 21, 30, 36] and the monograph [13]. Apart from interesting theoretical considerations, the motivation to study such function spaces comes from applications to fluid dynamics [1, 2, 34], image processing [11], PDE and the calculus of variation [3, 16, 18, 20, 28, 35, 47].

In a recent effort to complete the picture of the variable exponent Lebesgue and Sobolev spaces, Almeida and Samko [4] and Gurka, Harjulehto and Nekvinda [19] introduced variable exponent Bessel potential spaces $\mathcal{L}^{\alpha,p(\cdot)}$ with constant $\alpha \in \mathbb{R}$. As in the classical case, this space coincides with the Lebesgue/Sobolev space for integer α. There was taken a step further by Xu [44, 45, 46], who considered Besov $B^{\alpha}_{p(\cdot),\,q}$ and Triebel–Lizorkin $F^{\alpha}_{p(\cdot),\,q}$ spaces with variable p, but fixed q and α.

Along a different line of inquiry, Leopold [25, 26, 27] studied pseudo-differential operators with symbols of the type $\langle \xi^{m(x)} \rangle$, and defined related function spaces of Besov-type with variable smoothness, $B^{m(\cdot)}_{p,p}$. In fact, Beauzamy [7] had studied similar ΨDEs already in the beginning of the 70s. Function spaces of variable smoothness have recently been studied by Besov [8, 9, 10]: he generalized Leopold's work by considering both Triebel–Lizorkin spaces $F^{\alpha(\cdot)}_{p,\,q}$ and Besov spaces $B^{\alpha(\cdot)}_{p,\,q}$ in $\mathbb{R}^n$. By way of application, Schneider and Schwab [39] used $B^{m(\cdot)}_{2,2}(\mathbb{R})$ in the analysis of certain Black–Scholes equations. For further considerations of ΨDEs, we refer to Hoh [22] and references therein.

Integrating the above mentioned spaces into a single larger scale promises similar gains and simplifications as were seen in the constant exponent case in the 60s and 70s with the advent of the full Besov and Triebel–Lizorkin scales. Most of the advantages of unification

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do not occur with only one index variable: for instance, traces or Sobolev embeddings cannot be covered in this case, since they involve an interaction between integrability and smoothness. To tackle this, Diening, Hästö and Roudenko [15] introduced Triebel–Lizorkin spaces with all three indices variable, $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and showed that they behaved nicely with respect to trace. Subsequently, Vybíral [43] proved Sobolev (Jawerth) type embeddings in these spaces; they were also studied by Kempka [23]. These studies were all restricted to bounded exponents p and q.

Vybíral [43] and Kempka [23] also considered Besov spaces $B^{\alpha(\cdot)}_{p(\cdot),q}$—note that only the case of constant q was included. This is quite natural, since the norm in the Besov space is usually defined via the iterated space $\ell^q(L^p)$ so that the space integration in L^p is done first, followed by the sum over frequency scales in ℓ^q. Therefore, it is not obvious how q could depend on x, which has already been integrated out. It is the purpose of the present paper to propose a method making this dependence possible and thus completing the unification process in the variable integrability-smoothness case by introducing the Besov space $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ with all three indices variable.

Our space includes the previously mentioned spaces of Besov-type, as well as the Hölder–Zygmund space $C^{\alpha(\cdot)}$. As in the constant exponent case, it is possible to consider unbounded exponents p and q in the Besov space case, while for the Triebel–Lizorkin space one needs p to be bounded. Another advantage of the Besov space for constant exponent is its simplicity compared to the Triebel–Lizorkin space; for instance, the latter requires vector-valued maximal and multiplier theorems, whereas the simple scalar case suffices in the Besov case. Unfortunately, this is not true for the generalization with variable q (this is to be expected, see Remark 4.2 for a discussion). We will nevertheless see that working in the Besov space is relatively simple once some basic tools have been established for dealing in the "iterated" space $\ell^{q(\cdot)}(L^{p(\cdot)})$ in Sections 3 and 4.

We then define the Besov space $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ in Section 5 and give several basic properties establishing the soundness of our definition. In Section 6 we prove elementary embeddings between Besov and Triebel–Lizorkin spaces, as well as Sobolev embeddings in the Besov scale. In Section 7 we show that our scale includes the variable order Hölder–Zygmund space as a special case: $B^{\alpha(\cdot)}_{\infty,\infty} = C^{\alpha(\cdot)}$ for $0 < \alpha < 1$. In Section 8 we give an alternative characterization of the Besov space by means of approximations by analytic functions.

Before starting our main presentation with some conventions and results on semimodular and variable exponent spaces, we point out one possible interesting avenue for future research which might be opened by this work: real interpolation. So far, complex interpolation has been considered in the variable exponent context in [13, 14]. Real interpolation, however, is more difficult in this setting. Using standard notation, we have, for constant exponents,

$$(L^{p_0}, L^{p_1})_{\theta,q} = L^{p_\theta,q},$$

where $1/p_\theta := \theta/p_0 + (1-\theta)/p_1$ and $L^{p_\theta,q}$ is the Lorenz space. To obtain interpolation of Lebesgue spaces one simply chooses $q = p_\theta$. Although details have not been presented anywhere as best we know, it seems that there are no major difficulties in letting p_0 and p_1 be variable here, i.e.

$$(L^{p_0(\cdot)}, L^{p_1(\cdot)})_{\theta,q} = L^{p_\theta(\cdot),q},$$

where p_θ is defined point-wise by the same formula as before. However, this time we do not obtain an interpolation result in Lebesgue spaces, since we cannot set the constant q equal to the function p_θ. In fact, the role of q in the real interpolation method is quite similar to the role of q in the Besov space $B^{\alpha}_{p,q}$. Therefore, we hope that the approach introduced in this paper for Besov spaces with variable q will also allow us to generalize real interpolation properly to the variable exponent context. Another interesting challenge is to extend extrapolation [12] to the setting of Besov spaces.

2. Preliminaries

In this section we introduce some conventions and notation, and state some basic results. For the latter we refer to [13, Chapters 1–3].

We use c as a generic positive constant, i.e. a constant whose value may change from appearance to appearance. The expression $f \approx g$ means that $\frac{1}{c} g \leqslant f \leqslant cg$ for some suitably independent constant c. By χ_A we denote the characteristic function of $A \subset \mathbb{R}^n$. By $\operatorname{supp} f$ we denote the support of the function f, i.e. the closure of its zero set. The notation $X \hookrightarrow Y$ denotes continuous embeddings from X to Y.

Modular spaces. The spaces studied in this paper fit into the framework of so-called semimodular spaces. For an exposition of these concepts we refer to the monographs [13, 31]. We recall the following definition:

Definition 2.1. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $\varrho\colon X \to [0,\infty]$ is called a *semimodular* on X if the following properties hold:

(1) $\varrho(0) = 0$.
(2) $\varrho(\lambda f) = \varrho(f)$ for all $f \in X$ and $|\lambda| = 1$.
(3) $\varrho(\lambda f) = 0$ for all $\lambda > 0$ implies $f = 0$.
(4) $\lambda \mapsto \varrho(\lambda f)$ is left-continuous on $[0,\infty)$ for every $f \in X$.

A semimodular ϱ is called a *modular* if

(5) $\varrho(f) = 0$ implies $f = 0$.

A semimodular ϱ is called *continuous* if

(6) for every $f \in X$ the mapping $\lambda \mapsto \varrho(\lambda f)$ is continuous on $[0,\infty)$.

A semimodular ϱ can be additionally qualified by the term *(quasi)convex*. This means, as usual, that

$$\varrho(\theta f + (1-\theta)g) \leqslant A[\theta\varrho(f) + (1-\theta)\varrho(g)],$$

for all $f, g \in X$; here $A = 1$ in the convex case, and $A \in [1,\infty)$ in the quasiconvex case.

Once we have a semimodular in place, we obtain a normed space in a standard way:

Definition 2.2. If ϱ is a (semi)modular on X, then

$$X_\varrho := \big\{x \in X \colon \exists \lambda > 0 \ \varrho(\lambda x) < \infty\big\}$$

is called a *(semi)modular space.*

Theorem 2.3. *Let ϱ be a (quasi)convex semimodular on X. Then X_ϱ is a (quasi)normed space with the* Luxemburg (quasi)norm *given by*

$$\|x\|_\varrho := \inf\Big\{\lambda > 0 \colon \varrho\Big(\frac{1}{\lambda}x\Big) \leqslant 1\Big\}.$$

For simplicity we will refer to semimodulars as modulars except when special clarity is needed; similarly, we later drop the word "quasi".

One key method for dealing with the somewhat complicated definition of a norm is the following relationship which follows from the definition and left-continuity: $\varrho(f) \leqslant 1$ if and only if $\|f\|_\varrho \leqslant 1$.

Spaces of variable integrability. The *variable exponents* that we consider are always measurable functions on $\mathbb{R}^n$ with range $(c,\infty]$ for some $c>0$. We denote the set of such functions by $\mathcal{P}_0$. The subset of variable exponents with range $[1,\infty]$ is denoted by $\mathcal{P}$. For $A\subset\mathbb{R}^n$ and $p\in\mathcal{P}_0$ we denote $p_A^+=\operatorname{ess\,sup}_A p(x)$ and $p_A^-=\operatorname{ess\,inf}_A p(x)$; we abbreviate $p^+=p_{\mathbb{R}^n}^+$ and $p^-=p_{\mathbb{R}^n}^-$.

The function φ_p is defined as follows:

$$\varphi_p(t)=\begin{cases} t^p & \text{if } p\in(0,\infty),\\ 0 & \text{if } p=\infty \text{ and } t\leqslant 1,\\ \infty & \text{if } p=\infty \text{ and } t>1.\end{cases}$$

The convention $1^\infty=0$ is adopted in order that φ_p be left-continuous. In what follows we write t^p instead of $\varphi_p(t)$, with this convention implied. The variable exponent modular is defined by

$$\varrho_{p(\cdot)}(f):=\int_{\mathbb{R}^n}\varphi_{p(x)}(|f(x)|)\,dx.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}$ and its norm $\|f\|_{p(\cdot)}$ are defined by the modular as explained in the previous subsection. The *variable exponent Sobolev space* $W^{k,p(\cdot)}$ is the subspace of $L^{p(\cdot)}$ consisting of functions f whose distributional k-th order derivative exists and satisfies $|D^k f|\in L^{p(\cdot)}$ with norm

$$\|f\|_{W^{k,p(\cdot)}}=\|f\|_{p(\cdot)}+\|D^k f\|_{p(\cdot)}.$$

We say that $g:\mathbb{R}^n\to\mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g\in C^{\log}_{\rm loc}$, if there exists $c_{\log}>0$ such that

$$|g(x)-g(y)|\leqslant\frac{c_{\log}}{\log(e+1/|x-y|)}$$

for all $x,y\in\mathbb{R}^n$. We say that g is *globally* log-*Hölder continuous*, abbreviated $g\in C^{\log}$, if it is locally log-Hölder continuous and there exists $g_\infty\in\mathbb{R}$ such that

$$|g(x)-g_\infty|\leqslant\frac{c_{\log}}{\log(e+|x|)}$$

for all $x\in\mathbb{R}^n$. The notation $\mathcal{P}^{\log}$ is used for those variable exponents $p\in\mathcal{P}$ with $\frac{1}{p}\in C^{\log}$. The class $\mathcal{P}_0^{\log}$ is defined analogously. If $p\in\mathcal{P}^{\log}$, then convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:

$$\|\varphi*f\|_{p(\cdot)}\leqslant c\,\|\varphi\|_1\|f\|_{p(\cdot)}.$$

3. The mixed Lebesgue-sequence space

In this section we introduce a generalization of the iterated function space $\ell^q(L^{p(\cdot)})$ for the case of variable q, which allows us to define Besov spaces with variable q in Section 5. We give a general but quite strange looking definition for the mixed Lebesgue-sequence space modular. This is not strictly an iterated function space—indeed, it cannot be, since then there would be no space variable left in the outer function space. To motivate our definition, we show that it has several sensible properties (Examples 3.2 and 3.4) and that it concurs with the iterated space when q is constant (Proposition 3.3). Then we show that our modular in fact is a semimodular in the sense defined in the previous section and conclude that it defines a normed space.

Definition 3.1. Let $p, q \in \mathcal{P}_0$. The *mixed Lebesgue-sequence space* $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((f_\nu)_\nu\big) := \sum_\nu \inf\Big\{\lambda_\nu > 0 \,\Big|\, \varrho_{p(\cdot)}\Big(f_\nu/\lambda_\nu^{\frac{1}{q(\cdot)}}\Big) \leqslant 1\Big\}.$$

Here we use the convention $\lambda^{1/\infty} = 1$. The norm is defined from this as usual:

$$\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\Big\{\mu > 0 \,\Big|\, \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\tfrac{1}{\mu}(f_\nu)_\nu\big) \leqslant 1\Big\}.$$

If $q^+ < \infty$, then

$$\inf\Big\{\lambda > 0 \,\Big|\, \varrho_{p(\cdot)}\Big(f/\lambda^{\frac{1}{q(\cdot)}}\Big) \leqslant 1\Big\} = \big\||f|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Since the right-hand side expression is much simpler, we use this notation to stand for the left-hand side even when $q^+ = \infty$. For instance, we often use the notation

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((f_\nu)_\nu\big) = \sum_\nu \big\||f_\nu|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}}$$

for the modular.

The norm in $\ell^{q(\cdot)}(L^{p(\cdot)})$ is usually quite complicated to calculate. Here are some examples where it is possible to simplify its expression.

Example 3.2. Suppose that $p \equiv \infty$. Then

$$\varrho_{\ell^{q(\cdot)}(L^\infty)}\big((f_\nu)_\nu\big) = \sum_\nu \inf\Big\{\lambda_\nu > 0 \,\Big|\, \varrho_\infty\Big(f_\nu/\lambda_\nu^{\frac{1}{q(\cdot)}}\Big) \leqslant 1\Big\}.$$

Now $\varrho_\infty(g) \leqslant 1$ if and only if $|g| \leqslant 1$ almost everywhere. Thus $|f_\nu|/\lambda_\nu^{\frac{1}{q(\cdot)}} \leqslant 1$ a.e., hence $\lambda_\nu \geqslant \operatorname{ess\,sup}_x |f_\nu(x)|^{q(x)}$. It follows that

$$\varrho_{\ell^{q(\cdot)}(L^\infty)}\big((f_\nu)_\nu\big) = \sum_\nu \operatorname*{ess\,sup}_x |f_\nu(x)|^{q(x)}.$$

Note how the case $q(x) = \infty$ is included by the convention $t^\infty = \infty\chi_{(1,\infty)}(t)$.

Another considerable simplification occurs when q is a constant. In this case $\ell^q(L^{p(\cdot)})$ is really an iterated function space in the sense that we take the ℓ^q-norm of $L^{p(\cdot)}$-norms as we now show. This also justifies the notation $\ell^{q(\cdot)}(L^{p(\cdot)})$ even though this is not in general an iterated space.

Proposition 3.3. *If $q \in (0, \infty]$ is constant, then*

$$\big\|(f_\nu)_\nu\big\|_{\ell^q(L^{p(\cdot)})} = \Big\|\big\|f_\nu\big\|_{p(\cdot)}\Big\|_{\ell^q}.$$

Proof. Suppose first that $q \in (0, \infty)$. Since q is constant,

$$\big\||f_\nu|^q\big\|_{\frac{p(\cdot)}{q}} = \big\|f_\nu\big\|_{p(\cdot)}^q$$

and thus

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((f_\nu)_\nu\big) = \sum_\nu \big\|f_\nu\big\|_{p(\cdot)}^q = \Big\|\big\|f_\nu\big\|_{p(\cdot)}\Big\|_{\ell^q}^q$$

from which the claim follows.

In the case $q = \infty$, we find

$$\varrho_{\ell^\infty(L^{p(\cdot)})}\big((f_\nu)_\nu\big) = \sum_\nu \inf\Big\{\lambda_\nu > 0 \,\Big|\, \varrho_{p(\cdot)}\big(f_\nu/\lambda_\nu^0\big) \leqslant 1\Big\}.$$

Here the infimum is zero, unless at least one of the sets over which it is taken is empty, in which case it is infinite. Therefore, the inequality in the definition of the norm,

$$\big\|(f_\nu)_\nu\big\|_{\ell^\infty(L^{p(\cdot)})} = \inf\Big\{\mu > 0 \,\Big|\, \varrho_{\ell^\infty(L^{p(\cdot)})}\big(\tfrac{(f_\nu)_\nu}{\mu}\big) \leqslant 1\Big\},$$

holds if and only if μ is such that $\varrho_{p(\cdot)}\big(f_\nu/\mu\big) \leqslant 1$ for every ν, which means that

$$\inf \mu = \sup\big\{\|f_\nu\|_{p(\cdot)}\big\} = \big\|\|f_\nu\|_{p(\cdot)}\big\|_{\ell^\infty}. \qquad \square$$

Example 3.4. Let us then consider what the norm looks like when $(f_\nu) = (f, 0, 0 \ldots)$. We evaluate the modular:

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\tfrac{1}{\mu}(f_\nu)_\nu\big) = \inf\bigg\{\lambda > 0 \,\bigg|\, \varrho_{p(\cdot)}\Big(\tfrac{1}{\mu} f/\lambda^{\frac{1}{q(\cdot)}}\Big) \leqslant 1\bigg\}.$$

By the definition of the norm, we need to find the infimum of $\mu > 0$ such that the modular of $\frac{1}{\mu}(f_\nu)_\nu$ is at most one:

$$\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\mu > 0 \,\middle|\, \inf\bigg\{\lambda > 0 \,\bigg|\, \varrho_{p(\cdot)}\Big(\tfrac{1}{\mu} f/\lambda^{\frac{1}{q(\cdot)}}\Big) \leqslant 1\bigg\} \leqslant 1\right\}.$$

In order to choose a small μ, we should make λ as big as possible. But the final inequality says that $\lambda \leqslant 1$. Setting $\lambda = 1$, we see that

$$\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\Big\{\mu > 0 \,\Big|\, \varrho_{p(\cdot)}\big(\tfrac{1}{\mu} f\big) \leqslant 1\Big\} = \|f\|_{p(\cdot)}.$$

Thus we see that the values of q have no influence on the value of $\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ when the sequence has just one non-zero entry, just as in the constant exponent case.

So far we have proved various results about the modular $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$. However, now it is time to investigate properly in what sense it is a modular in terms of Definition 2.1.

Proposition 3.5. *Let $p, q \in \mathcal{P}_0$. Then $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a semimodular. Additionally,*

(a) *it is a modular if $p^+ < \infty$; and*
(b) *it is continuous if $p^+, q^+ < \infty$.*

Proof. We need to check properties (1)–(4) of Definition 2.1 and properties (5)–(6) under the appropriate additional assumptions. Properties (1) and (2) are clear. To prove (3), we suppose that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\lambda(f_\nu)_\nu\big) = 0$$

for all $\lambda > 0$. Clearly, $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((0, \ldots, 0, \lambda f_{\nu_0}, 0, \ldots)\big) \leqslant \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\lambda(f_\nu)_\nu\big) = 0$. Thus it follows from Example 3.4 that $\|f_{\nu_0}\|_{p(\cdot)} = 0$, and so $f = 0$. If p is bounded, then the same argument implies (5).

To prove the left-continuity we start by noting that $\mu \mapsto \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\mu(f_\nu)_\nu\big)$ in non-decreasing. By relabeling the function if necessary, we see that it suffices to show that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\mu(f_\nu)_\nu\big) \nearrow \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((f_\nu)_\nu\big)$$

as $\mu \nearrow 1$. We assume that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((f_\nu)_\nu\big) < \infty;$$

the other case is similar. We fix $\varepsilon > 0$ and choose $N > 0$ such that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((f_\nu)_\nu\big) - \varepsilon < \sum_{\nu=0}^{N} \inf\Big\{\lambda_\nu > 0 \,\Big|\, \varrho_{p(\cdot)}\Big(f_\nu/\lambda_\nu^{\frac{1}{q(\cdot)}}\Big) \leqslant 1\Big\}.$$

By the left-continuity of $\mu \mapsto \varrho_{p(\cdot)}(\mu f)$, we then choose $\mu^* < 1$ such that

$$\sum_{\nu=0}^{N} \inf\Big\{\lambda_\nu > 0 \,\Big|\, \varrho_{p(\cdot)}\Big(f_\nu/\lambda_\nu^{\frac{1}{q(\cdot)}}\Big) \leqslant 1\Big\} - \varepsilon < \sum_{\nu=0}^{N} \inf\Big\{\lambda_\nu > 0 \,\Big|\, \varrho_{p(\cdot)}\Big(\mu f_\nu/\lambda_\nu^{\frac{1}{q(\cdot)}}\Big) \leqslant 1\Big\}$$

for all $\mu \in (\mu^*, 1)$. Then $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((f_\nu)_\nu\big) < \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\mu(f_\nu)_\nu\big) + 2\varepsilon$ in the same range, which proves (4). When $q^+ < \infty$, a similar argument reduces (6) to the continuity of $\varrho_{p(\cdot)}$, which holds when $p^+ < \infty$. □

Normally, we would have shown that the modular is quasiconvex as part of the previous theorem. Then Theorem 2.3 would immediately imply that the modular in $\ell^{q(\cdot)}(L^{p(\cdot)})$ defines a quasinorm. Unfortunately, we do not know whether the modular is quasiconvex when $q^+ = \infty$. Therefore, we prove the quasiconvexity of the norm directly; we do this in two steps, beginning with the true convexity. Notice that our assumption when q is non-constant is not as expected. We also do not know if it is necessary.

Theorem 3.6. *Let $p, q \in \mathcal{P}$. If either $\frac{1}{p} + \frac{1}{q} \leqslant 1$ point-wise, or q is a constant, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm.*

Proof. Theorem 2.3 implies all the other claims, except the convexity. If $p \in \mathcal{P}$ and $q \in [1, \infty]$ is a constant, then by Proposition 3.3, the convexity follows directly from the convexity of the modulars in ℓ^q and $L^{p(\cdot)}$.

Thus it remains only to consider $\frac{1}{p} + \frac{1}{q} \leqslant 1$ and to show that

$$\big\|(f_\nu)_\nu + (g_\nu)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant \big\|(f_\nu)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} + \big\|(g_\nu)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Let $\lambda > \|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ and $\mu > \|(g_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$. Then the claim follows from left-continuity if we show that

$$\left\|\frac{(f_\nu)_\nu + (g_\nu)_\nu}{\lambda + \mu}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant 1.$$

Moving to the modular, we get the equivalent condition

$$\sum_\nu \left\|\left|\frac{f_\nu + g_\nu}{\lambda + \mu}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

with our usual convention regarding the case $p/q = 0$. Since

$$\sum_\nu \left\|\left|\frac{f_\nu}{\lambda}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1 \quad \text{and} \quad \sum_\nu \left\|\left|\frac{g_\nu}{\mu}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

the claim follows provided we show that

$$\left\|\left|\frac{f_\nu + g_\nu}{\lambda + \mu}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \frac{\lambda}{\lambda + \mu}\left\|\left|\frac{f_\nu}{\lambda}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} + \frac{\mu}{\lambda + \mu}\left\|\left|\frac{g_\nu}{\mu}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}$$

for every ν. Fix now one ν. Denote the norms on the right-hand side of the previous inequality by σ and τ. Then what we need to show reads

$$\int_{\mathbb{R}^n} \Big|\frac{f_\nu + g_\nu}{\lambda+\mu}\Big|^{p(x)} \Big(\frac{\lambda\sigma+\mu\tau}{\lambda+\mu}\Big)^{-\frac{p(x)}{q(x)}}\,dx \leqslant 1. \tag{3.7}$$

We use Hölder's inequality (with two-point atomic measure and weights (λ,μ)) as follows:

$$\begin{aligned}|f_\nu|+|g_\nu| &= \lambda\sigma^{\frac{1}{q(x)}}\frac{|f_\nu|/\lambda}{\sigma^{1/q(x)}} + \mu\tau^{\frac{1}{q(x)}}\frac{|g_\nu|/\mu}{\tau^{1/q(x)}}\\ &\leqslant (\lambda+\mu)^{1-\frac{1}{p(x)}-\frac{1}{q(x)}}(\lambda\sigma+\mu\tau)^{\frac{1}{q(x)}}\bigg(\lambda\Big(\frac{|f_\nu|/\lambda}{\sigma^{1/q(x)}}\Big)^{p(x)} + \mu\Big(\frac{|g_\nu|/\mu}{\tau^{1/q(x)}}\Big)^{p(x)}\bigg)^{\frac{1}{p(x)}}.\end{aligned}$$

With this, we obtain

$$\Big|\frac{f_\nu + g_\nu}{\lambda+\mu}\Big|^{p(x)} \Big(\frac{\lambda\sigma+\mu\tau}{\lambda+\mu}\Big)^{-\frac{p(x)}{q(x)}} \leqslant \frac{\lambda}{\lambda+\mu}\Big(\frac{|f_\nu|/\lambda}{\sigma^{1/q(x)}}\Big)^{p(x)} + \frac{\mu}{\lambda+\mu}\Big(\frac{|g_\nu|/\mu}{\tau^{1/q(x)}}\Big)^{p(x)}.$$

Integrating the inequality over $\mathbb{R}^n$ and taking into account that σ is the norm of f_ν/λ and τ the norm of g_ν/μ gives us (3.7), which completes the proof. □

Then we consider the quasinorm case.

Theorem 3.8. *If $p,q\in\mathcal{P}_0$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a quasinorm on $\ell^{q(\cdot)}(L^{p(\cdot)})$.*

Proof. By Theorem 2.3, we only need to cosinder quasiconvexity. Let $r\in(0,\frac{1}{2}\min\{p^-,q^-,2\}]$ and define $\tilde p = p/r$ and $\tilde q = q/r$. Then clearly $\frac{1}{\tilde p}+\frac{1}{\tilde q}\leqslant 1$. Thus we obtain by the previous theorem that

$$\begin{aligned}\big\|(f_\nu)_\nu + (g_\nu)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} &= \big\|\,|(f_\nu)_\nu + (g_\nu)_\nu|^r\big\|_{\ell^{\tilde q(\cdot)}(L^{\tilde p(\cdot)})}^{\frac{1}{r}}\\ &\leqslant \big\|\,(|f_\nu|^r)_\nu + (|g_\nu|^r)_\nu\big\|_{\ell^{\tilde q(\cdot)}(L^{\tilde p(\cdot)})}^{\frac{1}{r}}\\ &\leqslant \Big(\big\|\,(|f_\nu|^r)_\nu\big\|_{\ell^{\tilde q(\cdot)}(L^{\tilde p(\cdot)})} + \big\|\,(|g_\nu|^r)_\nu\big\|_{\ell^{\tilde q(\cdot)}(L^{\tilde p(\cdot)})}\Big)^{\frac{1}{r}}\\ &= \Big(\big\|\,(f_\nu)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r + \big\|\,(g_\nu)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r\Big)^{\frac{1}{r}}\\ &\leqslant 2^{\frac{1}{r}-1}\Big(\big\|\,(f_\nu)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} + \big\|\,(g_\nu)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big),\end{aligned}$$

which completes the proof. □

Surprisingly, the condition $p,q\geqslant 1$ is not sufficient to guarantee that the modular $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ be convex! Although it is not true that the modular $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is never convex when q is non-constant, the following example shows that it may be only quasiconvex for arbitrarily small oscillations of q and for arbitrarily large p^-.

Note that the example deals only with sequences having a single non-zero entry. In Example 3.4 we saw that the sequence norm $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ equals the $L^{p(\cdot)}$-norm in this case, so that the triangle inequality holds even though the modular is not convex. We do not know if there exists an example of when $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is not convex and $p,q\geqslant 1$. (Recall that the convexity of the modular is sufficient but not necessary for the convexity of the norm.)

Example 3.9. Consider $(f_v) = (f, 0, 0, \ldots)$ and $(g_v) = (g, 0, 0, \ldots)$. Let $p \in [1, \infty)$ be a constant. Fix two disjoint unit cubes Q_1 and Q_2. Let $a, b \in (0, \infty)$ and $q_1, q_2 \in [1, \infty)$, suppose that $q|_{Q_1} = q_1$ and $q|_{Q_2} = q_2$, and define $f = a^{1/q_1}\chi_{Q_1}$ and $g = b^{1/q_2}\chi_{Q_2}$.

Since q is constant when f is non-zero, we conclude by Proposition 3.3 that

$$\varrho_{\ell^{q(\cdot)}(L^p)}\big((f_\nu)_\nu\big) = \varrho_{\ell^{q_1}(L^p(Q_1))}\big((f_\nu)_\nu\big) = \|a^{1/q_1}\chi_{Q_1}\|_p^{q_1} = a.$$

Similarly, $\varrho_{\ell^{q(\cdot)}(L^p)}\big((g_\nu)_\nu\big) = b$. Then we consider the modular of $\frac{1}{2}(f+g)$:

$$\varrho_{\ell^{q(\cdot)}(L^p)}\big(\tfrac{1}{2}(f_\nu + g_\nu)_\nu\big) = \inf\left\{\lambda > 0 \,\middle|\, \varrho_p\Big(\tfrac{1}{2}(f+g)/\lambda^{\frac{1}{q(\cdot)}}\Big) \leqslant 1\right\}.$$

The condition in the infimum translates to

$$1 \geqslant \int_{\mathbb{R}^n} \left(\frac{f+g}{2\lambda^{1/q(x)}}\right)^p dx = \tfrac{1}{2^p}\int_{\mathbb{R}^n} \Big(\frac{a}{\lambda}\Big)^{\frac{p}{q_1}}\chi_{Q_1} + \Big(\frac{b}{\lambda}\Big)^{\frac{p}{q_2}}\chi_{Q_2}\,dx = \tfrac{1}{2^p}\Big(\frac{a}{\lambda}\Big)^{\frac{p}{q_1}} + \tfrac{1}{2^p}\Big(\frac{b}{\lambda}\Big)^{\frac{p}{q_2}}.$$

Since the right hand side is continuous and decreasing in λ, we see that there exists a unique $\lambda_0 > 0$ for which equality holds. This number is the value of the modular of $\frac{1}{2}(f+g)$. Therefore the convexity inequality for the modular,

$$\varrho_{\ell^{q(\cdot)}(L^p)}\big(\tfrac{1}{2}(f_\nu + g_\nu)_\nu\big) \leqslant \tfrac{1}{2}\big[\varrho_{\ell^{q(\cdot)}(L^p)}\big((f_\nu)_\nu\big) + \varrho_{\ell^{q(\cdot)}(L^p)}\big((g_\nu)_\nu\big)\big],$$

can be written as

$$\lambda_0 \leqslant \frac{a+b}{2} \quad \text{where} \quad \Big(\frac{a}{\lambda_0}\Big)^{\frac{p}{q_1}} + \Big(\frac{b}{\lambda_0}\Big)^{\frac{p}{q_2}} = 2^p.$$

Let us denote $x := a/\lambda_0$ and $y := b/\lambda_0$. Then the convexity condition becomes

$$2 \leqslant x + y \quad \text{when} \quad x^{\frac{p}{q_1}} + y^{\frac{p}{q_2}} = 2^p.$$

By monotonicity, we may reformulate this as follows:

$$x^{\frac{p}{q_1}} + y^{\frac{p}{q_2}} \leqslant 2^p \quad \text{when} \quad 2 = x + y. \tag{3.10}$$

Thus we need to look for the maximum of $x^{\frac{p}{q_1}} + (2-x)^{\frac{p}{q_2}}$ on $[0, 2]$.

Suppose first that $p = 1$. Then (3.10) holds with equality at $x = y = 1$, but this is not a maximum if $q_1 \neq q_2$. Thus we see that the inequality $x^{1/q_1} + y^{1/q_2} \leqslant 2$ does not hold in this case, which means that the modular is non-convex for arbitrarily small $|q_1 - q_2|$.

On the other hand, fix $p > 1$ and choose $q_1 = 1$. Then we can choose $x \in (0, 2)$ so large that $2^p - x^{p/q_1} = 1/2$. Since $y = 2 - x > 0$, we can choose q_2 so large that $y^{p/q_2} > 1/2$. Thus we see that there exists q_1 and q_2 for every p such that (3.10) does not hold.

We end the section by explicitly stating the open problem regarding the triangle inequality.

Open problem 3.11. Suppose that $p, q \in \mathcal{P}$. Is $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ a norm on $\ell^{q(\cdot)}(L^{p(\cdot)})$?

4. The maximal operator in the mixed Lebesgue-sequence space

Despite its title, this section is actually mostly about how to work around the maximal operator. Recall that the Hardy-Littlewood maximal operator M is defined on L^1_{loc} by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy,$$

where $B(x, r)$ denotes the ball with center $x \in \mathbb{R}^n$ and radius $r > 0$. Although the maximal operator has often proved to be very useful in analysis, it is not well suited to the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$:

Example 4.1. Let us take, for instance, the space $\ell^{q(\cdot)}(L^2)$. Let q, q_1, q_2, Q_1 and Q_2 be as in Example 3.9, and let $f_\nu := a_\nu \chi_{Q_1}$ for constants $a_\nu > 0$. Then

$$\varrho_{\ell^{q(\cdot)}(L^2)}\big((f_\nu)_\nu\big) = \sum_\nu \big\| \, |f_\nu|^{q(\cdot)} \big\|_{\frac{2}{q(\cdot)}} = \sum_\nu a_\nu^{q_1}$$

and

$$\varrho_{\ell^{q(\cdot)}(L^2)}\big(\lambda(Mf_\nu)_\nu\big) \geqslant \sum_\nu \big\| |\lambda c\, a_\nu\, \chi_{Q_2}|^{q(\cdot)} \big\|_{\frac{2}{q(\cdot)}} = c \sum_\nu (\lambda a_\nu)^{q_2}.$$

(The constant c depends on the distance between Q_1 and Q_2, but is always positive.) If $q_1 > q_2$, then we can choose the sequence such that $(a_\nu)_\nu \in \ell^{q_1} \setminus \ell^{q_2}$. But then

$$\varrho_{\ell^{q(\cdot)}(L^2)}\big((f_\nu)_\nu\big) < \infty \quad \text{whereas} \quad \varrho_{\ell^{q(\cdot)}(L^2)}\big(\lambda(Mf_\nu)_\nu\big) = \infty \quad \text{for every } \lambda > 0.$$

Thus we see that $M : \ell^{q(\cdot)}(L^2) \not\to \ell^{q(\cdot)}(L^2)$.

Remark 4.2. This example shows that $\ell^{q(\cdot)}(L^{p(\cdot)})$ does not enjoy one key feature of iterated function spaces, namely inheritance of properties from the constituent spaces. Upon closer reflection, this is not so surprising. In the case $\ell^{q}(L^{p(\cdot)})$, the boundedness of the maximal operator, for instance, is inherited, since the outer norm functions on the inner norm in a global fashion. In the case $\ell^{q(\cdot)}(L^{p(\cdot)})$, this is exactly what we want to avoid, since the global approach would necessarily preclude us from considering q which depends on the local space variable. Thus we see that this undesirable property is a direct consequence of the local character of our function space.

The previous example showed that the maximal function is not going to be a good tool in the variable exponent space $\ell^{q(\cdot)}(L^{p(\cdot)})$. Similarly, it was found in [15] that the vector-valued maximal inequality never holds in the iterated function space $L^{p(\cdot)}(\ell^{q(\cdot)})$ when q is non-constant. As in the variable index Triebel–Lizorkin case [15], we use instead so-called η-functions, which have appropriate scaling. The function which we call η is defined on $\mathbb{R}^n$ by

$$\eta_{\nu,m}(x) := \frac{2^{n\nu}}{(1+2^\nu|x|)^m}$$

with $\nu \in \mathbb{N}$ and $m > 0$. Note that $\eta_{\nu,m} \in L^1$ when $m > n$ and that $\|\eta_{\nu,m}\|_1 = c_m$ is independent of ν. We next present some useful lemmas from [15].

Lemma 4.3 (Lemma 6.1, [15])**.** *If $\alpha \in C^{\log}_{\mathrm{loc}}$, then there exists $d \in (n, \infty)$ such that if $m > d$, then*

$$2^{\nu\alpha(x)}\eta_{\nu,2m}(x-y) \leqslant c\, 2^{\nu\alpha(y)}\eta_{\nu,m}(x-y)$$

with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $\nu \in \mathbb{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$2^{\nu\alpha(x)}\eta_{\nu,2m} * f(x) \leqslant c\, \eta_{\nu,m} * (2^{\nu\alpha(\cdot)}f)(x).$$

Remark 4.4. For most properties of the space, Lemma 4.3 is the only property of the smoothness that we need. In recent years also spaces with constant p and q, but more general smoothness functions $\beta_\nu(x)$ have been considered, see e.g. [10, 23]. These are covered by most of our results provided only the previous lemma holds for them.

The next lemma tells us that in most circumstances two convolutions are as good as one.

Lemma 4.5 (Lemma A.3, [15]). *For $\nu_0, \nu_1 \geqslant 0$ and $m > n$, we have*

$$\eta_{\nu_0,m} * \eta_{\nu_1,m} \approx \eta_{\min\{\nu_0,\nu_1\},\, m}$$

with the constant depending only on m and n.

The set $\mathcal{S}$ denotes the usual Schwartz space of rapidly decreasing complex-valued functions and $\mathcal{S}'$ denotes the dual space of tempered distributions. We denote the Fourier transform of φ by $\hat{\varphi}$. The next lemma often allows us to deal with exponents which are smaller than 1. It is Lemma A.6 in [15].

Lemma 4.6 ("The r-trick"). *Let $r > 0$, $\nu \geqslant 0$ and $m > n$. Then there exists $c = c(r, m, n) > 0$ such that*

$$|g(x)| \leqslant c\left(\eta_{\nu,m} * |g|^r(x)\right)^{1/r}, \quad x \in \mathbb{R}^n$$

for all $g \in \mathcal{S}'$ with $\operatorname{supp}\hat{g} \subset \{\xi : |\xi| \leqslant 2^{\nu+1}\}$

Let us then prove one more lemma about η-functions which shows that they are well suited also for mixed Lebesgue sequence spaces, and hence Besov spaces as well.

Lemma 4.7. *Let $p, q \in \mathcal{P}^{\log}$. For $m > n$, there exists $c > 0$ such that*

$$\|(\eta_{\nu,2m} * f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c\,\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Proof. By a scaling argument, we see that it suffices to consider the case $\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = 1$ and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_\nu \big\||c\,\eta_{\nu,2m} * f_\nu|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 2 \quad \text{whenever} \quad \sum_\nu \big\||f_\nu|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}} = 1.$$

This clearly follows from the inequality

$$\big\||c\,\eta_{\nu,2m} * f_\nu|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \big\||f_\nu|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-\nu} =: \delta,$$

which we proceed to prove. The claim can be reformulated as showing that

$$\big\|\delta^{-1}\,|c\,\eta_{\nu,2m} * f_\nu|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

which is equivalent to

$$\big\|\delta^{-\frac{1}{q(\cdot)}} c\,\eta_{\nu,2m} * f_\nu\big\|_{p(\cdot)} \leqslant 1.$$

Since $1/q$ is log-Hölder continuous and $\delta \in [2^{-\nu}, 1 + 2^{-\nu}]$, we can move $\delta^{-\frac{1}{q(\cdot)}}$ inside the convolution by Lemma 4.3: $\delta^{-1/q(\cdot)}|\eta_{\nu,2m} * f_\nu| \leqslant c\,|\eta_{\nu,m} * (\delta^{-1/q(\cdot)} f_\nu)|$. Since convolution is bounded in $L^{p(\cdot)}$ when $p \in \mathcal{P}^{\log}$, we obtain

$$\big\|\delta^{-\frac{1}{q(\cdot)}} c\,\eta_{\nu,2m} * f_\nu\big\|_{L^{p(\cdot)}} \leqslant \big\|c\,\eta_{\nu,m} * (\delta^{-\frac{1}{q(\cdot)}} f_\nu)\big\|_{L^{p(\cdot)}} \leqslant \big\|\delta^{-\frac{1}{q(\cdot)}} f_\nu\big\|_{L^{p(\cdot)}}$$

with an appropriate choice of $c > 0$. Now the right-hand side is less than or equal to one if and only if

$$\big\||\delta^{-\frac{1}{q(\cdot)}} f_\nu|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

which follows immediately from the definition of δ. □

In the previous lemma we required that $p, q \geqslant 1$. This restriction can often be circumvented by the r-trick combined with the following identity, which follows directly from the definition:

$$\big\|(f_\nu)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \big\|(|f_\nu|^r)_\nu\big\|^{\frac{1}{r}}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}.$$

5. The definition of the Besov space

We use a Fourier approach to the Besov and Triebel–Lizorkin space. For this we need some general definitions, well-known from the constant exponent case.

Definition 5.1. We say a pair (φ, Φ) is *admissible* if $\varphi, \Phi \in \mathcal{S}$ satisfy

- $\operatorname{supp} \hat\varphi \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leqslant |\xi| \leqslant 2\}$ and $|\hat\varphi(\xi)| \geqslant c > 0$ when $\frac{3}{5} \leqslant |\xi| \leqslant \frac{5}{3}$,
- $\operatorname{supp} \hat\Phi \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$ and $|\hat\Phi(\xi)| \geqslant c > 0$ when $|\xi| \leqslant \frac{5}{3}$.

We set $\varphi_\nu(x) := 2^{\nu n}\varphi(2^\nu x)$ for $\nu \in \mathbb{N}$ and $\varphi_0(x) := \Phi(x)$.

We always denote by φ_ν and ψ_ν admissible functions in the sense of the previous definition. Usually, the Besov space is defined using the functions φ_ν; when this is not the case, it will be explicitly marked, e.g. $\|\cdot\|^{\psi}_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$.

Using the admissible functions (φ, Φ) we can define the norms

$$\|f\|_{F^\alpha_{p,q}} := \Big\|\big\|2^{\nu\alpha}\, \varphi_\nu * f\big\|_{\ell^q}\Big\|_p \quad \text{and} \quad \|f\|_{B^\alpha_{p,q}} := \Big\|\big\|2^{\nu\alpha}\, \varphi_\nu * f\big\|_p\Big\|_{\ell^q},$$

for constants $\alpha \in \mathbb{R}$ and $p, q \in (0, \infty]$ (excluding $p = \infty$ for the F-scale). The Triebel–Lizorkin space $F^\alpha_{p,q}$ and the Besov space $B^\alpha_{p,q}$ consist of all distributions $f \in \mathcal{S}'$ for which $\|f\|_{F^\alpha_{p,q}} < \infty$ and $\|f\|_{B^\alpha_{p,q}} < \infty$, respectively. It is well-known that these spaces do not depend on the choice of the initial system (φ, Φ) (up to equivalence of quasinorms). Further details on the classical theory of these spaces can be found in the books of Triebel [40, 41]; see also [42] for recent developments.

Definition 5.2. Let φ_ν be as in Definition 5.1. For $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}_0$, the *Besov space* $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathcal{S}'$ such that

$$\|f\|^{\varphi}_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \big\|(2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

In the case of $p = q$ we use the notation $B^{\alpha(\cdot)}_{p(\cdot)} := B^{\alpha(\cdot)}_{p(\cdot),p(\cdot)}$.

To the Besov space we can also associate the following modular:

$$\varrho^{\varphi}_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}(f) := \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big((2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu\big),$$

which can be used to define the norm. By Proposition 3.3 we directly obtain the following simplification in the case when q is constant:

Corollary 5.3. *If q is a constant, then*

$$\|f\|^{\varphi}_{B^{\alpha(\cdot)}_{p(\cdot),q}} = \Big\|\big\|2^{\nu\alpha(\cdot)}\,\varphi_\nu * f\big\|_{p(\cdot)}\Big\|_{\ell^q}.$$

An important special case of the Besov space is when $p = q$. In this case we show that the Besov space agrees with the corresponding Triebel–Lizorkin space studied in [15]. This space is defined via the norm

$$\|f\|^{\varphi}_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \Big\|\big\|2^{\nu\alpha(\cdot)}\,\varphi_\nu * f\big\|_{\ell^{q(\cdot)}}\Big\|_{p(\cdot)}.$$

Notice that there is no difficulty with q depending on the space variable x here, since the $\ell^{q(\cdot)}$-norm is inside the $L^{p(\cdot)}$-norm.

Proposition 5.4. *Let $p \in \mathcal{P}_0$ and $\alpha \in L^\infty$. Then $B^{\alpha(\cdot)}_{p(\cdot)} = F^{\alpha(\cdot)}_{p(\cdot)}$.*

Proof. The claim follows from the following calculation:

$$\begin{aligned}
\varrho^{\varphi}_{B^{\alpha(\cdot)}_{p(\cdot)}}(f) &= \sum_\nu \big\| |2^{\nu\alpha(\cdot)}\, \varphi_\nu * f|^{p(\cdot)} \big\|_1 = \sum_\nu \int_{\mathbb{R}^n} |2^{\nu\alpha(x)}\, \varphi_\nu * f(x)|^{p(x)}\, dx \\
&= \int_{\mathbb{R}^n} \sum_\nu |2^{\nu\alpha(x)}\, \varphi_\nu * f(x)|^{p(x)}\, dx \\
&= \int_{\mathbb{R}^n} \big\| 2^{\nu\alpha(x)}\, \varphi_\nu * f(x) \big\|^{p(x)}_{\ell^{p(x)}}\, dx = \varrho^{\varphi}_{F^{\alpha(\cdot)}_{p(\cdot)}}(f).
\end{aligned}$$ □

So far we have not considered whether the space given by Definition 5.2 depends on the choice of (φ, Φ). Therefore, the previous result has to be understood in the sense that the Besov space defined from a certain (φ, Φ) equals the Triebel–Lizorkin space defined by *the same* φ. This is not entirely satisfactory. In [15] it was shown that the Triebel–Lizorkin space is independent of the basis functions, essentially assuming that $p, q, \alpha \in \mathcal{P}_0^{\log} \cap L^\infty$. We prove now a corresponding result for the Besov space, but with more general assumptions; namely we allow $p, q \in \mathcal{P}_0^{\log}$ to be unbounded, and assume of $\alpha \in L^\infty$ only local log-Hölder continuity.

Theorem 5.5. *Let $p, q \in \mathcal{P}_0^{\log}$ and $\alpha \in C^{\log}_{\rm loc} \cap L^\infty$. Then the space $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the admissible basis functions φ_ν, i.e. different functions yield equivalent quasinorms.*

Proof. Let (φ, Φ) and (ψ, Ψ) be two pairs of admissible functions. By symmetry, it suffices to prove that

$$\|f\|^{\varphi}_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c\, \|f\|^{\psi}_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Define $K := \{-1, 0, 1\}$. Following classical lines, and using that $\hat{\varphi}_\nu \hat{\psi}_\mu = 0$ when $|\mu - \nu| > 1$, we have

$$\varphi_\nu * f = \sum_{k \in K} \varphi_\nu * \psi_{\nu+k} * f.$$

Fix $r \in (0, \min\{1, p^-\})$ and $m > n$ large. Since $|\varphi_\nu| \leqslant c\, \eta_{\nu,2m/r}$, with $c > 0$ independent of ν, we obtain

$$|\varphi_\nu * \psi_{\nu+k} * f| \leqslant c\, \eta_{\nu,2m/r} * |\psi_{\nu+k} * f| \leqslant c\, \eta_{\nu,2m/r} * \Big(\eta_{\nu+k,2m} * |\psi_{\nu+k} * f|^r \Big)^{1/r},$$

where in the second inequality we used the r-trick. By Minkowski's integral inequality (with exponent $1/r > 1$) and Lemma 4.5 we further obtain

$$|\varphi_\nu * \psi_{\nu+k} * f|^r \leqslant c\, \big[\eta_{\nu,2m/r} * \eta^{1/r}_{\nu+k,2m} \big]^r * |\psi_{\nu+k} * f|^r \approx \eta_{\nu+k,2m} * |\psi_{\nu+k} * f|^r.$$

This, together with Lemma 4.3 and Lemma 4.7, gives

$$\begin{aligned}
\big\|\big(2^{\nu\alpha(\cdot)}\,\varphi_\nu * f\big)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} &= \big\|\big(2^{\nu\alpha(\cdot)r}|\varphi_\nu * f|^r\big)_\nu\big\|^{1/r}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})} \\
&\leqslant c\sum_{k\in K}\Big\|\big(2^{\nu\alpha(\cdot)r}\,\eta_{\nu+k,2m} * |\psi_{\nu+k} * f|^r\big)_\nu\Big\|^{1/r}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})} \\
&\leqslant c\sum_{k\in K}\Big\|\big(\eta_{\nu+k,m} * (2^{\nu\alpha(\cdot)r}|\psi_{\nu+k} * f|^r)\big)_\nu\Big\|^{1/r}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})} \\
&\leqslant c\sum_{k\in K}\Big\|\big(2^{\nu\alpha(\cdot)r}|\psi_{\nu+k} * f|^r\big)_\nu\Big\|^{1/r}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}. \\
&= c\sum_{k\in K}\big\|\big(2^{\nu\alpha(\cdot)}\psi_{\nu+k} * f\big)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.
\end{aligned}$$

By the shift invariance of the mixed Lebesgue sequence space, the last sum equals $3\,\|f\|^{\psi}_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$, which completes the proof. □

Although one would obviously like to work in the variable index Besov space independent of the choice of basis functions φ_ν, the assumptions needed in the previous theorem are quite strong in the sense that many of the later results work under much weaker assumptions. In the interest of clarity, we state those results only with the assumptions actually needed in their proofs. They should then be understood to hold with any particular choice of basis functions. For simplicity, we will not explicitly include the dependence on φ, thus omitting φ in the notation of the norm and modular.

Remark 5.6. Recently, Schneider [37, 38] studied Besov spaces of *varying smoothness* B^{s,s_0}_p, were the function $x \mapsto s(x)$ determines the smoothness point-wise and s_0 is a constant determining the smoothness globally. These spaces are supposed to classify the smoothness behavior of a function in the neighborhood of each point. Nevertheless, they follow a different line of investigation and apparently cannot be included in our scale.
Roughly speaking, another way of generalizing the classical scale $B^s_{p,q}$ is to replace the constant smoothness parameter s by appropriate functions or sequences. Function spaces with *generalized smoothness* have been considered in the literature from different points of view. The paper [17] gives an unified treatment of such spaces following the Fourier analytical approach. In that paper several references can be found including historical remarks on spaces of generalized smoothness.

6. Embeddings

The following theorem gives basic embeddings between Besov spaces and Triebel–Lizorkin spaces.

Theorem 6.1. *Let $\alpha, \alpha_0, \alpha_1 \in L^\infty$ and $p, q_0, q_1 \in \mathcal{P}_0$.*

(i) *If $q_0 \leqslant q_1$, then*
$$B^{\alpha(\cdot)}_{p(\cdot),\,q_0(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),\,q_1(\cdot)}.$$

(ii) *If $(\alpha_0-\alpha_1)^- > 0$, then*
$$B^{\alpha_0(\cdot)}_{p(\cdot),\,q_0(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p(\cdot),\,q_1(\cdot)}.$$

(iii) *If* $p^+, q^+ < \infty$, *then*

$$B^{\alpha(\cdot)}_{p(\cdot),\min\{p(\cdot),q(\cdot)\}} \hookrightarrow F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),\max\{p(\cdot),q(\cdot)\}}.$$

Proof. Assume that $q_0 \leqslant q_1$. We note that $\lambda^{\frac{1}{q_0(x)}} \leqslant \lambda^{\frac{1}{q_1(x)}}$ when $\lambda \leqslant 1$. By the definition it follows that

$$\varrho_{B^{\alpha(\cdot)}_{p(\cdot),\,q_0(\cdot)}}(f/\mu) \geqslant \varrho_{B^{\alpha(\cdot)}_{p(\cdot),\,q_1(\cdot)}}(f/\mu)$$

for every $\mu > 0$, which implies (i).

By (i),

$$B^{\alpha_0(\cdot)}_{p(\cdot),\,q_0(\cdot)} \hookrightarrow B^{\alpha_0(\cdot)}_{p(\cdot),\,q_0^+} \quad \text{and} \quad B^{\alpha_1(\cdot)}_{p(\cdot),\,q_1^-} \hookrightarrow B^{\alpha_1(\cdot)}_{p(\cdot),\,q_1(\cdot)}.$$

Therefore, it suffices to prove (ii) for constant exponents q_0^+ and q_1^-, which we denote again by $q_0, q_1 \in (0,\infty]$ for simplicity. Then the proof is similar to the constant exponent situation. Indeed,

$$\Big\|\big\|2^{\nu\alpha_1(\cdot)}\,\varphi_\nu * f\big\|_{p(\cdot)}\Big\|_{\ell^{q_1}} \leqslant c_1 \Big\|\big\|2^{\nu\alpha_0(\cdot)}\,\varphi_\nu * f\big\|_{p(\cdot)}\Big\|_{\ell^{\infty}} \leqslant c_1 \Big\|\big\|2^{\nu\alpha_0(\cdot)}\,\varphi_\nu * f\big\|_{p(\cdot)}\Big\|_{\ell^{q_0}}$$

with $c_1^{q_1} = \sum_{\nu\geqslant 0} 2^{-\nu q_1(\alpha_0-\alpha_1)^-} < \infty$.

To prove the first embedding in (iii), let $r := \min\{p, q\}$ and $f_\nu(x) := 2^{\nu\alpha(x)}\,|\varphi_\nu * f(x)|$. We assume that $\varrho_{B^{\alpha(\cdot)}_{p(\cdot),\,r(\cdot)}}(f) \leqslant 1$. Then it suffices to show that $\varrho_{F^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}}(f) \leqslant c$. Since $\ell^{r(x)} \hookrightarrow \ell^{q(x)}$, we obtain

$$\varrho_{p(\cdot)}\big(\|f_\nu\|_{\ell^{q(x)}}\big) \leqslant \varrho_{p(\cdot)}\big(\|f_\nu\|_{\ell^{r(x)}}\big) = \int_{\mathbb{R}^n} \Big(\sum_\nu f_\nu^{r(x)}\Big)^{\frac{p(x)}{r(x)}} dx = \varrho_{\frac{p(\cdot)}{r(\cdot)}}\Big(\sum_\nu f_\nu^{r(\cdot)}\Big).$$

Thus it suffices to show that the right hand side is bounded by a constant, which follows if the corresponding norm is bounded. Using the triangle inequality, we obtain just this:

$$\Big\|\sum_\nu f_\nu^{r(\cdot)}\Big\|_{\frac{p(\cdot)}{r(\cdot)}} \leqslant \sum_\nu \big\|f_\nu^{r(\cdot)}\big\|_{\frac{p(\cdot)}{r(\cdot)}} = \varrho_{B^{\alpha(\cdot)}_{p(\cdot),\,r(\cdot)}}(f) \leqslant 1.$$

For the second embedding in (iii), we use a similar derivation, with $s = \max\{p, q\}$. We assume that $\varrho_{F^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}}(f) \leqslant 1$. Then we estimate the modular in the Besov space with a reverse triangle inequality which holds since $p/s \leqslant 1$:

$$\varrho_{B^{\alpha(\cdot)}_{p(\cdot),\,s(\cdot)}}(f) = \sum_\nu \big\|f_\nu^{s(\cdot)}\big\|_{\frac{p(\cdot)}{s(\cdot)}} \leqslant \Big\|\sum_\nu f_\nu^{s(\cdot)}\Big\|_{\frac{p(\cdot)}{s(\cdot)}} = \big\|\|f_\nu\|^{s(\cdot)}_{\ell^{s(\cdot)}}\big\|_{\frac{p(\cdot)}{s(\cdot)}}.$$

Since p/s is bounded, the right hand side is bounded if and only if the corresponding modular is bounded. In fact,

$$\varrho_{\frac{p(\cdot)}{s(\cdot)}}\big(\|f_\nu\|^{s(\cdot)}_{\ell^{s(\cdot)}}\big) = \int_{\mathbb{R}^n} \|f_\nu\|^{p(x)}_{\ell^{s(x)}}\,dx = \varrho_{F^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}}(f) \leqslant 1,$$

so we are done. □

We next consider embeddings of Sobolev-type which trade smoothness for integrability. For constant exponents it is well-known that

$$B^{\alpha_0}_{p_0,q} \hookrightarrow B^{\alpha_1}_{p_1,q} \tag{6.2}$$

if $\alpha_0 - \frac{n}{p_0} = \alpha_1 - \frac{n}{p_1}$, where $0 < p_0 \leqslant p_1 \leqslant \infty$, $0 < q \leqslant \infty$, $-\infty < \alpha_1 \leqslant \alpha_0 < \infty$ (see e.g. [40, Theorem 2.7.1]). This is a consequence of certain Nikolskii inequalities for entire analytic functions of exponential type (cf. [40, p. 18] for constant exponents), which we now generalize to the variable exponent setting.

For constant p, the proof of (6.2) can be done via dilation arguments (see [40], Remarks 1 and 4 on pp. 18 and 23). With the r-trick we overcome the problems with dilations in the variable exponent case.

Lemma 6.3. *Let $p_1, p_0, q \in \mathcal{P}_0$ with $\alpha - n/p_1$ and $1/q$ locally* log*-Hölder continuous. If $p_1 \geqslant p_0$, then there exists $c > 0$ such that*

$$\big\| |c\, 2^{\nu\alpha(\cdot)} g|^{q(\cdot)} \big\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leqslant \Big\| |2^{\nu(\alpha(\cdot)+\frac{n}{p_0(\cdot)}-\frac{n}{p_1(\cdot)})} g|^{q(\cdot)} \Big\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-\nu}$$

for all $\nu \in \mathbb{N}_0$ and $g \in L^{p_0(\cdot)} \cap \mathcal{S}'$ with supp $\hat{g} \subset \{\xi : |\xi| \leqslant 2^{\nu+1}\}$ *such that the norm on the right hand side is at most one.*

Proof. Let us denote $\beta := \alpha - n/p_1$ and

$$\lambda := \big\| |2^{\nu(\beta(\cdot)+\frac{n}{p_0(\cdot)})} g|^{q(\cdot)} \big\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-\nu}.$$

Note that the assumption on the norm implies that $\lambda \in [2^{-\nu}, 1+2^{-\nu}]$. Using the r-trick and Lemma 4.3, we get

$$\lambda^{-\frac{r}{q(x)}} 2^{\nu r\beta(x)} |g(x)|^r \leqslant c\, \lambda^{-\frac{r}{q(x)}} 2^{\nu r\beta(x)} \,(\eta_{\nu,2m} * |g|^r)(x) \leqslant c\, \eta_{\nu,m} * \big(\lambda^{-\frac{1}{q(\cdot)}} 2^{\nu\beta(\cdot)} |g|\big)^r (x)$$

for large m. Fix $r \in (0, p_0^-)$ and set $s = p_0/r \in \mathcal{P}_0$. An application of Hölder's inequality with exponent s yields

$$\lambda^{-\frac{1}{q(x)}} 2^{\nu\beta(x)} |g(x)| \leqslant c\, \big\| 2^{-\frac{\nu n}{s(\cdot)}} \eta_{\nu,m}(x - \cdot) \big\|_{s'(\cdot)}^{1/r} \, \big\| \lambda^{-\frac{1}{q(\cdot)}} 2^{\nu(\beta(\cdot)+\frac{n}{p_0(\cdot)})} g \big\|_{p_0(\cdot)}.$$

The second norm on the right hand side is bounded by 1 due to the choice of λ. To show that the first norm is also bounded, we investigate the corresponding modular:

$$\varrho_{s'(\cdot)}\big(2^{-\frac{\nu n}{s(\cdot)}} \eta_{\nu,m}(x-\cdot)\big) = \int_{\mathbb{R}^n} 2^{\nu n}\,(1+2^\nu|x-y|)^{-ms'(y)}\, dy \leqslant \int_{\mathbb{R}^n} (1+|2^\nu x - z|)^{-m\,(s')^-}\, dz < \infty,$$

since $m\,(s')^- > n$. Now with the appropriate choice of $c_0 \in (0,1]$, we find that

$$\begin{aligned}\big(c_0\, \lambda^{-\frac{1}{q(x)}} 2^{\nu\alpha(x)} |g(x)|\big)^{p_1(x)} &= c_0^{p_0(x)} \Big[c_0 \frac{2^{\nu\beta(x)}|g(x)|}{\lambda^{-1/q(x)}}\Big]^{p_1(x)-p_0(x)} \big(\lambda^{-\frac{1}{q(x)}} 2^{\nu(\beta(x)+\frac{n}{p_0(x)})}\, |g(x)|\big)^{p_0(x)} \\ &\leqslant \big(\lambda^{-\frac{1}{q(x)}} 2^{\nu(\beta(x)+\frac{n}{p_0(x)})}\, |g(x)|\big)^{p_0(x)}.\end{aligned}$$

Integrating this inequality over $\mathbb{R}^n$ and taking into account the definition of λ gives us the claim. □

Applying the previous lemma, we obtain the following generalization of (6.2).

Theorem 6.4 (Sobolev inequality)**.** *Let $p_0, p_1, q \in \mathcal{P}_0$ and $\alpha_0, \alpha_1 \in L^\infty$ with $\alpha_0 \geqslant \alpha_1$. If $1/q$ and*

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$$

are locally log*-Hölder continuous, then*

$$B^{\alpha_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Proof. Suppose without loss of generality that the $B^{\alpha_0(\cdot)}_{p_0(\cdot),q(\cdot)}$-modular of a function is less than 1. Then an application of the previous lemma with $\alpha(x) = \alpha_1(x)$ and $g = \varphi_\nu * f$, shows that the $B^{\alpha_1(\cdot)}_{p_1(\cdot),q(\cdot)}$-modular is bounded by a constant. □

Corollary 6.5. *Let $p_0, p_1, q_0, q_1 \in \mathcal{P}_0$ and $\alpha_0, \alpha_1 \in L^\infty$ with $\alpha_0 \geqslant \alpha_1$. If*

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)} + \varepsilon(x)$$

is locally log*-Hölder continuous and $\varepsilon^- > 0$, then*

$$B^{\alpha_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}.$$

Proof. By Theorems 6.1(i) and 6.4,

$$B^{\alpha_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha_0(\cdot)}_{p_0(\cdot),\infty} \hookrightarrow B^{\alpha_1(\cdot)+\varepsilon(\cdot)}_{p_1(\cdot),\infty}.$$

We combine this with the embedding $B^{\alpha_1(\cdot)+\varepsilon(\cdot)}_{p_1(\cdot),\infty} \hookrightarrow B^{\alpha_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}$ from Theorem 6.1(ii) to conclude the proof. □

Remark 6.6. It suffices to assume uniform continuity in the previous corollary (and hence in Proposition 6.9) instead of log-Hölder continuity. This is achieved by choosing an auxiliary smoothness function $\tilde{\alpha}$ between α_0 and α_1 with the appropriate continuity modulus.

Let C_u be the space of all bounded uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup norm. Concerning embeddings into C_u, we have the following result.

Corollary 6.7. *Let $\alpha \in C^{\log}_{\text{loc}}$, $p \in \mathcal{P}^{\log}$ and $q \in \mathcal{P}_0$. If*

$$\alpha(x) - \tfrac{n}{p(x)} \geqslant \delta \max\{1 - \tfrac{1}{q(x)}, 0\}$$

for some fixed $\delta > 0$ and every $x \in \mathbb{R}^n$, then

$$B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow C_u.$$

Proof. Let $\gamma(x) := \alpha(x) - \frac{n}{p(x)}$. By Theorem 6.1(i), we may replace q with the larger exponent $\max\{1, \delta/(\delta - \gamma)\} \in \mathcal{P}^{\log}$. It then follows from Theorem 6.4 that

$$B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow B^{\gamma(\cdot)}_{\infty,q(\cdot)}.$$

Since $B^0_{\infty,1} \hookrightarrow C_u$ by classical results (e.g., [40, Proposition 2.5.7]), we will complete the proof by showing that

$$B^{\gamma(\cdot)}_{\infty,q(\cdot)} \hookrightarrow B^0_{\infty,1}.$$

Denote $f_\nu := \varphi_\nu * f$. The remaining embedding can be written, using homogeneity in the usual manner, as

$$\sum_\nu \sup_x |f_\nu| \leqslant c \quad \text{whenever} \quad \sum_\nu \sup_x |2^{\nu\gamma(x)} f_\nu|^{q(x)} \leqslant 1,$$

where we used the expression from Example 3.2 for the second modular. We choose x_ν such that $\sup_x |f_\nu| \leqslant 2\,|f_\nu(x_\nu)|$ for each ν. Then it follows from Young's inequality that

$$\sum_\nu \sup_x |f_\nu| \approx \sum_\nu |f_\nu(x_\nu)| \leqslant \sum_\nu |2^{\nu\gamma(x_\nu)} f_\nu(x_\nu)|^{q(x_\nu)} + 2^{-\nu\gamma(x_\nu)q'(x_\nu)} \leqslant 1 + \sum_\nu 2^{-\nu\delta} \leqslant c,$$

which completes the proof of the remaining embedding. □

Let $\mathcal{L}^{\alpha,p(\cdot)}$, $\alpha \in \mathbb{R}$, be the Bessel potential space modeled in $L^{p(\cdot)}$. It was shown in [15] that $F^\alpha_{p(\cdot),2} = \mathcal{L}^{\alpha,p(\cdot)}$ when $\alpha \geqslant 0$, $1 < p^- \leqslant p^+ < \infty$ and $p \in \mathcal{P}^{\log}$. Under the same assumptions on p, by Theorem 6.1 one gets the embedding

$$B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{L}^{\sigma,p(\cdot)}$$

for $\alpha^- > \sigma \geqslant 0$. In particular, we have $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow L^{p(\cdot)}$ if $\alpha^- > 0$ (cf. [4, Corollary 6.2] or [19, Theorem 6.1]). Next we derive a stronger version of this.

Let us define

$$\sigma_p(x) := n\Big(\frac{1}{\min\{1,p(x)\}} - 1\Big) \quad \text{and} \quad \overline{p}(x) := \max\{1,p(x)\}, \quad x \in \mathbb{R}^n. \tag{6.8}$$

If $\alpha - n/p = \alpha - \sigma_p - n/\overline{p}$ is log-Hölder continuous, $p, q \in \mathcal{P}_0$, $\alpha \in L^\infty$ and $(\alpha - \sigma_p)^- > 0$, then by Corollary 6.5 we get

$$B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow B^0_{\overline{p}(\cdot),1}.$$

We further conclude that

$$\|f\|_{\overline{p}(\cdot)} \leqslant \sum_{\nu \geqslant 0} \|\varphi_\nu * f\|_{\overline{p}(\cdot)} = \|f\|_{B^0_{\overline{p}(\cdot),1}} \leqslant c\, \|f\|_{B^{\alpha(\cdot)-\sigma_{p(\cdot)}}_{\overline{p}(\cdot),1}}.$$

This shows that under the above assumptions the elements from $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ are regular distributions. A discussion of such results for constant exponents can be found in [40, Section 2.5.3]; see also [33, Section 2.2.4]. In sum, we obtain the following result, without the assumptions $p^- > 1$ and $p^+ < \infty$.

Proposition 6.9. *Assume that $p, q \in \mathcal{P}_0$ and $\alpha \in L^\infty$ are such that $\alpha - n/p$ is* log-*Hölder continuous. Let σ_p and $\overline{p}$ be as in* (6.8)*. If $(\alpha - \sigma_p)^- > 0$, then*

$$B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow L^{\overline{p}(\cdot)}.$$

Let $p, q \in \mathcal{P}_0$ and $\alpha \in L^\infty$. Define $\alpha_0 := (\alpha - \frac{n}{p})^-$. Then $\alpha \geqslant \alpha_0 + \frac{n}{p} =: \alpha_1 \in L^\infty$. It is clear that $\alpha_1 - \frac{n}{p} = \alpha_0$ is log-Hölder continuous. Therefore we obtain by Theorem 6.4 that

$$B^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p(\cdot),\,\infty} \hookrightarrow B^{\alpha_1(\cdot)-\frac{n}{p(\cdot)}}_{\infty,\,\infty} = B^{\alpha_0}_{\infty,\,\infty} \hookrightarrow \mathcal{S}'.$$

Here the last embedding is just the well known constant exponent case [40, Theorem 2.3.3]. A similar argument gives the embedding of $\mathcal{S}$ into the variable index Besov space. Thus we obtain:

Theorem 6.10. *If $p, q \in \mathcal{P}_0$ and $\alpha \in L^\infty$, then*

$$\mathcal{S} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)} \hookrightarrow \mathcal{S}'.$$

Remark 6.11. As in the classical case (e.g. [40, Theorem 2.3.3]), using the previous theorem one can prove the completeness of the Besov space $B^{\alpha(\cdot)}_{p(\cdot),\,q}$, hence it is a (quasi)Banach space.

7. The Hölder–Zygmund space

In this section we show that our scale of Besov spaces includes also the Hölder–Zygmund spaces of continuous functions. This application requires in particular that we include the case of unbounded p and q.

We start by generalizing the definition of Hölder–Zygmund spaces to the variable order setting. Such spaces have been considered e.g. in [5, 6, 32].

Recall that C_u denotes the set of all bounded uniformly continuous functions.

Definition 7.1. Let $\alpha : \mathbb{R}^n \to (0,1]$. The *Zygmund space* $\mathcal{C}^{\alpha(\cdot)}$ consists of all $f \in C_u$ such that $\|f\|_{\mathcal{C}^{\alpha(\cdot)}} < \infty$, where

$$\|f\|_{\mathcal{C}^{\alpha(\cdot)}} := \|f\|_\infty + \sup_{x\in\mathbb{R}^n,\, h\in\mathbb{R}^n\setminus\{0\}} \frac{|\Delta_h^2 f(x)|}{|h|^{\alpha(x)}}.$$

For $\alpha < 1$, the *Hölder space* $C^{\alpha(\cdot)}$ is defined analogously but with the norm given by

$$\|f\|_{C^{\alpha(\cdot)}} := \|f\|_\infty + \sup_{x\in\mathbb{R}^n,\, h\in\mathbb{R}^n\setminus\{0\}} \frac{|\Delta_h^1 f(x)|}{|h|^{\alpha(x)}}.$$

Here Δ_h^j is the j-th order difference operator ($h \in \mathbb{R}^n$, $j \in \mathbb{N}$):

$$\Delta_h^1 f(x) = f(x+h) - f(x), \quad \Delta_h^{j+1} f = \Delta_h^1(\Delta_h^j f).$$

One can easily derive the point-wise inequality

$$\sup_h |h|^{-\alpha(x)} |\Delta_h^1 f(x)| \leqslant \frac{1}{2-2^{\alpha^+}} \sup_h |h|^{-\alpha(x)} |\Delta_h^2 f(x)|, \quad x \in \mathbb{R}^n.$$

Hence we have $\mathcal{C}^{\alpha(\cdot)} \hookrightarrow C^{\alpha(\cdot)}$ for $\alpha^+ < 1$. In fact, these two spaces coincide for such α, as in the classical case. This is one consequence of the following result.

Theorem 7.2. *For α locally* log*-Hölder continuous with $\alpha^- > 0$,*

$$B^{\alpha(\cdot)}_{\infty,\infty} = \mathcal{C}^{\alpha(\cdot)} \quad (\alpha \leqslant 1) \quad \textit{and} \quad B^{\alpha(\cdot)}_{\infty,\infty} = C^{\alpha(\cdot)} \quad (\alpha^+ < 1).$$

Combining this result with the embeddings from Section 6 as follows

$$W^{1,p(\cdot)} = F^1_{p(\cdot),2} \hookrightarrow B^1_{p(\cdot),\infty} \hookrightarrow B^{1-n/p(\cdot)}_{\infty,\infty} = C^{1-n/p(\cdot)},$$

we obtain the following result which extends [6, Theorem 7] to unbounded domains in the case of Euclidean spaces.

Corollary 7.3. *If $p \in \mathcal{P}^{\log}$ with $n < p^- \leqslant p^+ < \infty$, then*

$$W^{1,p(\cdot)} \hookrightarrow C^{1-n/p(\cdot)}.$$

By a similar argument, we can also obtain the embedding

$$B^{\alpha(\cdot)+n/p(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{C}^{\alpha(\cdot)}$$

in the case $\alpha^- > 0$ for $p, q \in \mathcal{P}_0^{\log}$ and $\alpha \in C^{\log}_{\mathrm{loc}}$. We then move on to the proof of the theorem itself.

Proof of Theorem 7.2. The proof is naturally divided into two parts. First we consider the claim that

$$\mathcal{C}^{\alpha(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{\infty,\infty} \quad (\alpha \leqslant 1) \quad \text{and} \quad C^{\alpha(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{\infty,\infty} \quad (\alpha^+ < 1).$$

We prove only the first embedding; the second is similar. We estimate the absolute value on the right hand side of

$$\|f\|_{B^{\alpha(\cdot)}_{\infty,\infty}} = \sup_\nu \sup_x |2^{\nu\alpha(x)} \varphi_\nu * f(x)|$$

by $\|f\|_{\mathcal{C}^{\alpha(\cdot)}}$. The term $\nu = 0$ is easily estimated in terms of $\|f\|_\infty$, so we consider in what follows $\nu > 0$.

Since the Besov space is independent of the choice of admissible φ, we may assume without loss of generality that $\varphi(-y) = \varphi(y)$. Then

$$\varphi_\nu * f(x) = \tfrac{1}{2}\int_{\mathbb{R}^n} \varphi_\nu(h)[f(x+h)+f(x-h)]\,dh = \tfrac{1}{2}\int_{\mathbb{R}^n} \varphi_\nu(h)\Delta_h^2 f(x-h)\,dh,$$

where we used the fact that $\int \varphi_\nu(y)\,dy = \widehat{\varphi_\nu}(0) = 0$ in the second step. By definition, $|\Delta_h^2 f(x-h)| \leqslant \|f\|_{\mathcal{C}^{\alpha(\cdot)}}|h|^{\alpha(x-h)}$. For small h, the log-Hölder continuity implies that $|h|^{\alpha(x-h)} \leqslant c\,|h|^{\alpha(x)}$. Thus we obtain

$$\begin{aligned}|\varphi_\nu * f(x)| &\leqslant c\int_{|h|<1} |\varphi_\nu(h)|\,|h|^{\alpha(x)}\,dh + c\int_{|h|\geqslant 1} |\varphi_\nu(h)|\,|h|^{\alpha^+}\,dh\\ &= c\int_{|h|<2^\nu} |\varphi(h)|\,|2^{-\nu}h|^{\alpha(x)}\,dh + c\int_{|h|\geqslant 2^\nu} |\varphi(h)|\,|2^{-\nu}h|^{\alpha^+}\,dh\\ &\leqslant c2^{-\nu\alpha(x)}\int_{\mathbb{R}^n} |\varphi(h)|\,[|h|^{\alpha^+}+|h|^{\alpha^-}]\,dh,\end{aligned}$$

where in the second step we used a change of variables. Since φ decays faster than any polynomial (as $\operatorname{supp}\hat{\varphi}$ is bounded), the integral on the right-hand side is finite, and so we are done.

We then move on to the second part of the proof of the theorem, and consider the claim

$$B^{\alpha(\cdot)}_{\infty,\infty} \hookrightarrow \mathcal{C}^{\alpha(\cdot)} \quad (\alpha \leqslant 1) \quad\text{and}\quad B^{\alpha(\cdot)}_{\infty,\infty} \hookrightarrow C^{\alpha(\cdot)} \quad (\alpha^+ < 1).$$

First we note that

$$\sup_{0<|h|\leqslant 1}\sup_x \frac{|\Delta_h^M f(x)|}{|h|^{\alpha(x)}} \leqslant 2^{\alpha^+}\sup_{k\geqslant 0}\sup_{|h|\leqslant 2^{-k}}\sup_x |2^{k\alpha(x)}\Delta_h^M f(x)|.$$

(We restrict ourselves to $|h| \leqslant 1$ since large h are easily handled.) For $a > 0$ and $M \geqslant 1$ there exists $c > 0$ such that

$$|\Delta_h^M(\varphi_\nu * f)(x)| \leqslant c\,\min\{1, 2^{(\nu-k)M}\}(\varphi_\nu^* f)_a(x),$$

for every $\nu, k \in \mathbb{N}_0$ and $|h| \leqslant 2^{-k}$, where $(\varphi_\nu^* f)_a(x) := \sup_y \frac{|\varphi_\nu * f(x-y)|}{1+|2^\nu y|^a}$ is the Peetre maximal function, cf. [40, (2.5.12/8)]. Since $f = \sum_\nu \varphi_\nu * f$ with convergence in L^∞, we can use the previous estimate to obtain

$$\begin{aligned}\sup_{|h|\leqslant 2^{-k}} |2^{k\alpha(x)}\Delta_h^M f(x)| &\leqslant c\sum_{\nu<k} 2^{(\nu-k)(M-\alpha(x))}\,2^{\nu\alpha(x)}(\varphi_\nu^* f)_a(x)\\ &\quad + c\sum_{\nu\geqslant k+1} 2^{(k-\nu)\alpha(x)}\,2^{\nu\alpha(x)}(\varphi_\nu^* f)_a(x)\end{aligned} \tag{7.4}$$

Therefore, we need to estimate $2^{\nu\alpha(x)}(\varphi_\nu^* f)_a(x)$. Let us denote $K := \sup_x 2^{\nu\alpha(x)}|\varphi_\nu * f(x)|$. Then

$$2^{\nu\alpha(x)}(\varphi_\nu^* f)_a(x) = \sup_y 2^{\nu\alpha(x)}\frac{|\varphi_\nu * f(x-y)|}{1+|2^\nu y|^a} \leqslant K\sup_y \frac{2^{\nu(\alpha(x)-\alpha(x-y))}}{1+|2^\nu y|^a}.$$

When $|y| < 2^{-\nu/2}$, it follows from the log-Hölder continuity of α that $\nu\,(\alpha(x)-\alpha(x-y)) \leqslant c$. When $|y| \geqslant 2^{-\nu/2}$, the right-hand side is bounded by $K2^{\nu(\alpha^+-\alpha^--a/2)}$, which remains bounded provided we choose $a > 2(\alpha^+-\alpha^-)$. Therefore we have shown that

$$2^{\nu\alpha(x)}(\varphi_\nu^* f)_a(x) \leqslant c\sup_x 2^{\nu\alpha(x)}|\varphi_\nu * f(x)| \leqslant c\,\|f\|_{B^{\alpha(\cdot)}_{\infty,\infty}}.$$

Using this in (7.4), we find that

$$\sup_{|h|\leqslant 2^{-k}} |2^{k\alpha(x)}\Delta_h^M f(x)| \leqslant c\left[\sum_{\nu<k} 2^{(\nu-k)(M-\alpha^+)} + \sum_{\nu\geqslant k+1} 2^{(k-\nu)\alpha^-}\right] \|f\|_{B^{\alpha(\cdot)}_{\infty,\infty}}.$$

If $M=1$, then we have assumed that $\alpha^+<1$; for $M=2$, $M-\alpha^+\geqslant 1$. Thus the terms in the brackets are bounded, so we have estimated the main part of the norm. Since we also have $\|f\|_\infty \leqslant c\,\|f\|_{B^{\alpha(\cdot)}_{\infty,\infty}}$ for $\alpha^->0$, the proof is complete. □

8. Characterization by approximations

The aim of this section is to characterize the elements from $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ in terms of Nikolskii representations involving sequences of entire analytic functions. Let

$$\mathcal{U}^{p(\cdot)} := \left\{(u_\nu)_\nu \subset \mathcal{S}'\cap L^{p(\cdot)} :\ \operatorname{supp}\hat{u}_\nu \subset \{\xi\,:\,|\xi|\leqslant 2^{\nu+1}\},\ \nu\in\mathbb{N}_0\right\}.$$

Theorem 8.1. *Let $p,q\in\mathcal{P}_0^{\log}$ and $\alpha\in C^{\log}_{\rm loc}\cap L^\infty$ with $\alpha^->0$. Then $f\in\mathcal{S}'$ belongs to $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ if and only if there exists $u=(u_\nu)_\nu\in\mathcal{U}^{p(\cdot)}$ such that*

$$f=\lim_{\nu\to\infty} u_\nu \quad \text{in} \quad \mathcal{S}' \tag{8.2}$$

and

$$\|f\|^u := \|u_0\|_{p(\cdot)} + \left\|\left(2^{\nu\alpha(\cdot)}\,(f-u_\nu)\right)_\nu\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}<\infty.$$

Moreover,

$$\|f\|^\star := \inf_u\ \|f\|^u$$

is an equivalent quasinorm in $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, where the infimum is taken over all possible representations $(u_\nu)_\nu\in\mathcal{U}^{p(\cdot)}$ satisfying (8.2).

Proof. First we show that $\|f\|^\star\leqslant c\,\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$. If $(\varphi_\nu)_\nu$ is an admissible system, then

$$u_\nu := \sum_{j=0}^{\nu}\varphi_j * f \to f \quad \text{in} \quad \mathcal{S}' \quad \text{as} \quad \nu\to\infty.$$

Thus $(u_\nu)_\nu\in\mathcal{U}^{p(\cdot)}$ and

$$\left(2^{\nu\alpha(\cdot)}(f-u_\nu)\right)_\nu = \sum_{j\geqslant 0} 2^{-j\alpha(\cdot)}\left(2^{(j+\nu)\alpha(\cdot)}\varphi_{j+\nu}*f\right)_\nu \quad \text{in } \mathcal{S}'.$$

Observe that $2^{-j\alpha(\cdot)}\leqslant 2^{-j\alpha^-}$ and that $\alpha^->0$ by assumption. Let $r\in(0,\frac12\min\{p,q,2\})$. Using the previous expression and the triangle inequality in the mixed Lebesgue-sequence space (Theorem 3.6), we obtain

$$\begin{aligned}
\left\|\left(2^{\nu\alpha(\cdot)}\,(f-u_\nu)\right)_\nu\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} &= \left\|\left|\sum_{j\geqslant 0} 2^{-j\alpha(\cdot)}\left(2^{(j+\nu)\alpha(\cdot)}\varphi_{j+\nu}*f\right)_\nu\right|^r\right\|^{\frac1r}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}\\
&\leqslant \left\|\sum_{j\geqslant 0} 2^{-jr\alpha(\cdot)}\left(2^{(j+\nu)r\alpha(\cdot)}|\varphi_{j+\nu}*f|^r\right)_\nu\right\|^{\frac1r}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}\\
&\leqslant \left(\sum_{j\geqslant 0} 2^{-jr\alpha^-}\left\|\left(2^{(j+\nu)r\alpha(\cdot)}|\varphi_{j+\nu}*f|^r\right)_\nu\right\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}\right)^{\frac1r}\\
&\leqslant c\left\|\left(2^{\nu\alpha(\cdot)}\varphi_\nu*f\right)_\nu\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},
\end{aligned}$$

where the last step follows from the invariance of the norm under shifts in the ν direction. Since $\|u_0\|_{p(\cdot)} = \|\varphi_0 * f\|_{p(\cdot)} \leqslant \|f\|_{B^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}}$, we have shown that

$$\|f\|^u \leqslant c\,\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}}.$$

Now we prove the opposite inequality. Let $(u_k)_k \in \mathcal{U}^{p(\cdot)}$ be such that $f = \lim_{k\to\infty} u_k$ and $\|f\|^u < \infty$. Then $\varphi_\nu * f = \sum_{k\geqslant -1} \varphi_\nu * (u_{\nu+k} - u_{\nu+k-1})$, $\nu \in \mathbb{N}_0$ (with $u_{-1} = 0$). Using the r-trick, with r as before, we find that

$$2^{\nu\alpha(x)}|\varphi_\nu * f| \leqslant 2^{\nu\alpha(x)} \sum_{k\geqslant -1} \big|\varphi_\nu * (u_{\nu+k} - u_{\nu+k-1})\big| \leqslant \sum_{k\geqslant -1} \big[\eta_{\nu,m} * (2^{\nu\alpha(\cdot)r}|u_{\nu+k} - u_{\nu+k-1}|^r)\big]^{\frac{1}{r}}.$$

Since $2^{\nu\alpha(\cdot)} \leqslant 2^{(\nu+k)\alpha(\cdot)}2^{-k\alpha^-}$, we obtain

$$\big\|\big(2^{\nu\alpha(\cdot)}\varphi_\nu * f\big)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c \sum_{k\geqslant -1} 2^{-k\alpha^-} \big\|\big(\eta_{\nu,m} * (2^{(\nu+k)\alpha(\cdot)r}|u_{\nu+k} - u_{\nu+k-1}|^r)\big)_\nu\big\|^{\frac{1}{r}}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}.$$

Then we can get rid of the function η by Lemma 4.7. Using

$$|u_{\nu+k} - u_{\nu+k-1}| \leqslant |f - u_{\nu+k}| + |f - u_{\nu+k-1}|,$$

we find that

$$\big\|\big(2^{\nu\alpha(\cdot)}\varphi_\nu * f\big)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c \sum_{k\geqslant -1} 2^{-k\alpha^-} \big\|\big(2^{(\nu+k)\alpha(\cdot)}(f - u_{\nu+k})\big)_\nu\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Using again the invariance of the sequence space with respect to shifts, we see that the left hand side can be estimated by a constant times $\|f\|^u$. Taking the infimum over u, we obtain $\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),\,q(\cdot)}} \leqslant c\,\|f\|^\star$. □

Remark 8.3. Compared to the proof given in [40, Theorem 2.5.3] for constant exponents, we used the r-trick to circumvent the use of Fourier multipliers. Consequently, our proof requires only the assumption $\alpha^- > 0$, while the stronger assumption $\alpha > \sigma_p$ is needed in [40] even in the constant exponent case.

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