

# CAPACITIES IN GENERALIZED ORLICZ SPACES

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ABSTRACT. In this paper basic properties of both Sobolev and relative capacities are studied in generalized Orlicz spaces. The capacities are compared with each other and the Hausdorff measure. As an application, the existence of quasicontinuous representative of generalized Orlicz functions is proved.

## 1. INTRODUCTION

In the calculus of variations one studies existence and properties of solutions to minimization problems such as

$$\min_{u \in W^{1,1}} \int F(x, |\nabla u|) dx.$$

Classical techniques, by De Giorgi and Moser, cover both the linear case and the  $p$ -growth case, where  $F(t) \approx t^p$ . Marcellini [19] developed the theory of  $(p, q)$ -growth, which is based on the growth assumption  $t^p - 1 \lesssim F(x, t) \lesssim t^q + 1$ ,  $q > p$ . Zhikov [23] studied such minimizers as models of anisotropic materials and also observed that they exhibit the so-called Lavrentiev phenomenon whereby minimizers do not have improved regularity and may even be discontinuous.

In the variable exponent case,  $F(x, t) \approx t^{p(x)}$ , the change in the anisotropy (growth rate) is gradual owing to the continuity of  $p$ . For instance, in electrorheological fluid dynamics, where the anisotropy depends on the smooth electrical field, this is a reasonable assumption [22]. In other situations, such as composite materials, a more clear-cut transition is better. To this end, Baroni, Colombo and Mingione [1, 2, 3, 8] have developed a regularity theory of the double phase functional  $F(x, t) = t^p + a(x)t^q$ ,  $q > p$ , which has the property that the growth rate changes abruptly from  $p$  to  $q$  in the sets  $\{a = 0\}$  and  $\{a > 0\}$  (see also [4, 5, 7, 9, 21]). Recently, we were able to generalize their first step, showing Hölder continuity, to the general  $\Phi$ -growth case [16]. Also other results have recently been obtained for partial differential equations with generalized Orlicz growth, cf. [12, 13, 15, 16].

A different approach to differential equations is based on (nonlinear) potential theory. The foundation of nonlinear potential theory includes general notions of a Radon measure, a capacity and generalized functions. The sets of capacity zero are the exceptional sets for representatives of the function. In this paper we give basic properties of both Sobolev capacity and relative capacity in the generalized Orlicz setting. We follow the general framework of [18] and its previous adaption to the variable exponent setting [10, Chapter 10]. The results can be applied e.g. in the study of boundary behavior of solutions to PDE.

We consider general  $\Phi$ -functions which need not be convex. Many of the proofs in this paper follow a standard pattern, since they do not depend on the exact form of the integrand used in the definition of capacity. We have therefore omitted or abbreviated several proofs (e.g. Section 6). Nevertheless, it is necessary to check these basic building blocks in order to proceed with constructing the theory, since the results are not covered by earlier

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2010 *Mathematics Subject Classification.* 46E35.

*Key words and phrases.* Sobolev capacity, relative capacity, Generalized Orlicz space, Musielak–Orlicz spaces, nonstandard growth, variable exponent, double phase.

results, merely similar. There are also some proofs which are new, namely Theorem 3.5, Example 4.3 and Proposition 6.2. Furthermore, this general setting clarifies the necessity of various assumptions for different properties (e.g. Theorem 3.2). In particular, we find that the relative capacity is Choquet also for non-convex  $\Phi$ -function, whereas convexity is needed for the Sobolev capacity.

It should be noted that Ohno and Shimomura [20] have recently studied (Sobolev) capacity in the generalized Orlicz case. However, they consider the capacity in a metric measure space setting with Hajlasz gradients. These results therefore work in the Euclidean setting only when the maximal operator is bounded, since the Hajlasz gradient corresponds to  $M(|\nabla f|)$ .

The outline of the paper is as follows. We start by introducing our notation and basic definitions. Then we study Sobolev capacity and compare it with the Hausdorff measure. Next we derive existence of a quasicontinuous representative of generalized Orlicz function. Finally, we study relative capacity and compare it with the Sobolev capacity.

## 2. PRELIMINARIES

We study spaces of functions defined in  $\mathbb{R}^n$  or opens sets  $\Omega \subset \mathbb{R}^n$ .

A real-valued function is *L-almost increasing*,  $L \geq 1$ , if  $Lf(s) \geq f(t)$  for  $s > t$ . So a 1-almost increasing function is increasing. *L-almost decreasing* is defined analogously.

**Definition 2.1.** We say that  $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$  is a *weak  $\Phi$ -function*, and write  $\varphi \in \Phi_w(\Omega)$ , if the following conditions hold.

- For every  $t \in [0, \infty)$  the function  $x \mapsto \varphi(x, t)$  is measurable and for every  $x \in \Omega$  the function  $t \mapsto \varphi(x, t)$  is non-decreasing and left-continuous.
- $\varphi(x, 0) = \lim_{t \rightarrow 0^+} \varphi(x, t) = 0$  and  $\lim_{t \rightarrow \infty} \varphi(x, t) = \infty$  for every  $x \in \Omega$ .
- There exists  $L \geq 1$  such that  $t \mapsto \frac{\varphi(x, t)}{t}$  is *L-almost increasing* in  $(0, \infty)$ , for every  $x \in \Omega$ .

If  $\varphi \in \Phi_w(\Omega)$  is convex with respect to the second variable, then it is called a *convex  $\Phi$ -function*, and we write  $\varphi \in \Phi_c(\Omega)$ .

If there exists  $\beta \in (0, 1]$  such that  $\varphi(x, \beta) \leq 1$  and  $\varphi(x, 1/\beta) \geq 1$  for all  $x \in \Omega$ , then we say that (A0) holds. Further, we say that  $\varphi \in \Phi_w(\Omega)$  satisfies

(aInc) $_p$  if there exists  $L \geq 1$  such that  $t \mapsto \frac{\varphi(x, t)}{t^p}$  is *L-almost increasing* in  $(0, \infty)$  for every  $x \in \Omega$ ,

(aInc) if there exist  $p > 1$  such that (aInc) $_p$  holds,

(aDec) $_q$  if there exists  $L \geq 1$  such that  $t \mapsto \frac{\varphi(x, t)}{t^q}$  is *L-almost decreasing* in  $(0, \infty)$  for every  $x \in \Omega$ ,

(aDec) if there exist  $q > 1$  such that (aDec) $_q$  holds.

The corresponding conditions with  $L = 1$  are denoted by (Inc) and (Dec). Note that the definition of weak  $\Phi$ -function includes the assumption (aInc) $_1$ .

**Definition 2.2.** Let  $\varphi \in \Phi_w(\Omega)$  and define the *modular*  $\varrho_{\varphi(\cdot)}$  for  $f \in L^0(\Omega)$  by

$$\varrho_{\varphi(\cdot)}(f) := \int_{\Omega} \varphi(x, |f(x)|) dx.$$

The *generalized Orlicz space*, also called Musielak–Orlicz space, is defined as the set

$$L^{\varphi(\cdot)}(\Omega) := \left\{ f \in L^0(\Omega) : \lim_{\lambda \rightarrow 0^+} \varrho_{\varphi(\cdot)}(\lambda f) = 0 \right\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{\varphi(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_{\varphi(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

For  $\varphi \in \Phi_c(\Omega)$ ,  $L^{\varphi(\cdot)}(\Omega)$  is a Banach space [10, Theorem 2.3.13.].

**Definition 2.3.** A function  $f \in L^{\varphi(\cdot)}(\Omega)$  belongs to Sobolev space  $W^{1,\varphi(\cdot)}(\Omega)$ , if its weak partial derivatives  $\partial_1 f, \dots, \partial_n f$  exist and belong to  $L^{\varphi(\cdot)}(\Omega)$ , that is,

$$W^{1,\varphi(\cdot)}(\Omega) := \{f \in W_{loc}^{1,1}(\Omega) : f, |\nabla f| \in L^{\varphi(\cdot)}(\Omega)\}.$$

We define a semimodular on  $W^{1,\varphi(\cdot)}(\Omega)$  by

$$\varrho_{1,\varphi(\cdot)}(f) := \varrho_{\varphi(\cdot)}(f) + \varrho_{\varphi(\cdot)}(|\nabla f|).$$

which induces a quasinorm by  $\|f\|_{1,\varphi(\cdot)} := \inf \{\lambda > 0 : \varrho_{1,\varphi(\cdot)}(\frac{f}{\lambda}) \leq 1\}$ .

**Lemma 2.4.** Let  $\varphi \in \Phi_c(\Omega)$  satisfy (aInc) and (aDec), and let  $f_j, g_j \in W^{1,\varphi(\cdot)}(\Omega)$  for  $j = 1, 2, \dots$ . Assume further that the sequence  $(\varrho_{1,\varphi(\cdot)}(g_j))_{j=1}^{\infty}$  is bounded. If  $\varrho_{1,\varphi(\cdot)}(f_j - g_j) \rightarrow 0$  as  $j \rightarrow \infty$ , then

$$|\varrho_{1,\varphi(\cdot)}(f_j) - \varrho_{1,\varphi(\cdot)}(g_j)| \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

*Proof.* Since  $\varphi$  is convex and satisfies (aDec) it satisfies (Dec) for some possible larger exponent  $q$  by Lemma 2.6 of [16]. Let  $\lambda \in (0, 1)$ . By convexity and (Dec) we obtain

$$\begin{aligned} \varphi(x, |f_j|) &= \varphi(x, |f_j - g_j + g_j|) \leq \varphi(x, \frac{\lambda}{\lambda} |f_j - g_j| + \frac{1-\lambda}{1-\lambda} |g_j|) \\ &\leq \lambda \varphi(x, \frac{1}{\lambda} |f_j - g_j|) + (1-\lambda) \varphi(x, \frac{1}{1-\lambda} |g_j|) \\ &\leq \lambda^{1-q} \varphi(x, |f_j - g_j|) + (1-\lambda)^{1-q} \varphi(x, |g_j|). \end{aligned}$$

Next we subtract  $\varphi(x, g_j)$  from both sides and integrate over  $\Omega$ . Hence

$$\varrho_{\varphi(\cdot)}(f_j) - \varrho_{\varphi(\cdot)}(g_j) \leq \lambda^{1-q} \varrho_{\varphi(\cdot)}(f_j - g_j) + ((1-\lambda)^{1-q} - 1) \varrho_{\varphi(\cdot)}(g_j).$$

Note that the choice  $\lambda = \frac{1}{2}$  implies that  $(\varrho_{\varphi(\cdot)}(f_j))$  is a bounded sequence.

Swapping  $f_j$  and  $g_j$  gives a similar inequality, and combining the inequalities, we find that that

$$|\varrho_{\varphi(\cdot)}(f_j) - \varrho_{\varphi(\cdot)}(g_j)| \leq \lambda^{1-q} \varrho_{\varphi(\cdot)}(f_j - g_j) dx + ((1-\lambda)^{1-q} - 1)(\varrho_{\varphi(\cdot)}(f_j) + \varrho_{\varphi(\cdot)}(g_j)).$$

The same argument can be applied also to the weak gradient. Hence

$$|\varrho_{1,\varphi(\cdot)}(f_j) - \varrho_{1,\varphi(\cdot)}(g_j)| \leq \lambda^{1-q} \varrho_{1,\varphi(\cdot)}(f_j - g_j) + ((1-\lambda)^{1-q} - 1)(\varrho_{1,\varphi(\cdot)}(f_j) + \varrho_{1,\varphi(\cdot)}(g_j)).$$

Let  $\varepsilon > 0$  be given. Since  $\varrho_{1,\varphi(\cdot)}(f_j) + \varrho_{1,\varphi(\cdot)}(g_j) \leq 2c$ , we can choose  $\lambda$  so small that

$$((1-\lambda)^{1-q} - 1)(\varrho_{1,\varphi(\cdot)}(f_j) + \varrho_{1,\varphi(\cdot)}(g_j)) \leq \frac{\varepsilon}{2}.$$

We can then choose  $j_0$  so large that  $\lambda^{1-q} \varrho_{1,\varphi(\cdot)}(f_j - g_j) \leq \frac{\varepsilon}{2}$  when  $j \geq j_0$  and it follows that  $|\varrho_{1,\varphi(\cdot)}(f_j) - \varrho_{1,\varphi(\cdot)}(g_j)| \leq \varepsilon$ .  $\square$

The first claim of the following lemma has been proved in [15, Lemma 4.4]. The proof for the second claim is similar.

**Lemma 2.5.** Let  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy (A0) and let  $E \subset \mathbb{R}^n$  be bounded.

- (a) If  $\varphi$  satisfies (aInc) $_p$ , then  $L^{\varphi(\cdot)}(E) \hookrightarrow L^p(E)$ .
- (b) If  $\varphi$  satisfies (aDec) $_q$ , then  $L^q(E) \hookrightarrow L^{\varphi(\cdot)}(E)$ .

**Lemma 2.6.** Let  $\Omega \subset \mathbb{R}^n$ . Let  $\varphi \in \Phi_w(\Omega)$  satisfy (aInc) $_p$  and (aDec) $_q$ . Then

$$\|f\|_{\varphi(\cdot)} \leq \max \left\{ (L \varrho_{\varphi(\cdot)}(f))^{\frac{1}{q}}, (L \varrho_{\varphi(\cdot)}(f))^{\frac{1}{p}} \right\}$$

where  $L$  is the maximum of the constants from (aInc) and (aDec).

*Proof.* If  $\varrho_{\varphi(\cdot)}(f) = 0$ , then  $\|f\|_{\varphi(\cdot)} = 0$  and the claim holds.

Let  $\varrho_{\varphi(\cdot)}(f) > 0$  and assume first that  $L\varrho_{\varphi(\cdot)}(f) \leq 1$ . Then (aDec) gives that

$$\varphi\left(x, \frac{|f(x)|}{(L\varrho_{\varphi(\cdot)}(f))^{\frac{1}{q}}}\right) \leq L(L\varrho_{\varphi(\cdot)}(f))^{-1}\varphi(x, |f(x)|) = (\varrho_{\varphi(\cdot)}(f))^{-1}\varphi(x, |f(x)|).$$

Integrating over  $\Omega$ , we find that  $\varrho_{\varphi(\cdot)}(f/(L\varrho_{\varphi(\cdot)}(f))^{1/q}) \leq 1$ , which yields  $\|f\|_{\varphi(\cdot)} \leq (L\varrho_{\varphi(\cdot)}(f))^{\frac{1}{q}}$ . If  $L\varrho_{\varphi(\cdot)}(f) > 1$  we similarly use (aInc) to conclude that  $\|f\|_{\varphi(\cdot)} \leq (L\varrho_{\varphi(\cdot)}(f))^{\frac{1}{p}}$ . The claim follows from these two cases.  $\square$

### 3. SOBOLEV CAPACITY

We define a set of test-functions for the capacity of the set  $E$  by

$$S_{1,\varphi(\cdot)}(E) := \{f \in W^{1,\varphi(\cdot)}(\mathbb{R}^n) : f \geq 1 \text{ in an open set containing } E \text{ and } f \geq 0\}.$$

The *generalized Orlicz  $\varphi(\cdot)$ -capacity* of  $E$  is defined by

$$C_{\varphi(\cdot)}(E) := \inf_{f \in S_{1,\varphi(\cdot)}(E)} \int_{\mathbb{R}^n} \varphi(x, f) + \varphi(x, |\nabla f|) dx.$$

We now prove the following properties for the set function  $E \mapsto C_{\varphi(\cdot)}(E)$ . We emphasize that these do not require the convexity of  $\varphi$ .

**Proposition 3.1.** *Let  $\varphi \in \Phi_w(\mathbb{R}^n)$ .*

- (S1)  $C_{\varphi(\cdot)}(\emptyset) = 0$ .
- (S2) *If  $E_1 \subset E_2 \subset \mathbb{R}^n$ , then  $C_{\varphi(\cdot)}(E_1) \leq C_{\varphi(\cdot)}(E_2)$ .*
- (S3) *If  $E \subset \mathbb{R}^n$ , then  $C_{\varphi(\cdot)}(E) = \inf_{U \supset E \text{ open}} C_{\varphi(\cdot)}(U)$ .*
- (S4) *If  $E_1, E_2 \subset \mathbb{R}^n$ , then  $C_{\varphi(\cdot)}(E_1 \cup E_2) + C_{\varphi(\cdot)}(E_1 \cap E_2) \leq C_{\varphi(\cdot)}(E_1) + C_{\varphi(\cdot)}(E_2)$ .*
- (S5) *If  $K_1 \supset K_2 \supset \dots$  are compact, then  $\lim_{i \rightarrow \infty} C_{\varphi(\cdot)}(K_i) = C_{\varphi(\cdot)}\left(\bigcap_{i=1}^{\infty} K_i\right)$ .*

*Proof.* (S1) follows from testing with  $f = 0$ . Since a test-function for  $E_2$  is also a test-function for  $E_1$ , (S2) follows from the infimum in the definition of capacity.

If  $U \supset E$ , then  $C_{\varphi(\cdot)}(E) \leq C_{\varphi(\cdot)}(U)$  by (S2). Thus,

$$C_{\varphi(\cdot)}(E) \leq \inf_{U \supset E \text{ open}} C_{\varphi(\cdot)}(U).$$

For the opposite inequality, let  $\varepsilon > 0$  and choose  $f \in S_{1,\varphi(\cdot)}(E)$  such that

$$\varrho_{1,\varphi(\cdot)}(f) \leq C_{\varphi(\cdot)}(E) + \varepsilon.$$

Denote  $V := \text{int}\{f \geq 1\}$ . Then  $f \in S_{1,\varphi(\cdot)}(V)$ , and so

$$\inf_{U \supset E \text{ open}} C_{\varphi(\cdot)}(U) \leq C_{\varphi(\cdot)}(V) \leq \varrho_{1,\varphi(\cdot)}(f) \leq C_{\varphi(\cdot)}(E) + \varepsilon.$$

This implies (S3) as  $\varepsilon \rightarrow 0^+$ .

Let  $\varepsilon > 0$ . Choose  $f_j \in S_{1,\varphi(\cdot)}(E_j)$  such that

$$\varrho_{1,\varphi(\cdot)}(f_j) \leq C_{\varphi(\cdot)}(E_j) + \varepsilon \quad \text{for } j \in \{1, 2\}.$$

Since  $\max\{f_1, f_2\} \geq 1$  in an open set containing  $E_1 \cup E_2$ , it follows that  $\max\{f_1, f_2\} \in S_{1,\varphi(\cdot)}(E_1 \cup E_2)$ . Similarly,  $\min\{f_1, f_2\} \in S_{1,\varphi(\cdot)}(E_1 \cap E_2)$ .

The lattice property of Sobolev functions [10, Proposition 8.1.9] implies that  $\nabla \max\{f_1, f_2\} = \nabla f_1$  and  $\nabla \min\{f_1, f_2\} = \nabla f_2$  almost everywhere in  $A = \{f_1 \geq f_2\}$ . Therefore

$$\begin{aligned} & \int_A \varphi(x, |\nabla \max\{f_1, f_2\}|) dx + \int_A \varphi(x, |\nabla \min\{f_1, f_2\}|) dx \\ &= \int_A \varphi(x, |\nabla f_1|) dx + \int_A \varphi(x, |\nabla f_2|) dx. \end{aligned}$$

The same argument also gives the equality in the complement of  $A$ , i.e.  $\{f_1 < f_2\}$ . Hence

$$\varrho_{\varphi(\cdot)}(\nabla \max\{f_1, f_2\}) + \varrho_{\varphi(\cdot)}(\nabla \min\{f_1, f_2\}) = \varrho_{\varphi(\cdot)}(\nabla f_1) + \varrho_{\varphi(\cdot)}(\nabla f_2)$$

almost everywhere. An analogous results holds without the derivative  $\nabla$ .

Since  $\max\{f_1, f_2\} \in S_{1, \varphi(\cdot)}(E_1 \cup E_2)$ , it follows from the definition that  $C_{\varphi(\cdot)}(E_1 \cup E_2) \leq \varrho_{1, \varphi(\cdot)}(\max\{f_1, f_2\})$ , and similarly  $C_{\varphi(\cdot)}(E_1 \cap E_2) \leq \varrho_{1, \varphi(\cdot)}(\min\{f_1, f_2\})$ . Combining this with the conclusion of the previous paragraph, we find that

$$\begin{aligned} C_{\varphi(\cdot)}(E_1 \cup E_2) + C_{\varphi(\cdot)}(E_1 \cap E_2) &\leq \varrho_{1, \varphi(\cdot)}(\max\{f_1, f_2\}) + \varrho_{1, \varphi(\cdot)}(\min\{f_1, f_2\}) \\ &= \varrho_{1, \varphi(\cdot)}(f_1) + \varrho_{1, \varphi(\cdot)}(f_2) \\ &\leq C_{\varphi(\cdot)}(E_1) + C_{\varphi(\cdot)}(E_2) + 2\varepsilon. \end{aligned}$$

(S4) follows from this as  $\varepsilon \rightarrow 0^+$ .

It remains to prove (S5). Since  $\bigcap_{i=1}^{\infty} K_i \subset K_j$  for any  $j$ , the “ $\geq$ ”-inequality follows from (S2). To prove the opposite inequality, we choose an open  $U \supset \bigcap_{i=1}^{\infty} K_i$ . Since  $\bigcap_{i=1}^{\infty} K_i$  is compact and  $(K_j)$  is decreasing, there is a positive integer  $k$  such that  $K_i \subset U$  for all  $i \geq k$ . Then, again by (S2), we have

$$\lim_{i \rightarrow \infty} C_{\varphi(\cdot)}(K_i) \leq C_{\varphi(\cdot)}(U).$$

Taking the infimum of this inequality over such sets  $U$ , we obtain the claim by (S3).  $\square$

Notice that we need convexity for the next property.

**Theorem 3.2.** *Let  $\varphi \in \Phi_c(\mathbb{R}^n)$  satisfy (aInc) and (aDec) and  $E_1 \subset E_2 \subset \dots \subset \mathbb{R}^n$ . Then*

$$(S6) \quad \lim_{i \rightarrow \infty} C_{\varphi(\cdot)}(E_i) = C_{\varphi(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right).$$

*Proof.* Let us denote  $E := \bigcup_{i=1}^{\infty} E_i$ . By (S2),  $\lim_{i \rightarrow \infty} C_{\varphi(\cdot)}(E_i) \leq C_{\varphi(\cdot)}(E)$ .

Now to prove the opposite inequality, we may assume that  $\lim_{i \rightarrow \infty} C_{\varphi(\cdot)}(E_i) < \infty$ . Let  $f_i \in S_{1, \varphi(\cdot)}(E_i)$  and  $\varrho_{1, \varphi(\cdot)}(f_i) \leq C_{\varphi(\cdot)}(E_i) + 2^{-i}$  for every  $i \in \mathbb{N}$ .

The space  $L^{\varphi(\cdot)}(\mathbb{R}^n)$  is uniformly convex and reflexive [14, Theorem 1.3]. The same holds for  $W^{1, \varphi(\cdot)}$ , which is a closed subspace of  $(L^{\varphi(\cdot)})^{n+1}$ . By reflexivity, the bounded sequence  $(f_i)$  has a subsequence which converges weakly to a function  $f \in W^{1, \varphi(\cdot)}(\mathbb{R}^n)$ . It follows from the Banach-Saks theorem that

$$\frac{1}{m} \sum_{i=1}^m f_i \rightarrow f \quad \text{in } W^{1, \varphi(\cdot)}(\mathbb{R}^n) \quad \text{as } m \rightarrow \infty.$$

Let  $g_j := \frac{1}{j(j-1)} \sum_{i=j+1}^{j^2} f_i$ . Then

$$\frac{1}{j^2} \sum_{i=1}^{j^2} f_i - g_j = \left( \frac{1}{j^2} - \frac{1}{j(j-1)} \right) \sum_{i=1}^{j^2} f_i + \frac{1}{j(j-1)} \sum_{i=1}^j f_i$$

from which we get, by the triangle inequality, that

$$\left\| \frac{1}{j^2} \sum_{i=1}^{j^2} f_i - g_j \right\|_{1, \varphi(\cdot)} \leq \frac{1}{j-1} \left\| \frac{1}{j^2} \sum_{i=1}^{j^2} f_i \right\|_{1, \varphi(\cdot)} + \frac{1}{j-1} \left\| \frac{1}{j} \sum_{i=1}^j f_i \right\|_{1, \varphi(\cdot)} \rightarrow 0$$

as  $j \rightarrow \infty$ , so that  $g_j \rightarrow f$  in  $W^{1, \varphi(\cdot)}(\mathbb{R}^n)$ .

By the convexity of the modular and the choice of  $g_j$ , we obtain that

$$\varrho_{1, \varphi(\cdot)}(g_j) \leq \frac{1}{j(j-1)} \sum_{i=j+1}^{j^2} \varrho_{1, \varphi(\cdot)}(f_i) \leq \sup_{i \geq j+1} \varrho_{1, \varphi(\cdot)}(f_i) \leq \sup_{i \geq j+1} (C_{\varphi(\cdot)}(E_i) + 2^{-i}).$$

Now  $C_{\varphi(\cdot)}(E_i)$  increases in  $i$  whereas  $2^{-i}$  decreases. Hence

$$(3.3) \quad \varrho_{1, \varphi(\cdot)}(g_j) \leq \lim_{i \rightarrow \infty} C_{\varphi(\cdot)}(E_i) + 2^{-j}.$$

By considering a subsequence if necessary, we may assume that  $\|g_{j+1} - g_j\|_{1, \varphi(\cdot)} \leq 2^{-j}$ . Then  $h_j := g_j + \sum_{i=j}^{\infty} |g_{i+1} - g_i| \in W^{1, \varphi(\cdot)}(\mathbb{R}^n)$ . By definition of a test-function, there exists an open set  $U_i \supset E_i$  such that  $f_i \geq 1$  in  $U_i$ . As  $E_j$  is an increasing sequence, it follows that  $E_j \subset U_i$  for every  $j \leq i$ . In  $\bigcap_{i=j+1}^{j^2} U_j$ ,  $f_i \geq 1$  for all relevant  $i$ , so that  $g_j \geq 1$  in the same (open) set. Since  $h_j \geq \sup_{i \geq j} g_j$ , we have

$$E_i \subset \text{int}\{g_i \geq 1\} \subset \text{int}\{h_j \geq 1\}.$$

Taking the union over  $i$  of the previous inclusion, we find that  $h_j \geq 1$  in the open set  $\bigcup_{i=j}^{\infty} \text{int}\{g_i \geq 1\} \supset E$ , so that  $h_j \in S_{1, \varphi(\cdot)}(E)$ . Hence

$$(3.4) \quad C_{\varphi(\cdot)}(E) \leq \varrho_{1, \varphi(\cdot)}(h_j) \quad \text{for } j = 1, 2, \dots$$

Furthermore,

$$\|h_j - g_j\|_{1, \varphi(\cdot)} \leq \sum_{i=j}^{\infty} \|g_{i+1} - g_i\|_{1, \varphi(\cdot)} \leq \sum_{i=j}^{\infty} 2^{-i} = 2^{-j+1}$$

and, hence, by [10, Corollary 2.1.15],  $\varrho_{1, \varphi(\cdot)}(h_j - g_j) \leq \|h_j - g_j\|_{1, \varphi(\cdot)} \rightarrow 0$  as  $j \rightarrow \infty$ . Then, using Lemma 2.4, we also have  $|\varrho_{1, \varphi(\cdot)}(h_j) - \varrho_{1, \varphi(\cdot)}(g_j)| \rightarrow 0$  as  $j \rightarrow \infty$ , which we apply in (3.4) to obtain

$$C_{\varphi(\cdot)}(E) \leq \lim_{j \rightarrow \infty} \varrho_{1, \varphi(\cdot)}(h_j) = \lim_{j \rightarrow \infty} \varrho_{1, \varphi(\cdot)}(g_j)$$

and further (3.3) implies

$$C_{\varphi(\cdot)}(E) \leq \lim_{j \rightarrow \infty} \varrho_{1, \varphi(\cdot)}(g_j) \leq \lim_{j \rightarrow \infty} \left( \lim_{i \rightarrow \infty} C_{\varphi(\cdot)}(E_i) + 2^{-j} \right) = \lim_{i \rightarrow \infty} C_{\varphi(\cdot)}(E_i). \quad \square$$

In the next result we use a trick to get back to weak  $\Phi$ -functions despite an application of (S6).

**Theorem 3.5.** *Let  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy (aInc) and (aDec) and  $E_1, E_2, \dots \subset \mathbb{R}^n$ . Then*

$$(S7) \quad C_{\varphi(\cdot)} \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} C_{\varphi(\cdot)}(E_i).$$

*Proof.* Let  $\psi \in \Phi_c(\mathbb{R}^n)$  with  $\varphi \simeq \psi$  [17, Lemma 3.1]. Since  $\varphi$  satisfies (aDec)  $\varphi \simeq \psi$  yields  $\varphi \approx \psi$ . Denote  $F_k^m := \bigcup_{i=m}^k E_i$  for  $k = m, m+1, \dots$ . By induction on (S4), we obtain that

$$C_{\varphi(\cdot)}(F_k^m) = C_{\varphi(\cdot)}(\bigcup_{i=m}^k E_i) \leq \sum_{i=m}^k C_{\varphi(\cdot)}(E_i) \leq \sum_{i=m}^{\infty} C_{\varphi(\cdot)}(E_i).$$

The same inequality holds also for  $\psi$ . Now, using (S6) for  $(F_k)$ , we have

$$C_{\psi(\cdot)}(\cup_{k=m}^{\infty} E_k) = C_{\psi(\cdot)}(\cup_{k=m}^{\infty} F_k^m) = \lim_{k \rightarrow \infty} C_{\psi(\cdot)}(F_k^m) \leq \sum_{i=m}^{\infty} C_{\psi(\cdot)}(E_i)$$

Furthermore, for  $k = 1, 2, \dots$ , by (S4) and  $\varphi \approx \psi$ ,

$$C_{\varphi(\cdot)}(\cup_{j=1}^{\infty} E_j) \leq C_{\varphi(\cdot)}(F_k^1) + C_{\varphi(\cdot)}(\cup_{j=k+1}^{\infty} E_j) \leq C_{\varphi(\cdot)}(F_k^1) + c C_{\psi(\cdot)}(\cup_{j=k+1}^{\infty} E_j).$$

By the two estimates in the previous paragraph, we obtain that

$$C_{\varphi(\cdot)}(\cup_{j=1}^{\infty} E_j) \leq \sum_{i=1}^{\infty} C_{\varphi(\cdot)}(E_i) + c \sum_{i=k}^{\infty} C_{\psi(\cdot)}(E_i).$$

Since the sum is finite (otherwise, there is nothing to prove), the second term on the right hand side tends to zero as  $k \rightarrow \infty$ . This gives the claim.  $\square$

*Remark 3.6.* A set function satisfying the properties (S1), (S2) and (S7) is called an *outer measure*. This holds if  $\varphi \in \Phi_w(\Omega)$  satisfies (aInc) and (aDec). If  $\varphi$  is convex and satisfies (aInc) and (aDec), then it is a *Choquet capacity*, [6], i.e. it satisfies (S1), (S2), (S5) and (S6). Then, for every Borel set  $E \subset \Omega$ ,

$$C_{\varphi(\cdot)}(E) = \sup \{ C_{\varphi(\cdot)}(K) : K \text{ is compact and } K \subset E \}.$$

#### 4. SOBOLEV CAPACITY AND HAUSDORFF MEASURE

In this section, we discuss simple relations between the generalized Orlicz capacity and the Lebesgue and Hausdorff measures.

**Lemma 4.1.** *Let  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy (aDec) and (A0). Then every measurable set  $E \subset \mathbb{R}^n$  satisfies  $|E| \leq c C_{\varphi(\cdot)}(E)$ , where  $c > 1$  depends on (aDec) and (A0).*

*Proof.* Let  $f \in S_{1,\varphi(\cdot)}(E)$ . By (aDec), with exponent  $q$  and constant  $L$ , we conclude that  $\varphi(x, \frac{f(x)}{\beta}) \leq L\beta^{-q}\varphi(x, f(x))$ . We have  $\varphi(x, \frac{f(x)}{\beta}) \geq 1$  for  $f(x) \geq 1$  by (A0). Therefore

$$|E| \leq \int_{\mathbb{R}^n} \varphi\left(x, \frac{f}{\beta}\right) dx \leq L\beta^{-q} \int_{\mathbb{R}^n} \varphi(x, f) dx \leq L\beta^{-q} \varrho_{1,\varphi(\cdot)}(f).$$

Taking infimum over  $f \in S_{1,\varphi(\cdot)}(E)$ , we obtain the claim.  $\square$

**Proposition 4.2.** *Let  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy (A0),  $(aInc)_p$ ,  $p > 1$ , and (aDec). If  $C_{\varphi(\cdot)}(E) = 0$ , then  $C_p(E) = 0$ .*

*Proof.* Let  $B = B(0, R+1)$  and  $\eta \in C_0^\infty(B)$  be a cut-off function with  $\eta = 1$  in  $B(0, R)$ ,  $0 \leq \eta \leq 1$  and  $|\nabla\eta| \leq 2$ . Let  $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ . By Lemma 2.5, we obtain

$$\|f\eta\|_{L^p(\mathbb{R}^n)} = \|f\eta\|_{L^p(B)} \leq c\|\eta f\|_{L^{\varphi(\cdot)}(B)} \leq c\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}$$

where  $c$  depends on (aInc) and  $|B|$ . The same inequality holds for  $|\nabla f|$ . Now  $\nabla(f\eta) = \eta\nabla f + f\nabla\eta$ , and, thus  $|\nabla(f\eta)| \leq |\nabla f| + 2|f|$  by the assumptions on  $\eta$ . Since  $\|f\|_{W^{1,\varphi(\cdot)}(\mathbb{R}^n)} \approx \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}$  we obtain that

$$\|f\eta\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|f\eta\|_{L^p(\mathbb{R}^n)} + \|\nabla(f\eta)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{1,\varphi(\cdot)}(\mathbb{R}^n)}.$$

Since  $C_{\varphi(\cdot)}(E) = 0$ , we can choose  $f_i \in S_{1,\varphi(\cdot)}(E \cap B(0, R))$  with  $\varrho_{1,\varphi(\cdot)}(f_i) \rightarrow 0$ . Since  $\varphi$  satisfies (aDec),  $\|f_i\|_{W^{1,\varphi(\cdot)}(\mathbb{R}^n)} \rightarrow 0$  [10, Lemma 2.1.11]. Thus the above inequality implies that  $\|f_i\eta\|_{W^{1,p}(\mathbb{R}^n)} \rightarrow 0$ . Since  $f_i\eta$  is a test-function for the  $p$ -capacity of  $E \cap B(0, R)$ , we get  $C_p(E \cap B(0, R)) = 0$ , for every  $R > 0$ . Since  $E = \cup_{i=1}^{\infty} (E \cap B(0, i))$ , the claim follows by the subadditivity of the  $p$ -capacity.  $\square$

In the previous result the assumption  $(aInc)_p$  is natural, since  $p$  gives the capacity to compare with. However, the assumption  $(aDec)$  is surprising. The next example shows that it is nevertheless needed.

**Example 4.3.** Let  $\varphi(t) = 0$  when  $t \in [0, 11/10]$  and  $\varphi(t) = t - 11/10$  when  $t > 11/10$ . Then  $\varphi$  satisfies  $(aInc)$  with  $p = 1$  and  $(A0)$  with  $\beta = 10/21$ . Let  $B$  be an open ball with a radius one. Let  $f$  be a Lipschitz-continuous function that is one in  $B$ , zero in  $\mathbb{R}^n \setminus 2B$  and linear in  $2B \setminus B$ . Then  $|\nabla f| = 1$  in  $2B \setminus B$  and zero elsewhere. We obtain  $\varrho_{1,\varphi}(f) = 0$  and thus  $C_{\varphi(\cdot)}(B) = 0$ . On the other hand  $C_p(B) > 0$ .

The  $s$ -dimensional Hausdorff measure of a set  $E \subset \mathbb{R}^n$  is denoted by  $\mathcal{H}^s(E)$ .

**Corollary 4.4.** Let  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy  $(A0)$ ,  $(aInc)_p$  with  $p > 1$  and  $(aDec)$ .

- (1) If  $p \leq n$  and  $E \subset \mathbb{R}^n$  with  $C_{\varphi(\cdot)}(E) = 0$ , then  $\mathcal{H}^s(E) = 0$  for all  $s > n - p$ .
- (2) If  $p > n$ , then  $C_{\varphi(\cdot)}(E) = 0$  if and only if  $E = \emptyset$ , where  $E \subset \mathbb{R}^n$ .

*Proof.* If  $C_{\varphi(\cdot)}(E) = 0$ , we then obtain  $C_p(E) = 0$  by Proposition 4.2. This leads to the first claim by [11, Theorem 4, p. 156].

If  $p > n$ , then we may choose  $s = 0$  in (1) and so  $\mathcal{H}^0(E) = 0$ . Since  $\mathcal{H}^0$  is a counting measure,  $E$  must be an empty set. On the other hand,  $C_{\varphi(\cdot)}(\emptyset) = 0$ , by (S1).  $\square$

**Corollary 4.5.** Let  $\psi \in \Phi_w(\mathbb{R}^n)$  satisfy  $(A0)$ ,  $(aInc)$  and  $(aDec)_q$ . Let  $E \subset \mathbb{R}^n$  be bounded. If  $C_q(E) = 0$  or  $\mathcal{H}^{n-q}(E) < \infty$ , then  $C_{\psi(\cdot)}(E) = 0$ .

*Proof.* Let  $C_q(E) = 0$ . By Lemma 2.5,  $L^q(E) \hookrightarrow L^{\psi(\cdot)}(E)$ . The remaining proof follows the same procedure as in the proof of Proposition 4.2. If  $\mathcal{H}^{n-q}(E) < \infty$ , it follows from [11, Theorem 3, p. 154] that  $C_q(E) = 0$  and thus the claim follows from the first part.  $\square$

## 5. QUASICONTINUITY

In this section, we prove the existence of  $\varphi(\cdot)$ -quasicontinuous representatives of generalized Orlicz functions. A function  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$  is  $\varphi(\cdot)$ -quasicontinuous if for every  $\varepsilon > 0$  there exists an open set  $U$  with  $C_{\varphi(\cdot)}(U) < \varepsilon$  such that  $f$  restricted to  $\mathbb{R}^n \setminus U$  is continuous. We say that a claim holds  $\varphi(\cdot)$ -quasieverywhere if it holds except in a set of Sobolev  $\varphi(\cdot)$ -capacity zero.

In this section we assume the density of continuous functions. A sufficient condition for this can be found in Theorem 6.5 of [15].

**Lemma 5.1.** Let  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy  $(aInc)$  and  $(aDec)$ . Then, for each Cauchy sequence  $C(\mathbb{R}^n) \cap W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ , there is a subsequence which converges pointwise  $\varphi(\cdot)$ -quasieverywhere in  $\mathbb{R}^n$ . Moreover, the convergence is uniform outside a set of arbitrarily small Sobolev  $\varphi(\cdot)$ -capacity.

*Proof.* Let  $(f_i)$  be a Cauchy sequence in  $C(\mathbb{R}^n) \cap W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ . We assume without loss of generality, by considering a subsequence if necessary, that  $\|f_i - f_{i+1}\|_{1,\varphi(\cdot)} \leq 4^{-i}$  for every  $i = 1, 2, \dots$ . For  $i \in \mathbb{N}$  we denote  $g_i := 2^i |f_i - f_{i+1}| \in W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ ,  $E_i := \{g_i > 1\}$  and  $F_j := \cup_{i=j}^{\infty} E_i$ . Note that  $\|g_i\|_{1,\varphi(\cdot)} \leq 2^{-i}$ . We obtain by Corollary 2.1.15(a) of [10] that

$$C_{\varphi(\cdot)}(E_i) \leq \varrho_{1,\varphi(\cdot)}(g_i) \leq \|g_i\|_{1,\varphi(\cdot)} \leq 2^{-i}.$$

The subadditivity (S7) further implies,

$$C_{\varphi(\cdot)}(F_j) \leq \sum_{i=j}^{\infty} C_{\varphi(\cdot)}(E_i) \leq \sum_{i=j}^{\infty} 2^{-i} = 2^{1-j}.$$

Since  $(F_j)_{j=1}^\infty$  is decreasing and  $C_{\varphi(\cdot)}(F_j) \geq 0$ , the limit  $\lim_{j \rightarrow \infty} C_{\varphi(\cdot)}(F_j)$  exists. Since  $\bigcap_{j=1}^\infty F_j \subset F_k$  for all  $k = 1, 2, \dots$ , we obtain by (S2) that

$$C_{\varphi(\cdot)}(\bigcap_{j=1}^\infty F_j) \leq \lim_{j \rightarrow \infty} C_{\varphi(\cdot)}(F_j) \leq \lim_{j \rightarrow \infty} 2^{1-j} = 0.$$

Now  $(f_j)$  converges pointwise in  $\mathbb{R}^n \setminus \bigcap_{j=1}^\infty F_j$  and  $C_{\varphi(\cdot)}(\bigcap_{j=1}^\infty F_j) = 0$ , so the first claim of the lemma is proved. Moreover, for  $x \in \mathbb{R}^n \setminus F_j$  and  $k > l > j$ ,

$$|f_l(x) - f_k(x)| \leq \sum_{i=l}^{k-1} |f_i(x) - f_{i+1}(x)| \leq \sum_{i=l}^{k-1} 2^{-i} < 2^{1-l}.$$

Therefore, the convergence is uniform in  $\mathbb{R}^n \setminus F_j$ , and the second claim follows.  $\square$

The existence of the so-called  $\varphi(\cdot)$ -quasicontinuous representative follows from Lemma 5.1 by standard arguments (e.g. [10, Theorem 11.1.3]):

**Theorem 5.2.** *Let  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy (aInc) and (aDec). Assume that  $C(\mathbb{R}^n) \cap W^{1,\varphi(\cdot)}(\mathbb{R}^n)$  is dense in  $W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ . Then for each  $f \in W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ , there exists a  $\varphi(\cdot)$ -quasicontinuous function  $g \in W^{1,\varphi(\cdot)}(\mathbb{R}^n)$  such that  $f = g$  almost everywhere in  $\mathbb{R}^n$ .*

**Lemma 5.3.** *Let  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy (aDec). Suppose that  $f \in W^{1,\varphi(\cdot)}(\mathbb{R}^n)$  is a non-negative  $\varphi(\cdot)$ -quasicontinuous function with  $f \geq 1$  in  $E \subset \mathbb{R}^n$ . Then for every  $\varepsilon > 0$  there exists  $g \in S_{1,\varphi(\cdot)}(E)$  such that  $\varrho_{1,\varphi(\cdot)}(f - g) < \varepsilon$ .*

*Proof.* Let  $\varepsilon \in (0, 1)$  and  $\delta := \frac{\varepsilon}{2L2^q(1+L\varrho_{1,\varphi(\cdot)}(f))}$ , and let  $U \subset \mathbb{R}^n$  be an open set such that  $f$  restricted to  $\mathbb{R}^n \setminus U$  is continuous and  $C_{\varphi(\cdot)}(U) < \delta$ . Moreover, let us take  $h \in S_{1,\varphi(\cdot)}(U)$  such that  $\varrho_{1,\varphi(\cdot)}(h) < \delta$ , and denote  $g := (1 + \delta)f + h$ . Then  $g \in W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ . The set  $V := \{x \in \mathbb{R}^n \setminus U : (1 + \delta)f(x) > 1\} \cup U$  is open and contains  $E$ . Since  $g \geq 1$  in  $V$ ,  $g \in S_{1,\varphi(\cdot)}(E)$ . It remains to estimate  $\varrho_{1,\varphi(\cdot)}(f - g)$ .

Now,  $\varphi(x, h + \delta f) \leq \varphi(x, 2h) + \varphi(x, 2\delta f)$ . By (aDec)<sub>p</sub> and (aInc)<sub>1</sub> we have  $\varphi(x, 2\delta f) \leq L2^q\varphi(x, \delta f) \leq L^22^q\delta\varphi(x, f)$ , and so  $\varphi(x, h + \delta f) \leq L2^q(\varphi(x, h) + L\delta\varphi(x, f))$ . Therefore

$$\varrho_{\varphi(\cdot)}(f - g) = \varrho_{\varphi(\cdot)}(h + \delta f) \leq L2^q(\varrho_{\varphi(\cdot)}(h) + L\delta\varrho_{\varphi(\cdot)}(f)) < L2^q(1 + L\varrho_{\varphi(\cdot)}(f))\delta = \frac{1}{2}\varepsilon$$

by the assumption on  $\delta$ . An analogous inequality holds for the gradient, and so the claim follows.  $\square$

## 6. RELATIVE CAPACITY

In this section, we introduce relative  $\varphi(\cdot)$ -capacity, analogous to the relative  $p(\cdot)$ -capacity of variable exponent spaces in [10], taken with respect to an open set,  $\Omega$ , in  $\mathbb{R}^n$ .

**Definition 6.1.** Let  $\Omega \subset \mathbb{R}^n$ ,  $K \subset \Omega$  be compact, and  $\varphi \in \Phi_w(\Omega)$ . Denote

$$R_{\varphi(\cdot)}(K, \Omega) := \{f \in W^{1,\varphi(\cdot)}(\Omega) \cap C_0(\Omega) : f > 1 \text{ in } K \text{ and } f \geq 0\}.$$

We define  $\text{cap}_{\varphi(\cdot)}^*(K, \Omega) := \inf_{f \in R_{\varphi(\cdot)}(K, \Omega)} \varrho_{\varphi(\cdot)}(|\nabla f|)$ . Further, for  $U \subset \Omega$  open, we set

$$\text{cap}_{\varphi(\cdot)}(U, \Omega) := \sup_{K \subset U \text{ compact}} \text{cap}_{\varphi(\cdot)}^*(K, \Omega),$$

and, for an arbitrary set  $E \subset \Omega$ , we define

$$\text{cap}_{\varphi(\cdot)}(E, \Omega) := \inf_{U \supset E \text{ open}} \text{cap}_{\varphi(\cdot)}(U, \Omega).$$

The number  $\text{cap}_{\varphi(\cdot)}(E, \Omega)$  is called the *relative  $\varphi(\cdot)$ -capacity* of  $E$  with respect to  $\Omega$ .

In the next result we offer two alternate set of assumptions. It seems that (aDec) alone is not sufficient, although we have not been able to prove this.

**Proposition 6.2.** *If  $\varphi \in \Phi_w(\Omega)$  satisfies either (Dec) or (A0), then*

$$\text{cap}_{\varphi(\cdot)}^*(K, \Omega) = \inf_{f \in \tilde{R}_{\varphi(\cdot)}(K, \Omega)} \int_{\Omega} \varphi(x, |\nabla f|) dx,$$

where  $\tilde{R}_{\varphi(\cdot)}(K, \Omega) := \{f \in W^{1, \varphi(\cdot)}(\Omega) \cap C_0(\Omega) : f \geq 1 \text{ in } K \text{ and } f \geq 0\}$ .

*Proof.* Since  $R_{\varphi(\cdot)}(K, \Omega) \subset \tilde{R}_{\varphi(\cdot)}(K, \Omega)$ , the inequality “ $\geq$ ” is clear. Now, to prove the opposite inequality, we fix  $\varepsilon > 0$  and let  $f \in \tilde{R}_{\varphi(\cdot)}(K, \Omega)$ , be such that

$$\varrho_{\varphi(\cdot)}(|\nabla f|) \leq \inf_{g \in \tilde{R}_{\varphi(\cdot)}(K, \Omega)} \varrho_{\varphi(\cdot)}(|\nabla g|) + \varepsilon.$$

If (Dec) holds, we set  $g := (1 + \varepsilon)f$ . Then  $g > 1$  in  $K$ , so that

$$\text{cap}_{\varphi(\cdot)}^*(K, \Omega) \leq \varrho_{\varphi(\cdot)}(\nabla g) = \int_{\Omega} \varphi(x, (1 + \varepsilon)\nabla f) dx \leq (1 + \varepsilon)^q \int_{\Omega} \varphi(x, |\nabla f|) dx,$$

where we used (Dec) for the last inequality. The claim follows as  $\varepsilon \rightarrow 0^+$ .

If, on the other hand, (A0) holds, then we fix  $\eta \in C_0^\infty(\Omega)$  with  $\eta = 1$  in  $\text{supp } f$ . Let  $\lambda = \beta \|\nabla \eta\|_\infty^{-1}$ . We set  $g := f + \varepsilon' \lambda \eta \in R_{\varphi(\cdot)}(K, \Omega)$ . Since the supports of  $\nabla f$  and  $\nabla \eta$  are disjoint, we observe that

$$\varrho_{\varphi(\cdot)}(\nabla g) = \varrho_{\varphi(\cdot)}(\nabla f) + \varrho_{\varphi(\cdot)}(\varepsilon' \lambda \nabla \eta) \leq \varrho_{\varphi(\cdot)}(\nabla f) + L \varepsilon' \varrho_{\varphi(\cdot)}(\lambda \nabla \eta),$$

where (aInc)<sub>1</sub> has been used in the last inequality. The claim follows as  $\varepsilon, \varepsilon' \rightarrow 0^+$ .  $\square$

Next, we show that  $\text{cap}_{\varphi(\cdot)}^*(K, \Omega)$  and  $\text{cap}_{\varphi(\cdot)}(K, \Omega)$  are the same, that is, the relative capacity is well defined on compact sets.

**Proposition 6.3.** *Let  $\varphi \in \Phi_w(\Omega)$ . Then  $\text{cap}_{\varphi(\cdot)}^*(K, \Omega) = \text{cap}_{\varphi(\cdot)}(K, \Omega)$  for every compact  $K \subset \Omega$ .*

*Proof.* The inequality  $\text{cap}_{\varphi(\cdot)}^*(K, \Omega) \leq \text{cap}_{\varphi(\cdot)}(K, \Omega)$  follows directly from the definition.

Now, to prove the opposite inequality, fix  $\varepsilon > 0$  and let  $f \in R_{\varphi(\cdot)}(K, \Omega)$  be such that  $\varrho_{\varphi(\cdot)}(\nabla f) \leq \text{cap}_{\varphi(\cdot)}^*(K, \Omega) + \varepsilon$ . Then,  $f$  is greater than one in  $U := f^{-1}(1, \infty)$ , which is open since  $f \in C_0(\Omega)$ , and contains  $K$ . Thus,  $f$  is also a valid test-function for every compact set  $K' \subset U$ , so that

$$\text{cap}_{\varphi(\cdot)}(U, \Omega) = \sup_{K' \subset U} \text{cap}_{\varphi(\cdot)}^*(K', \Omega) \leq \varrho_{\varphi(\cdot)}(\nabla f) \leq \text{cap}_{\varphi(\cdot)}^*(K, \Omega) + \varepsilon.$$

The result follows from this as  $\varepsilon \rightarrow 0^+$ .  $\square$

Next, we show that the relative capacity has the same basic properties as the Sobolev capacity.

**Proposition 6.4.** *Let  $\Omega \subset \mathbb{R}^n$  be open and  $\varphi \in \Phi_w(\mathbb{R}^n)$ .*

(R1)  $\text{cap}_{\varphi(\cdot)}(\emptyset, \Omega) = 0$ .

(R2) *If  $E \subset E' \subset \Omega' \subset \Omega$ , then  $\text{cap}_{\varphi(\cdot)}(E, \Omega) \leq \text{cap}_{\varphi(\cdot)}(E', \Omega')$ .*

(R3) *If  $E \subset \Omega$ , then  $\text{cap}_{\varphi(\cdot)}(E, \Omega) = \inf_{U \supset E \text{ open}} \text{cap}_{\varphi(\cdot)}(U, \Omega)$ .*

(R4) *If  $E, F \subset \Omega$ , then*

$$\text{cap}_{\varphi(\cdot)}(E \cup F, \Omega) + \text{cap}_{\varphi(\cdot)}(E \cap F, \Omega) \leq \text{cap}_{\varphi(\cdot)}(E, \Omega) + \text{cap}_{\varphi(\cdot)}(F, \Omega).$$

(R5) *If  $K_1 \supset K_2 \supset \dots$  are compact, then  $\lim_{i \rightarrow \infty} \text{cap}_{\varphi(\cdot)}(K_i, \Omega) = \text{cap}_{\varphi(\cdot)}\left(\bigcap_{i=1}^{\infty} K_i, \Omega\right)$ .*

*Proof.* Properties (R1) and (R3) follow immediately from the definition. For proving property (R2), we observe that if  $E' \subset U \subset \Omega'$  then also  $E \subset U \subset \Omega$ . Thus

$$\text{cap}_{\varphi(\cdot)}(E', \Omega') = \inf_{E' \subset U \subset \Omega'} \text{cap}_{\varphi(\cdot)}(U, \Omega') \geq \text{cap}_{\varphi(\cdot)}(E, \Omega).$$

Properties (R4) and (R5) are proved exactly like (S4) and (S5).  $\square$

The proof of the following results follows from (R4) by standard arguments, see, e.g., [10, Lemma 10.2.5].

**Lemma 6.5.** *Suppose that  $\varphi \in \Phi_w(\Omega)$  and  $E_1, E_2, \dots, E_k \subset \Omega$ . Then,*

$$\text{cap}_{\varphi(\cdot)}\left(\bigcup_{i=1}^k E_i, \Omega\right) - \text{cap}_{\varphi(\cdot)}\left(\bigcup_{i=1}^k A_i, \Omega\right) \leq \sum_{i=1}^k \left(\text{cap}_{\varphi(\cdot)}(E_i, \Omega) - \text{cap}_{\varphi(\cdot)}(A_i, \Omega)\right),$$

whenever  $A_i \subset E_i, i = 1, 2, \dots, k$  and  $\text{cap}_{\varphi(\cdot)}\left(\bigcup_{i=1}^k E_i, \Omega\right) < \infty$ .

Furthermore, the previous lemma implies directly properties (R6) and (R7), see, e.g., [10, Theorem 10.2.6]. Note that these properties hold for the relative capacity without any extra assumptions on  $\varphi$ .

**Theorem 6.6.** *Let  $\varphi \in \Phi_w(\Omega)$ .*

(R6) *If  $E_1 \subset E_2 \subset \dots \subset \Omega$ , then  $\lim_{i \rightarrow \infty} \text{cap}_{\varphi(\cdot)}(E_i, \Omega) = \text{cap}_{\varphi(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i, \Omega\right)$ .*

(R7) *If  $E_i \subset \Omega, i = 1, 2, \dots$ , then  $\text{cap}_{\varphi(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i, \Omega\right) \leq \sum_{i=1}^{\infty} \text{cap}_{\varphi(\cdot)}(E_i, \Omega)$ .*

*Remark 6.7.* A set function satisfying the properties (R1), (R2), (R5) and (R6) is called a *Choquet capacity* [6]. They imply for every Borel set  $E \subset \Omega$  that

$$\text{cap}_{\varphi(\cdot)}(E, \Omega) = \sup \left\{ \text{cap}_{\varphi(\cdot)}(K, \Omega) : K \text{ is compact and } K \subset E \right\}.$$

## 7. RELATIONSHIP BETWEEN CAPACITIES

**Lemma 7.1.** *Assume that  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfies (A0) and  $(aDec)_q$ . If  $\Omega$  is bounded and  $K \subset \Omega$  is compact, then,*

$$C_{\varphi(\cdot)}(K) \leq C \max \left\{ \text{cap}_{\varphi(\cdot)}(K, \Omega)^{\frac{1}{q}}, \text{cap}_{\varphi(\cdot)}(K, \Omega) \right\},$$

where the constant  $C$  depends on the dimension,  $|\Omega|$  and the constants in (A0) and  $(aDec)$ .

*Proof.* We may assume that  $\text{cap}_{\varphi(\cdot)}(K, \Omega) < \infty$ . Let  $f \in R_{\varphi(\cdot)}(K, \Omega)$  with  $\varrho_{\varphi(\cdot)}(|\nabla f|) < 2 \text{cap}_{\varphi(\cdot)}(K, \Omega)$ . Extend  $f$  by zero outside of  $\Omega$  and set  $g := \min\{1, f\}$ . Since  $f \in C_0(\Omega)$  and  $f > 1$  in the compact set  $K$ ,  $U = \{f > 1\} \supset K$  is open. Hence  $g \in S_{1, \varphi(\cdot)}(K)$  and so

$$(7.2) \quad C_{\varphi(\cdot)}(K) \leq \int_{\mathbb{R}^n} \varphi(x, g) + \varphi(x, |\nabla g|) dx \leq \int_{\mathbb{R}^n} \varphi(x, g) + \varphi(x, |\nabla f|) dx.$$

Using  $0 \leq g \leq 1$ ,  $(aInc)_1$ ,  $(aDec)_q$  and (A0), we obtain

$$\varphi(x, g(x)) \leq Lg(x)\varphi(x, 1) \leq L^2g(x)\beta^{-q}\varphi(x, \beta) \leq L^2g(x)\beta^{-q}.$$

Integrating over the bounded set  $\Omega$  and using the classical Poincaré inequality, we find that

$$\int_{\Omega} \varphi(x, g) dx \leq C \int_{\Omega} g dx \leq C \int_{\Omega} f dx \leq C \int_{\Omega} |\nabla f| dx.$$

Then the embedding  $L^{\varphi(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$  (Lemma 2.5) and Lemma 2.6 for the function  $|\nabla f|$  give that

$$\int_{\Omega} \varphi(x, g) dx \leq C \max\{\varrho_{\varphi(\cdot)}(|\nabla f|)^{\frac{1}{q}}, \varrho_{\varphi(\cdot)}(|\nabla f|)\}.$$

Combining this with (7.2) and taking the infimum over  $f$ , we obtain the claim.  $\square$

In the next proof the Choquet property for Sobolev capacity is needed and hence we assume that  $\varphi$  is convex.

**Theorem 7.3.** *Assume that  $\varphi \in \Phi_c(\mathbb{R}^n)$  satisfies (A0) and (aDec) $_q$ . If  $\Omega$  is bounded and  $E \subset \Omega$ , then*

$$C_{\varphi(\cdot)}(E) \leq C \max\{\text{cap}_{\varphi(\cdot)}(E, \Omega)^{\frac{1}{q}}, \text{cap}_{\varphi(\cdot)}(E, \Omega)\},$$

where the constant  $C$  depends on the dimension,  $|\Omega|$ , and the constants in (A0) and (aDec).

*Proof.* We may assume that  $\text{cap}_{\varphi(\cdot)}(E, \Omega) < \infty$ . By the definition of  $\text{cap}_{\varphi(\cdot)}(E, \Omega)$ , there exist open sets  $U_i \supset E$  with  $\text{cap}_{\varphi(\cdot)}(U_i, \Omega) \rightarrow \text{cap}_{\varphi(\cdot)}(E, \Omega)$ , as  $i \rightarrow \infty$ . Let  $U := \bigcap_{i=1}^{\infty} U_i$ . Then  $U$  is a Borel set, and hence by the Choquet property (Remark 3.6),

$$C_{\varphi(\cdot)}(E) \leq C_{\varphi(\cdot)}(U) = \sup_{K \subset U} C_{\varphi(\cdot)}(K),$$

where the supremum is taken over all compact sets  $K \subset U$ . By Lemma 7.1, we obtain

$$\begin{aligned} C_{\varphi(\cdot)}(E) &\leq \sup_K C_{\varphi(\cdot)}(K) \leq C \sup_{K \subset U} \max\{\text{cap}_{\varphi(\cdot)}(K, \Omega)^{\frac{1}{q}}, \text{cap}_{\varphi(\cdot)}(K, \Omega)\} \\ &\leq C \max\{\text{cap}_{\varphi(\cdot)}(U_i, \Omega)^{\frac{1}{q}}, \text{cap}_{\varphi(\cdot)}(U_i, \Omega)\}. \end{aligned}$$

The claim follows from this as  $i \rightarrow \infty$ .  $\square$

From the above result, we can conclude that  $C_{\varphi(\cdot)}(E) = 0$  if  $\text{cap}_{\varphi(\cdot)}(E, \Omega) = 0$ . To get the converse implication, we first prove the following results.

**Lemma 7.4.** *Let  $\varphi \in \Phi_w(\Omega)$  satisfy (aDec). Assume that  $C(\mathbb{R}^n) \cap W^{1, \varphi(\cdot)}(\mathbb{R}^n)$  is dense in  $W^{1, \varphi(\cdot)}(\mathbb{R}^n)$ . If  $K \subset \mathbb{R}^n$  is compact, then*

$$C_{\varphi(\cdot)}(K) = \inf_{f \in S_{1, \varphi(\cdot)}^c(K)} \int_{\mathbb{R}^n} \varphi(x, f) + \varphi(x, |\nabla f|) dx,$$

where  $S_{1, \varphi(\cdot)}^c(K) = S_{1, \varphi(\cdot)}(K) \cap C(\mathbb{R}^n)$ .

*Proof.* Since  $C_{\varphi(\cdot)}(K)$  is defined as the infimum over the larger set  $S_{1, \varphi(\cdot)}(K)$ , the inequality “ $\leq$ ” is clear.

Suppose  $f \in S_{1, \varphi(\cdot)}(K)$ , with  $0 \leq f \leq 1$  and choose functions  $f_j \in C(\mathbb{R}^n) \cap W^{1, \varphi(\cdot)}(\mathbb{R}^n)$  converging to  $f$  in  $W^{1, \varphi(\cdot)}(\mathbb{R}^n)$ , with  $0 \leq f_j \leq 1$ . Choose a bounded neighborhood  $U$  of  $K$  such that  $f = 1$  in  $U$ . Let  $\eta \in C^1(\mathbb{R}^n)$ ,  $0 \leq \eta \leq 1$  with  $\eta = 1$  in  $\mathbb{R}^n \setminus U$  and  $\eta = 0$  in a neighborhood of  $K$ .

Let  $g_j := 1 - (1 - f_j)\eta$  and note that  $g_j \geq 0$  since  $0 \leq f_j, \eta \leq 1$ . We find that  $f - g_j = f - 1 + \eta - \eta f_j = (f - f_j)\eta + (1 - \eta)(f - 1) = (f - f_j)\eta$ , as  $f = 1$  in  $U$  and  $\eta = 1$  in  $\mathbb{R}^n \setminus U$ . Since  $\eta \in W^{1, \infty}(\mathbb{R}^n)$  and  $f_j \rightarrow f$  in  $W^{1, \varphi(\cdot)}(\mathbb{R}^n)$ , we find that  $g_j \rightarrow f$  in  $W^{1, \varphi(\cdot)}(\mathbb{R}^n)$ . Further,  $\eta = 0$  in a neighborhood of  $K$ , so  $g_j = 1$  in a neighborhood of  $K$ . Thus  $g_j \in S_{1, \varphi(\cdot)}^c(K)$ . This and  $g_j \rightarrow f$  imply the “ $\geq$ ” inequality.  $\square$

**Proposition 7.5.** *Let  $\Omega \subset \mathbb{R}^n$  be bounded and  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy (aDec). Assume that  $C(\mathbb{R}^n)$  is dense in  $W^{1, \varphi(\cdot)}(\mathbb{R}^n)$ . If  $E \subset \Omega$  with  $C_{\varphi(\cdot)}(E) = 0$ , then  $\text{cap}_{\varphi(\cdot)}(E, \Omega) = 0$ .*

*Proof.* Let  $K \subset \Omega$  be compact with  $C_{\varphi(\cdot)}(K) = 0$ . By Lemma 7.4 we may choose a sequence of functions  $(f_i)$  from  $S_{1,\varphi(\cdot)}^c(K)$  such that  $\|f_i\|_{W^{1,\varphi(\cdot)}(\mathbb{R}^n)} \rightarrow 0$ . Let  $\eta \in C_0^\infty(\Omega)$  be a cut-off function that equals two in  $K$ . Since  $f_i \in S_{1,\varphi(\cdot)}^c(K)$ , it is easy to conclude that  $\eta f_i > 1$  in  $K$  and  $\eta f_i \geq 0$ , hence  $\eta f_i \in R_{\varphi(\cdot)}(K, \Omega)$ . Since modular convergence and norm convergence are equivalent [10, Theorem 2.1.11], we obtain

$$\text{cap}_{\varphi(\cdot)}(K, \Omega) \leq \varrho_{\varphi(\cdot)}(\nabla(\eta f_i)) \leq \varrho_{\varphi(\cdot)}(c f_i) + \varrho_{\varphi(\cdot)}(c |\nabla f_i|) \rightarrow 0.$$

Hence the claim holds for compact sets.

By (S3) there exists a sequence of open sets  $U_i \supset E$  with  $C_{\varphi(\cdot)}(U_i) \rightarrow 0$  as  $i \rightarrow \infty$ . Let  $U := \bigcap_{i=1}^\infty U_i \cap \Omega$ . Then,  $U$  is a Borel set containing  $E$  which satisfies  $C_{\varphi(\cdot)}(U) = 0$ . By the Choquet property of the relative capacity, we obtain

$$\text{cap}_{\varphi(\cdot)}(E, \Omega) \leq \text{cap}_{\varphi(\cdot)}(U, \Omega) = \sup_K \text{cap}_{\varphi(\cdot)}(K, \Omega),$$

where the supremum is taken over all compact sets  $K \subset U$ . By the first part of the proof,  $\text{cap}_{\varphi(\cdot)}(K, \Omega) = 0$ , and the claim follows.  $\square$

If  $E \subset B$  and  $\Omega = 2B$  for a ball  $B$ , then the Lipschitz constant of the cut-off function  $\eta$  in the previous proof can be chosen to be  $c(\text{diam } B)^{-1}$ . Then an similar proof gives the following quantitative version of the previous result, cf. [10, Theorem 10.3.5].

**Theorem 7.6.** *Let  $B \subset \mathbb{R}^n$  be a ball and  $\varphi \in \Phi_w(\mathbb{R}^n)$  satisfy  $(aDec)_q$ . Assume that  $C(\mathbb{R}^n)$  is dense in  $W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ . For  $E \subset B$ ,*

$$\text{cap}_{\varphi(\cdot)}(E, 2B) \leq C \left(1 + \max\{\text{diam}(B), \text{diam}(B)^{-q}\}\right) C_{\varphi(\cdot)}(E)$$

where the constant  $C$  depends on  $q$  and  $L$ .

#### ACKNOWLEDGMENT

We thank the referee for some useful comments which improved the clarity of the presentation. Debangana Baruah was supported financially by the University of Turku Graduate School MATTI-program.

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