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RESEARCH ARTICLE

Comparative Gromov hyperbolicity results for the hyperbolic and quasihyperbolic metrics

Peter Hästö[a*], Ana Portilla[b†], José M. Rodríguez[c†] and Eva Tourís [c†]

[a] Department of Mathematical Sciences, P.O. Box 3000, FI-90014 University of Oulu, Finland; `peter.hasto@helsinki.fi`
[b] St. Louis University (Madrid Campus), Avenida del Valle 34, 28003 Madrid, Spain; `portillaa@madrid.slu.edu`
[c] Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain; `jomaro@math.uc3m.es`, `etouris@math.uc3m.es`

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In this article we investigate the Gromov hyperbolicity of Denjoy domains equipped with the hyperbolic or the quasihyperbolic metric. The focus are on comparative or decomposition results, which allow us to reduce the question of whether a given domain is Gromov hyperbolic to a series of questions concerning simpler domains. We also give several concrete examples of applications of the results.

Keywords: Poincaré metric; hyperbolic metric; quasihyperbolic metric; Gromov hyperbolic; Denjoy domain

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1. Introduction

In the 1980s M. Gromov, cf. [15], introduced a notion of abstract hyperbolic spaces, which have thereafter been studied and developed by many authors, e.g. [9, 10, 22, 31]. Initially, the research was mainly centered on hyperbolic group theory; lately researchers have shown an increasing interest in more direct studies of spaces endowed with metrics used in geometric function theory, e.g. [4, 6, 8, 16, 20, 21]. One of the primary questions is naturally whether a particular metric space is hyperbolic in the sense of Gromov or not. A classical example of a Gromov hyperbolic space is a Riemannian manifold with sectional curvature $K \leqslant -k^2 < 0$.

Gromov hyperbolicity of the quasihyperbolic metric was studied in M. Bonk, J. Heinonen and P. Koskela [8] and a geometric characterization in terms of a slice condition was given by Z. Balogh and S. Buckley in [4]. The Gromov hyperbolicity of the Poincaré hyperbolic metric is not as well understood, although several intrinsic results were obtained in [3] and [23]–[30], in particular in the case of Denjoy domains. The reason for studying Denjoy domains is that, on the one hand, they

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are a very general type of Riemann surfaces, and, on the other hand, their symmetry simplifies their study. This kind of surfaces are becoming more and more important in geometric function theory (see e.g. [1, 2, 11, 13]).

One way to approach the Gromov hyperbolicity with the hyperbolic metric is from the quasihyperbolic metric. For instance, one may ask whether there exists a domain such that one of these metrics is Gromov hyperbolic and the other is not. Since h_Ω and k_Ω are not in general quasi-isometric, it is natural to expect that this be the case. In [18] we studied this question in Denjoy domains and came to the surprising conclusion that in fact h_Ω is Gromov hyperbolic if and only if k_Ω is. Whether or not this is true in general domains remains an open question. A more precise description of these results is given in Section 3.

In this paper we give several results of a comparative type: suppose that $\mathbb{C} \setminus E$ and $\mathbb{C} \setminus F$ are Gromov hyperbolic. Does it follow that $\mathbb{C} \setminus (E \cup F)$ is? We also obtain the following comparative result with a very simple statement: if Ω is a non-Gromov hyperbolic Denjoy domain and $E \subset \mathbb{R} \cap \Omega$ is closed, then $\Omega \setminus E$ is not Gromov hyperbolic. We will also show how such results are useful when determining whether a particular domain is Gromov hyperbolic. To get started we review some notation and definitions in the next section and some useful tools in the following section.

In [19] we pursue similar questions from a different point of view and prove for general domains with the quasihyperbolic metric results which were proved in [14] for the hyperbolic metric.

2. Definitions and notation

By $\mathbb{H}$ we denote the upper half plane, $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. For $D \subset \mathbb{C}$ we denote by ∂D and $\overline{D}$ its boundary and closure, respectively. For $z \in D \subsetneq \mathbb{C}$ we denote by $\delta_D(z)$ the distance to the boundary, $\min_{a\in\partial D} |z - a|$. We denote by c and C generic constants which can change their value from line to line and even in the same line. We say that an inequality holds *quantitatively*, if it holds with a constant depending only on the constants in the assumptions.

Recall that a domain $\Omega \subset \mathbb{C}$ is said to be of hyperbolic type if it has at least two finite boundary points. The universal cover of such domain is the unit disk $\mathbb{D}$. In Ω we can define the Poincaré or *hyperbolic metric*, i.e. the metric obtained by pulling back the density $ds = 2|dz|/(1 - |z|^2)$ of the unit disk. The *quasihyperbolic metric* is the distance induced by the density $1/\delta_\Omega(z)$. By k_Ω and h_Ω we denote the quasihyperbolic and hyperbolic distance in Ω, respectively.

Length (of a curve) will be denoted by the symbol $\ell_{d,\Omega}$, where d is the metric with respect to which length is measured. The subscript "Eucl" is used to denote the length with respect to the Euclidean metric. Also, as most of the proofs apply to both the quasihyperbolic and the Poincaré metrics, we use the symbol κ as a "dummy metric" symbol, which stands for either k or h.

A geodesic metric space (X, d) is said to be *Gromov δ-hyperbolic*, if

$$d(w, [x, z] \cup [z, y]) \leqslant \delta$$

for every $x, y, z \in X$, corresponding geodesic segments $[x, y]$, $[y, z]$ and $[x, z]$, and some $w \in [x, y]$. If this inequality holds, we also say that the geodesic triangle is *δ-thin*, so Gromov hyperbolicity can be reformulated as requiring that all geodesic triangles are thin.

A *Denjoy domain* $\Omega \subset \mathbb{C}$ is a domain whose boundary is contained in the real axis. Hence, it satisfies $\Omega \cap \mathbb{R} = \cup_{n\in\Lambda}(a_n, b_n)$, where Λ is a countable index set, $\{(a_n, b_n)\}_{n\in\Lambda}$ are pair-wise disjoint, and $a_n, b_n \in \mathbb{R} \cup \{-\infty, \infty\}$. When studying Gromov hyperbolicity, we may restrict ourselves to the case where Λ is countably infinite, since if Λ is finite then h_Ω and k_Ω are easily seen to be Gromov hyperbolic by [17, Proposition 3.5].

Let Ω be a Denjoy domain. Then we have $\Omega \cap \mathbb{R} = \cup_{n\geqslant 0}(a_n, b_n)$ for some suitable disjoint intervals. We say that a curve in Ω is a *fundamental geodesic* if it is a geodesic joining (a_0, b_0) and (a_n, b_n), $n > 0$, which is contained in the closed halfplane $\overline{\mathbb{H}} = \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\}$.

A function between two metric spaces $f : X \longrightarrow Y$ is an (a, b)-*quasi-isometry*, $a \geqslant 1$, $b \geqslant 0$, if

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leqslant d_Y(f(x_1), f(x_2)) \leqslant a\, d_X(x_1, x_2) + b\,,$$

for every $x_1, x_2 \in X$. If there exists a constant c such that $d_Y(f(X), y) \leqslant c$ for every $y \in Y$, we say that X and Y are *quasi-isometric.*

3. Summary of previous results

In this section we recall the main results from [17, 18] which are needed in this paper.

We start with the following intrinsic characterization of Gromov hyperbolicity of Denjoy domains. It turns out that for comparative results, this characterization is often more useful than the extrinsic one that follows.

Theorem 3.1 [18, Theorem 4.3]: *Let Ω be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$, and denote by κ_Ω the Poincaré or quasihyperbolic metric in Ω. Let us fix $x_n \in (a_n, b_n)$ for each $n \geqslant 0$. Then κ_Ω is Gromov hyperbolic if and only if*

$$\sup_{n\geqslant 0}\ \sup_{y\in(0,R)}\ \kappa_\Omega(x_n + iy, \mathbb{R}) < \infty,$$

quantitatively, where $R := \sup |x_m - x_n|$.

Note that if Ω in the previous theorem is such that $\limsup_{n\to\infty} |a_n| = \infty$, then $R = \infty$ and the supremum is taken over all $y \in (0, \infty)$.

Then we move to the extrinsic characterization of Gromov hyperbolicity of Denjoy domains. We use the convention that $\min\{b_m - x, x - a_m\} = x - a_m$ if $b_m = \infty$, similarly for $a_m = -\infty$.

Theorem 3.2 [18, Theorem 5.2]: *With the notation of Theorem 3.1, κ_Ω is Gromov hyperbolic (quantitatively) if and only if there exists $L \geqslant 1$ such that for $n \geqslant 0$ and every $y \in (0, R)$ there exists $m \geqslant 0$ and $x \in (a_m, b_m)$ such that*

$$\tfrac{1}{L}|x_n - x| \leqslant y \leqslant L \min\{b_m - x, x - a_m\}.$$

The following is the main theorem of [18]. It is well-known that if two geodesic metric spaces are quasi-isometric, then either both are Gromov hyperbolic or neither is. Beardon's and Pommerenke's well-known result (see [5]), implies that there is not in general a constant such that $k_\Omega \leqslant c h_\Omega$, whereas the upper bound $h_\Omega \leqslant 2k_\Omega$ always holds. Therefore it is quite surprising that h_Ω is Gromov hyperbolic if and only if k_Ω is when Ω is Denjoy domain, which we now proceed to show. It remains an open question whether this type of equivalence holds in more general domains.

Theorem 3.3 [18, Theorem 4.6]**:** *Let Ω be a Denjoy domain. Then k_Ω is Gromov hyperbolic if and only if h_Ω is, quantitatively.*

Using the previously mentioned results as well as the results of [17] we were able to obtain the following very concrete description of Gromov hyperbolicity which covers many cases.

Theorem 3.4 [18, Theorem 5.5]**:** *Let Ω be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$, $b_0 > 0$, $b_n \leqslant a_{n+1}$ for every n, and $\liminf_{n\to\infty} b_n/a_n > 1$.*

(1) If $a_0 = -\infty$, then h_Ω and k_Ω are Gromov hyperbolic.
(2) If $a_0 > -\infty$, then h_Ω and k_Ω are Gromov hyperbolic if and only if

$$\limsup_{n\to\infty} \frac{a_{n+1}}{b_n} < \infty.$$

4. Comparative results

In this section we consider the problem of joining the boundaries of two domains. If each of the original boundaries gave rise to a Gromov hyperbolic space, what about their union?

As an application of Theorem 3.1 result we easily derive our first comparative result:

Corollary 4.1: *Let Ω be a Denjoy domain, and denote by κ_Ω the Poincaré or quasihyperbolic metric in Ω. Let E be a closed set in the real line. If $E \subset \Omega$ and κ_Ω is not Gromov hyperbolic, then $\kappa_{\Omega\setminus E}$ is not Gromov hyperbolic.*

Proof: If $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$, fix $x_n \in (a_n, b_n) \setminus E$ for each $n \geqslant 0$ and let R be as in Theorem 3.1. As κ_Ω is not Gromov hyperbolic, Theorem 3.1 gives that

$$\sup_{n\geqslant 0}\sup_{y\in(0,R)} \kappa_\Omega(x_n + iy, \mathbb{R}) = \infty.$$

Since $\kappa_\Omega \leqslant \kappa_{\Omega\setminus E}$ and $x_n \in \Omega \setminus E$ for every $n \geqslant 0$, the corresponding supremum for $\kappa_{\Omega\setminus E}$ is also infinity. Consequently, Theorem 3.1 gives that $\kappa_{\Omega\setminus E}$ is not Gromov hyperbolic. □

For the more sophisticated results we rely on the following types of decompositions.

Definition 4.2: Let (X, d) be a metric space, and let $\{X_n\}_n \subseteq X$ be a family of geodesic metric spaces such that $\eta_{nm} := X_n \cap X_m$ are compact sets. Further, assume that for any n and m the set $X \setminus \eta_{nm}$ is not connected, and that a and b are in different components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$, with $m \neq n$. If there exists positive constants c_1 and c_2 such that $\operatorname{diam}_{X_n}(\eta_{nm}) \leqslant c_1$ for every n, m, and $d_{X_n}(\eta_{nm}, \eta_{nk}) \geqslant c_2$ for every n and $m \neq k$, we say that $\{X_n\}_n$ is a (c_1, c_2)*-tree decomposition* of X.

Theorem 4.3 [Theorem 2.4, [28] and Theorem 2.9, [23]]**:** *Let us consider a metric space X and a family of geodesic metric spaces $\{X_n\}_n \subseteq X$ which is a (c_1, c_2)-tree decomposition of X. Then X is δ-hyperbolic quantitatively if and only if there exists a constant δ' such that X_n is δ'-hyperbolic for every n.*

Let us start by looking at the effects on Gromov hyperbolicity of joining boundary components of Gromov hyperbolic spaces which are uniformly separated. The

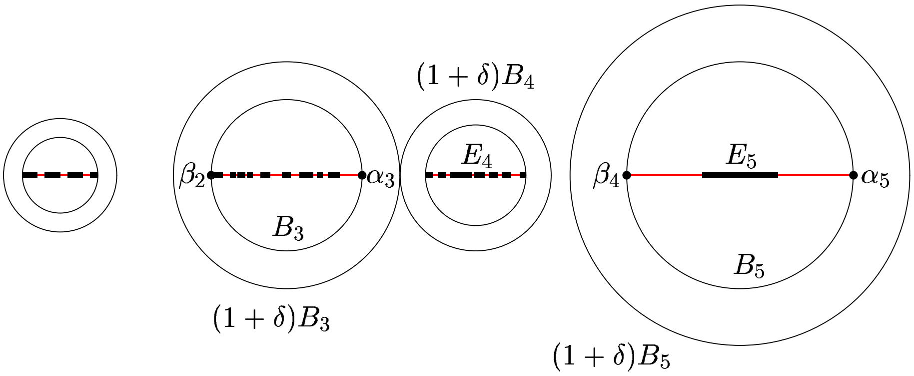

Figure 1. The setup of Theorem 4.4: the segments $[\beta_{k-1}, \alpha_k]$ are shown in red, the sets E_k with black think lines

setup of the following theorem is shown in Figure 1. If D is a closed subset of Ω, we denote by $k_{\Omega|D}$ the inner metric in D defined by δ_Ω, that is

$$k_{\Omega|D}(z,w) := \inf\big\{\ell_{k,\Omega}(\gamma) : \ \gamma \subset D \text{ is a curve joining } z \text{ and } w\big\}.$$

It is clear that $\ell_{k,\Omega|D}(\gamma) = \ell_{k,\Omega}(\gamma)$ for every curve $\gamma \subset D$ and $k_{\Omega|D}(z,w) \geqslant k_\Omega(z,w)$.

Theorem 4.4: *Consider a Denjoy domain* Ω *and suppose that* $\cup_{n=0}^{\infty}(\alpha_n, \beta_n) \subseteq \Omega$, *with* $\beta_n \leqslant \alpha_{n+1}$ *and* $\alpha_n, \beta_n \in \partial\Omega$. *Let* B_n *be the ball with* (β_{n-1}, α_n) *as diametrical chord. Assume that* $\delta \in (0,1)$ *is such that the dilated balls* $(1+\delta)B_n$ *are disjoint. Define* $E_n := [\beta_{n-1}, \alpha_n] \cap \partial\Omega$ *for* $n \geqslant 1$. *Then* (Ω, κ_Ω) *is* δ*-Gromov hyperbolic quantitatively if and only if* $(\mathbb{C} \setminus E_n, \kappa_{\mathbb{C}\setminus E_n})$ *is* δ'*-Gromov hyperbolic for every* $n \geqslant 1$ *and* $(\mathbb{C} \setminus \cup_n[\beta_{n-1}, \alpha_n], \kappa_{\mathbb{C}\setminus\cup_n[\beta_{n-1},\alpha_n]})$ *is Gromov hyperbolic.*

Proof: By Theorem 3.3 it suffices to consider the quasihyperbolic metric, and we assume that $\kappa_\Omega = k_\Omega$. For simplicity we set $\epsilon = \frac{1}{3}\delta$.

Let $\Omega_n := (1+2\epsilon)\overline{B_n} \cap \Omega$ and $\Omega_\infty := \mathbb{C} \setminus \cup_m(1+2\epsilon)B_m$, and define $\Omega'_n := (1+3\epsilon)B_n \cap \Omega$ and $\Omega'_\infty := \mathbb{C} \setminus \cup_m(1+\epsilon)\overline{B_m}$. We denote by X_n the metric space $(\Omega_n, k_{\Omega'_n|\Omega_n})$, for $n \in \mathbb{N} \cup \{\infty\}$.

Suppose that $x \in \Omega_n$ for $n < \infty$. Since $\Omega'_n \subset \mathbb{C} \setminus E_n$, it is clear that $\delta_{\Omega'_n}(x) \leqslant \delta_{\mathbb{C}\setminus E_n}(x)$. Suppose that a strict inequality holds. Then $\delta_{\Omega'_n}(x) = \delta_{(1+3\epsilon)B_n}(x) \geqslant \epsilon \operatorname{diam}_{\mathrm{Eucl}} B_n/2$. On the other hand, $\delta_{\mathbb{C}\setminus E_n}(x) \leqslant (1+2\epsilon) \operatorname{diam} B_n$. Hence we conclude that

$$\frac{\epsilon/2}{1+2\epsilon}\, \delta_{\mathbb{C}\setminus E_n}(x) \leqslant \delta_{\Omega'_n}(x) \leqslant \delta_{\mathbb{C}\setminus E_n}(x).$$

Since the densities in $\mathbb{C} \setminus E_n$ and Ω'_n are equivalent in Ω_n, we conclude that X_n is Gromov hyperbolic if and only if $(\Omega_n, k_{\mathbb{C}\setminus E_n|\Omega_n})$ is. Note that $Y_n = (\mathbb{C} \setminus (1+2\epsilon)\overline{B_n}, k_{\mathbb{C}\setminus E_n|\mathbb{C}\setminus(1+2\epsilon)\overline{B_n}})$ is Gromov hyperbolic, since if $t \geqslant 1+2\epsilon$ then

$$\begin{aligned} \ell_{k,\mathbb{C}\setminus E_n}(\partial(tB_n)) &\leqslant \int_{\partial(tB_n)} \frac{|dz|}{d_{\mathrm{Eucl}}(z, B_n)} \leqslant \frac{\pi t \operatorname{diam}_{\mathrm{Eucl}} B_n}{(t-1)\frac{1}{2}\operatorname{diam}_{\mathrm{Eucl}} B_n} \\ &= \frac{2\pi t}{t-1} \leqslant \frac{\pi(1+2\epsilon)}{\epsilon}, \end{aligned}$$

and Y_n thus is quasi-isometric to a cylinder. Since X_n, Y_n constitute a $(\pi(1+2\epsilon)/\epsilon, 1)$-tree decomposition of $\mathbb{C} \setminus E_n$, Theorem 4.3 gives that $(\Omega_n, k_{\mathbb{C}\setminus E_n|\Omega_n})$ is

Gromov hyperbolic if and only if $(\mathbb{C}\setminus E_n, k_{\mathbb{C}\setminus E_n})$ is. Consequently, X_n is Gromov hyperbolic if and only if $(\mathbb{C}\setminus E_n, k_{\mathbb{C}\setminus E_n})$ is.

A similar argument shows that X_∞ is quasi-isometric to $(\Omega_\infty, k_{\mathbb{C}\setminus\cup_n[\beta_{n-1},\alpha_n]})$, which is also assumed to be Gromov hyperbolic.

The intersections η_{nm}, corresponding to the tree decomposition $\{X_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ of Ω, are empty, except when $m=\infty$, in which case $\eta_{n,\infty}=\partial\big((1+2\epsilon)B_n\big)$. These sets have bounded quasihyperbolic diameter, and are separated from each other. Hence the claim follows from Theorem 4.3. □

Next we want to combine boundary components which are not uniformly separated. This is a more subtle question. For instance, let Ω be the Denjoy domain with $a_0=-\infty$, $b_0=0$ and

$$\liminf_{n\to\infty}\frac{b_n}{a_n}>1 \quad\text{and}\quad \limsup_{n\to\infty}\frac{a_{n+1}}{b_n}=\infty.$$

Let $\Omega'=\mathbb{C}\setminus(-\infty,-1]$. Then Ω and Ω' are Gromov hyperbolic, but $\Omega\cap\Omega'$ is not, by Theorem 3.4.

Definition 4.5: A closed set $E\subset\mathbb{R}$ is said to be *permeable at* $+\infty$ if $\mathbb{R}^+\setminus E$ is unbounded and there exists $a\in\mathbb{R}$ such that $(-\infty,a)\cap E=\emptyset$ and $\kappa_{\mathbb{C}\setminus((-\infty,a]\cup E)}$ is Gromov hyperbolic. *Permeable at* $-\infty$ is defined similarly.

The reason for the terminology of the previous definition is the following. Suppose that $E\subset\mathbb{R}$ is permeable at $+\infty$. Choose $x_0\in\mathbb{R}\setminus((-\infty,a]\cup E)$. Since $\mathbb{R}^+\setminus E$ is unbounded, $R=\infty$ in Theorem 3.1. Therefore there exist intervals $(a_m,b_m)\subset\mathbb{R}\setminus((-\infty,a]\cup E)$ of arbitrarily large length by Theorem 3.2, since y in the theorem can be chosen as large as we wish. Further, these intervals accumulate at $+\infty$, so the boundary has many holes near infinite, and hence is reasonably called permeable.

Theorem 4.6: *Let* $E\subset[1,\infty)$ *and* $F\subset(-\infty,-1]$ *be closed subsets. Then*

(1) $\mathbb{C}\setminus(E\cup F)$ *is Gromov hyperbolic quantitatively if* $\mathbb{C}\setminus E$ *and* $\mathbb{C}\setminus F$ *are Gromov hyperbolic and at least one of* E *and* F *is permeable.*

(2) $\mathbb{C}\setminus E$ *and* $\mathbb{C}\setminus F$ *are Gromov hyperbolic quantitatively if* $\mathbb{C}\setminus(E\cup F)$ *is Gromov hyperbolic.*

Proof: As before it suffices to consider the quasihyperbolic metric, by Theorem 3.3. Using an auxiliary dilation and translation, we may assume that $-1\in F$ and $1\in E$. We denote $\Omega':=\mathbb{C}\setminus((-\infty,-1]\cup E)$. Assume first that $\mathbb{C}\setminus E$ and $\mathbb{C}\setminus F$ are Gromov hyperbolic and that E is permeable. It follows from permeability that $k_{\Omega'}$ is Gromov hyperbolic. Hence there exist constants such that

$$\sup_{y>0,x\in\mathbb{R}\cap\Omega'} k_{\Omega'}(x+iy,\mathbb{R})<c_E \quad\text{and}\quad \sup_{y\in(0,|x|),x\in\mathbb{R}\setminus F} k_{\mathbb{C}\setminus F}(x+iy,\mathbb{R})<c_F,$$

by Theorem 3.1. Let x_0 be some point in $\mathbb{R}\setminus(E\cup F)$. The first claim follows by Theorem 3.1 once we show that $\sup_{y>0}k_{\mathbb{C}\setminus(E\cup F)}(x_0+iy,\mathbb{R})$ is bounded independent of x_0.

If $x_0\geqslant 0$, then we simply use the inequality $k_{\mathbb{C}\setminus(E\cup F)}(x_0+iy,\mathbb{R})\leqslant k_{\Omega'}(x_0+iy,\mathbb{R})<c_E$. If $x_0<0$ and $y>|x_0|$, then we use

$$k_{\mathbb{C}\setminus(E\cup F)}(x_0+iy,\mathbb{R})\leqslant k_{\mathbb{H}}(x_0+iy,iy)+k_{\Omega'}(iy,\mathbb{R})<1+c_E.$$

So it remains to consider $x_0<0$ with $y\leqslant|x_0|$. Let γ be a geodesic joining x_0+iy and $\mathbb{R}$ in $\mathbb{C}\setminus F$ with $\ell_{k,\mathbb{C}\setminus F}(\gamma)\leqslant c_F$. If γ lies in the left half-plane, then

$l_{k,\mathbb{C}\setminus F}(\gamma) = l_{k,\mathbb{C}\setminus(E\cup F)}(\gamma)$, and hence $k_{\mathbb{C}\setminus F}(z,\mathbb{R}) = k_{\mathbb{C}\setminus(E\cup F)}(z,\mathbb{R}) \leqslant c_F$. Otherwise let γ' denote the part of γ in the left half-plane and let z' be the endpoint of γ' on the imaginary axis. Then

$$\begin{aligned} k_{\mathbb{C}\setminus(E\cup F)}(z,\mathbb{R}) &\leqslant l_{k,\mathbb{C}\setminus(E\cup F)}(\gamma') + k_{\mathbb{C}\setminus(E\cup F)}(z',\mathbb{R}) \\ &\leqslant l_{k,\mathbb{C}\setminus F}(\gamma') + k_{\mathbb{C}\setminus((-\infty,-1]\cup E)}(z',\mathbb{R}) \\ &\leqslant c_F + c_E, \end{aligned}$$

so this quantity is bounded in all cases, and hence $\mathbb{C}\setminus(E\cup F)$ is Gromov hyperbolic, by Theorem 3.1.

Assume now that $\mathbb{C}\setminus(E\cup F)$ is Gromov hyperbolic. Applying Corollary 4.1 with $\Omega = \mathbb{C}\setminus F$ implies that $\mathbb{C}\setminus F$ is Gromov hyperbolic. Similarly, $\mathbb{C}\setminus E$ is Gromov hyperbolic. □

Proposition 4.7: *Let us consider a Denjoy domain* Ω *such that* $\Omega\cap\mathbb{R} = \cup_{n=0}^{\infty}(a_n,b_n)$, *with* $a_0 = 0$, $b_n \leqslant a_{n+1}$ *and* $a_{n+1}-b_n \leqslant L\min\{b_{n+1}-a_{n+1},\ b_n-a_n\}$ *for a positive constant* L *and for all* $n \geqslant 0$. *Suppose that there exists a subsequence* $\{a_{n_k}\}$ *with* $n_1 \leqslant N$, $n_{k+1}-n_k \leqslant N$ *and* $a_{n_{k+1}} \geqslant K a_{n_k}$ *for a fixed* $K > 1$ *and every* k. *Then* (Ω, h_Ω) *and* (Ω, k_Ω) *are* δ*-Gromov hyperbolic, where* δ *only depends on* L, K *and* N.

Proof: Consider $n = n_k, n_k+1, \dots, n_{k+1}-1$. It is clear that

$$\sum_{n=n_k}^{n_{k+1}-1} (b_n - a_n) + (a_{n+1} - b_n) = a_{n_{k+1}} - a_{n_k} \geqslant \underbrace{(1-\tfrac{1}{K})a_{n_{k+1}}}_{=M}.$$

If $a_{n+1} - b_n > \frac{M}{2N}$ for some n with $n_k \leqslant n \leqslant n_{k+1}-1$, then the assumptions of the theorem imply that $b_n - a_n > \frac{M}{2NL}$, and we define n'_k to be this n. Otherwise $a_{n+1} - b_n < \frac{M}{2N}$ for all such n, and hence $b_n - a_n > \frac{M}{2N}$ for some n, which we take as n'_k.

Therefore in any case we have n'_k such that $n_k \leqslant n'_k \leqslant n_{k+1}-1$ and $b_{n'_k} - a_{n'_k} \geqslant C a_{n_{k+1}} \geqslant C a_{n'_k}$. We define $\alpha_k := a_{n'_k}$ and $\beta_k := b_{n'_k}$ and consider $\mathbb{C}\setminus\cup_k[\beta_{k-1},\alpha_k]$. Since $\beta_k \geqslant (1+C)\alpha_k$, the first part of Theorem 3.4 implies that the hyperbolic and quasihyperbolic metrics on this set are Gromov hyperbolic.

Set $E_k := [\beta_{k-1},\alpha_k]\cap\partial\Omega$ so that $\Omega = \mathbb{C}\setminus\cup_k E_k$. Each set E_k consists of at most $2N$ closed intervals; the condition $a_{n+1}-b_n \leqslant L\min\{b_{n+1}-a_{n+1},\ b_n-a_n\}$ implies that $\kappa_\Omega((a_n,b_n),(a_{n+1},b_{n+1}))$ is bounded by a constant which just depends on L. Therefore $\mathbb{C}\setminus E_k$ is Gromov hyperbolic with constant depending only on N and L by [17, Proposition 3.5]. Thus the conditions of Theorem 4.4 are satisfied, so $\mathbb{C}\setminus\cup_k E_k = \Omega\cup(-\infty,0)$ is Gromov hyperbolic. Moreover, this set is permeable at $+\infty$, hence Theorem 4.6 implies the claim. □

Taking $a_n = b_{n-1}$ for every $n > 0$, we obtain the following result.

Corollary 4.8: *Let us consider an increasing sequence of positive numbers* $\{a_n\}$ *and the Denjoy domain*

$$\Omega := \mathbb{C}\setminus\left((-\infty,0]\cup\bigcup_{n=1}^{\infty}\{a_n\}\right).$$

Suppose that there exists a subsequence $\{a_{n_k}\}$ *with* $n_1 \leqslant N$, $n_{k+1}-n_k \leqslant N$ *and*

$a_{n_{k+1}} \geqslant K a_{n_k}$ for a fixed $K > 1$ and every k. Then (Ω, h_Ω) and (Ω, k_Ω) are δ-Gromov hyperbolic, where δ depends only on K and N.

The following example shows that the condition in Corollary 4.8 is not necessary.

Example 4.9 Let $\Omega := \mathbb{C} \setminus \big((-\infty, 0] \cup \bigcup_{n=1}^{\infty}\{a_n\}\big)$ be a Denjoy domain, with $\{a_n\}$ the increasing sequence such that

$$\bigcup_{n=1}^{\infty}\{\alpha_n\} = \bigcup_{m=1}^{\infty}\{2^m, 2^m + 2^{-m}, 2^m + 2^{-m} + 2^{-m+1}, \dots, 2^m + 2^{-m} + \dots + 2^{-1}\}.$$

The subsequence $\{2^m\}$ satisfies $2^{m+1} \geqslant 2 \cdot 2^m$, but there are m points between 2^m and 2^{m+1}, i.e. that there is not a uniform bound for the number of points between two consecutive points in the subsequence $\{2^m\}$, so that the conditions of Corollary 4.8 are not satisfied.

However, Ω is hyperbolic: we choose $\beta_{m-1} = 2^m$ and $\alpha_m = 2^m + 1 - 2^{-m}$. Then $E_m := \partial\Omega \cap [\beta_{m-1}, \alpha_m] = \{2^m, 2^m + 2^{-m}, 2^m + 2^{-m} + 2^{-m+1}, \dots, 2^m + 2^{-m} + \dots + 2^{-1}\}$. It is clear that these intervals satisfy the conditions of Theorem 4.4 and by Theorem 3.4 $\mathbb{C} \setminus \cup_m [\beta_{m-1}, \alpha_m]$ is Gromov hyperbolic. Then it remains only to check the Gromov hyperbolicity of $\mathbb{C} \setminus E_m$. This follows easily from Theorem 3.2.

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