The Dirichlet Energy Integral and Variable Exponent Sobolev Spaces with Zero Boundary Values

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Abstract. We define and study variable exponent Sobolev spaces with zero boundary values. This allows us to prove that the Dirichlet energy integral has a minimizer in the variable exponent case. Our results are based on a Poincaré-type inequality, which we prove under a certain local jump condition for the variable exponent.

Keywords: Variable exponent Sobolev space, zero boundary values, Sobolev capacity, Poincaré inequality, Dirichlet energy integral

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1. Introduction

In the beginning of the 90's Kováčik and Rákosník [19] introduced variable exponent Lebesgue and Sobolev spaces. These spaces are special cases of so-called Orlicz-Musielak spaces, and in this form their investigation goes back a bit further, to Hudzik [14] and Musielak [21]. Since the end of the last decade Sobolev spaces with variable exponent have been studied by numerous people including Diening [3], Diening and Růžička [4], Edmunds and Rákosník [6, 7, 8], Edmunds and Meskhi [9], Fan, Shen, and Zhao [10], and Pick and Růžička [22] and the authors in [11, 12].

One area where these spaces have found applications is the study of electrorheological fluids, as described in the book of Růžička [25]. The same spaces appear also in the study of variational integrals with non-standard growth, see the papers by Zhikov [27], Maracellini [15], and Acerbi and Mingione [2].

The classical Dirichlet boundary value problem arises from a partial differential equation: if Ω is a domain in $\mathbb{R}^n$ and $w : \partial\Omega \to \mathbb{R}$ is a continuous function, then the problem is to find a continuous function $u : \overline{\Omega} \to \mathbb{R}$ so that the Laplace equation $-\Delta u = 0$ is satisfied on Ω and $u = w$ on $\partial\Omega$. The function w gives the boundary values of u. By Weyl's lemma, such a u is always a C^2-function on Ω, and hence the problem may be considered in the classical sense. Classical potential theory is based on the Laplace equation which is clearly linear.

The p-Dirichlet boundary value problem for fixed p, $1 < p < \infty$, is to find a continuous function u on $\overline{\Omega}$ so that the p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1.1}$$

is satisfied on Ω and $u = w$ on $\partial\Omega$. Even more generally, we search for a function $u \in W^{1,p}(\Omega)$ and the boundary values are given by $w \in W^{1,p}(\Omega)$ only in the Sobolev sense, that is, $u - w \in W_0^{1,p}(\Omega)$. The p-Laplace equation (1.1) is the Euler equation for the variational integral

$$\int_\Omega |\nabla u|^p \, dx \tag{1.2}$$

which is called the p-Dirichlet energy integral on Ω. In the borderline case $p = n$ the integral (1.2) is conformally invariant, and the solutions of (1.1) are central to the theory of quasiconformal and quasiregular mappings. In general, when $p \neq 2$, the equation (1.1) is nonlinear and it must be understood in the weak sense.

The Dirichlet energy integral in metric measure spaces has been explored by Shanmugalingam [26]. She proved the existance of a minimizer under certain geometric constraints on the measure. Moreover, under the condition that the space has many rectifiable curves, the solution is unique. These results are based on the definition of the first order Sobolev spaces with zero boundary values in metric spaces equipped with a Borel regular measure by Kilpeläinen, Kinnunen, and Martio [17]. These authors showed that many classical results, including completeness, lattice properties and removable sets, extend to the metric setting.

An alternate way of stating the p-Dirichlet problem is the so-called p-Dirichlet energy minimizing problem that has been studied by many authors, see the references in [13]. Acerbi and Mingione [2] have studied the existence and the regularity of minimizers of the $p(\cdot)$-Dirichlet energy integral

$$\int_\Omega |\nabla u(x)|^{p(x)} \, dx \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$. They assumed that the variable exponent $p : \Omega \to (1, \infty)$ is log-Hölder continuous and that the functions $u \in W^{1,1}(\Omega)$ have boundary values in the classical sense and showed that the minimizer is Hölder continuous.

Our approach to the $p(\cdot)$-Dirichlet energy integral (1.3) is different than that in [2] and parallels that of [26]. We study functions with boundary values in the Sobolev sense. Hence we minimize over functions belonging to the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$. A crucial question is to define Sobolev spaces with zero boundary values, that is, the spaces $W_0^{1,p(\cdot)}(\Omega)$. For that we use the Sobolev $p(\cdot)$-capacity introduced by the authors in [11] and adapt the definition from the metric setting. It is not known whether this definition gives the same class of functions as that based on the closure of C_0^∞-functions, but see Theorem 3.3.

Another result which is needed in the study of the $p(\cdot)$-Dirichlet energy integral is the Poincaré inequality; in this context we call it the $p(\cdot)$-Poincaré inequality. Surprisingly, in the variable exponent Sobolev spaces the Poincaré inequality has attracted virtually no attention previously. We give a mild condition for the variable exponent p that guarantees validity of the $p(\cdot)$-Poincaré inequality. Our condition is, in some sense, sharp.

Finally, we are prepared to study the $p(\cdot)$-Dirichlet energy integral. Let $w \in W^{1,p(\cdot)}(\Omega)$. We prove in Theorem 5.2 that if $1 < \operatorname{ess\,inf} p \leqslant \operatorname{ess\,sup} p < \infty$ and if p is not too discontinuous, then there exists a function $u \in W^{1,p(\cdot)}(\Omega)$ which minimizes the integral (1.3) with $u - w \in W_0^{1,p(\cdot)}(\Omega)$. The minimizer is unique up to a set of zero $p(\cdot)$-capacity (Theorem 5.3). Moreover, we show in Theorem 5.4 that the function u minimizes the $p(\cdot)$-Dirichlet energy if and only if

$$\int_\Omega p(x)|\nabla u(x) + \nabla w(x)|^{p(x)-2}(\nabla u(x) + \nabla w(x)) \cdot \nabla(v(x) - u(x))dx \geqslant 0$$

for every $v \in W_0^{1,p(\cdot)}(\Omega)$. Our results are parallel to the fixed exponent case, see [13, Section 5].

2. Sobolev $p(\cdot)$-capacity

We denote by $\mathbb{R}^n$ the Euclidean space of dimension $n \geqslant 2$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote the open ball with center x and radius r by $B(x, r)$. We will next introduce variable exponent Lebesgue and Sobolev spaces in $\mathbb{R}^n$; note that we nevertheless use the standard definitions of the spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$ in the fixed exponent case $p \geqslant 1$ with open $\Omega \subset \mathbb{R}^n$.

Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function (called the *variable exponent* on $\mathbb{R}^n$). Throughout this paper the function p always denotes a variable exponent; also, we define $p^+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$ and $p^- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$. We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ to consist of all measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $\varrho_{p(\cdot)}(\lambda u) = \int_{\mathbb{R}^n} |\lambda u(x)|^{p(x)} dx < \infty$ for some $\lambda > 0$. The function $\varrho_{p(\cdot)} : L^{p(\cdot)}(\mathbb{R}^n) \to [0, \infty)$ is called the *modular* of the space $L^{p(\cdot)}(\mathbb{R}^n)$. We define a norm, the so-called *Luxemburg norm*, on this space by the formula $\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}$. The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\mathbb{R}^n)$ is the space of measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that u and the absolute value of the distributional gradient $\nabla u = (\partial_1 u, \dots, \partial_n u)$ is in $L^{p(\cdot)}(\mathbb{R}^n)$. The function $\varrho_{1,p(\cdot)} : W^{1,p(\cdot)}(\mathbb{R}^n) \to [0, \infty)$ is defined by $\varrho_{1,p(\cdot)}(u) = \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(|\nabla u|)$. The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\mathbb{R}^n)$ a Banach space. For the basics of variable exponent spaces see [19].

Recall from [11, Section 3] the definition and basic properties of the Sobolev $p(\cdot)$-capacity. For $E \subset \mathbb{R}^n$ we denote

$$S_{p(\cdot)}(E) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^n) : u \geqslant 1 \text{ in an open set containing } E\}.$$

The *Sobolev $p(\cdot)$-capacity* of E is defined by

$$C_{p(\cdot)}(E) = \inf_{u \in S_{p(\cdot)}(E)} \varrho_{1,p(\cdot)}(u) = \inf_{u \in S_{p(\cdot)}(E)} \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) dx.$$

In case $S_{p(\cdot)}(E) = \emptyset$, we set $C_{p(\cdot)}(E) = \infty$. For arbitrary measurable exponents $p : \mathbb{R}^n \to [1, \infty)$ the set function $E \mapsto C_{p(\cdot)}(E)$ has the following properties, [11, Theorem 3.1]:

(i) $C_{p(\cdot)}(\emptyset) = 0$.

(ii) [Monotony] If $E_1 \subset E_2$, then $C_{p(\cdot)}(E_1) \leqslant C_{p(\cdot)}(E_2)$.

(iii) If E is a subset of $\mathbb{R}^n$, then

$$C_{p(\cdot)}(E) = \inf\{C_{p(\cdot)}(U) : E \subset U,\ U \text{ open}\}.$$

(iv) If E_1 and E_2 are subsets of $\mathbb{R}^n$, then

$$C_{p(\cdot)}(E_1 \cup E_2) + C_{p(\cdot)}(E_1 \cap E_2) \leqslant C_{p(\cdot)}(E_1) + C_{p(\cdot)}(E_2).$$

(v) If $K_1 \supset K_2 \supset \dots$ are compact, then

$$\lim_{i \to \infty} C_{p(\cdot)}(K_i) = C_{p(\cdot)}\left(\bigcap_{i=1}^{\infty} K_i \right).$$

If $1 < p^- \leqslant p^+ < \infty$, then the following additional properties hold, [11, Theorem 3.2]:

(vi) If $E_1 \subset E_2 \subset \dots$ are subsets of $\mathbb{R}^n$, then

$$\lim_{i \to \infty} C_{p(\cdot)}(E_i) = C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i \right).$$

(vii) [Subaddivity] If $E_i \subset \mathbb{R}^n$ for $i = 1, 2, \ldots$, then

$$C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} C_{p(\cdot)}(E_i).$$

This means that if $1 < p^- \leqslant p^+ < \infty$, then the set function $E \mapsto C_{p(\cdot)}(E)$ is an outer measure and a Choquet capacity, see [11, Corollary 3.3 and Corollary 3.4].

A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be $p(\cdot)$-*quasicontinuous* (in $\mathbb{R}^n$) if for every $\varepsilon > 0$ there exists an open set O with $C_{p(\cdot)}(O) < \varepsilon$ such that u is continuous in $\mathbb{R}^n \setminus O$. For a subset E of $\mathbb{R}^n$ we say that a claim holds $p(\cdot)$-*quasieverywhere in* E (or $p(\cdot)$-*q.e. in* E, for short) if it holds everywhere except in a set $N \subset E$ with $C_{p(\cdot)}(N) = 0$.

The variable exponent $p : \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the *density condition* in $\mathbb{R}^n$ if the class of smooth functions is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. A sufficient condition for the density condition is known, see [6, Theorem 1]. It was proven in [11, Theorem 5.2] that if p satisfies the density condition with $1 < p^- \leqslant p^+ < \infty$, then every $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ has a $p(\cdot)$-quasicontinuous representative in $\mathbb{R}^n$. In addition, the following uniqueness result holds for the $p(\cdot)$-quasicontinuous representatives. For the proof of (i) we refer to [16]; (ii) follows directly from (i), see [17, Remark 3.3].

LEMMA 2.1. *Let* $1 < p^+ \leqslant p^- < \infty$, *and let* u *and* v *be* $p(\cdot)$-*quasicontinuous functions in* $\mathbb{R}^n$. *Suppose that* $O \subset \mathbb{R}^n$ *is open.*

(i) *If* $u = v$ *almost everywhere in* O, *then* $u = v$ $p(\cdot)$-*quasieverywhere in* O.

(ii) *If* $u \leqslant v$ *almost everywhere in* O, *then* $u \leqslant v$ $p(\cdot)$-*quasieverywhere in* O.

We will next consider a Sobolev $p(\cdot)$-capacity in terms of $p(\cdot)$-quasicontinuous functions. For $E \subset \mathbb{R}^n$ and $1 < p^- \leqslant p^+ < \infty$ we denote

$$\widetilde{C}_{p(\cdot)}(E) = \inf_{u \in \widetilde{S}_{p(\cdot)}(E)} \varrho_{1,p(\cdot)}(u)$$

where

$$\begin{aligned} \widetilde{S}_{p(\cdot)}(E) \;=\; & \{u \in W^{1,p(\cdot)}(\mathbb{R}^n) : u \text{ is } p(\cdot)\text{-quasicontinuous} \\ & \text{and } u \geqslant 1 \; p(\cdot)\text{-q.e. in } E\}. \end{aligned}$$

Here we use the convention that $\widetilde{C}_{p(\cdot)}(E) = \infty$ if $\widetilde{S}_{p(\cdot)}(E) = \emptyset$.

THEOREM 2.2. *Let* $1 < p^- \leqslant p^+ < \infty$ *and* $E \subset \mathbb{R}^n$.

(i) *We have* $C_{p(\cdot)}(E) \leqslant \widetilde{C}_{p(\cdot)}(E)$.

(ii) *If p satisfies the density condition, then* $C_{p(\cdot)}(E) = \widetilde{C}_{p(\cdot)}(E)$.

Proof. We follow the idea of the proof of the corresponding result in metric measure spaces, [17, Theorem 3.4]. However, in the proof of (i) the variable exponent causes some extra work; on the other hand we do not need the density condition. To achieve the reverse inequality and hence (ii), we need the density condition, but then the proof is even simpler than the corresponding proof in the metric measure spaces.

For the proof of (i), let $v \in \widetilde{S}_{p(\cdot)}(E)$. By truncation, we may assume that $0 \leqslant v \leqslant 1$. Fix ε, $0 < \varepsilon < 1$, and choose an open set V with $C_{p(\cdot)}(V) < \varepsilon$ so that $v = 1$ on $E \setminus V$ and that $v|_{\mathbb{R}^n \setminus V}$ is continuous. Define $U = \{x \in \mathbb{R}^n \setminus V : v(x) > 1 - \varepsilon\} \cup V$ and observe that $E \setminus V \subset U \setminus V$. Choose $u \in S_{p(\cdot)}(V)$ such that $\varrho_{1,p(\cdot)}(u) < \varepsilon$ and that $0 \leqslant u \leqslant 1$. We define $w = v/(1-\varepsilon) + u$. Then $w \geqslant 1$ a.e. in $(U \setminus V) \cup V = U$, which is an open neighborhood of E and hence $w \in S_{p(\cdot)}(E)$. By [20, Lemma 1.1] we have, for every $\delta > 0$,

$$\begin{aligned}
\varrho_{p(\cdot)}(w) &= \int_{\mathbb{R}^n} \left| \frac{v(x)}{1-\varepsilon} + u(x) \right|^{p(x)} dx \\
&\leqslant (1+\delta)^{p^+-1} \int_{\mathbb{R}^n} \left| \frac{v(x)}{1-\varepsilon} \right|^{p(x)} dx + \left(1 + \frac{1}{\delta}\right)^{p^+-1} \int_{\mathbb{R}^n} |u(x)|^{p(x)}\, dx \\
&< \frac{(1+\delta)^{p^+-1}}{(1-\varepsilon)^{p^+}} \int_{\mathbb{R}^n} |v(x)|^{p(x)}\, dx + \left(1 + \frac{1}{\delta}\right)^{p^+-1} \varepsilon \\
&\leqslant \left(\frac{1+\delta}{1-\varepsilon}\right)^{p^+} \int_{\mathbb{R}^n} |v(x)|^{p(x)}\, dx + \left(1 + \frac{1}{\delta}\right)^{p^+} \varepsilon.
\end{aligned}$$

If we choose $\delta = \varepsilon^{\frac{1}{2p^+}}$, then

$$\left(\frac{1+\delta}{1-\varepsilon}\right)^{p^+} = \left(\frac{1+\varepsilon^{\frac{1}{2p^+}}}{1-\varepsilon}\right)^{p^+} \to 1$$

and

$$\left(1 + \frac{1}{\delta}\right)^{p^+} \varepsilon = \left(\varepsilon^{\frac{1}{p^+}} + \varepsilon^{\frac{1}{2p^+}}\right)^{p+} \to 0$$

as $\varepsilon \to 0$. Hence

$$\varrho_{p(\cdot)}(w) \leqslant \int_{\mathbb{R}^n} |v(x)|^{p(x)}\, dx = \varrho_{p(\cdot)}(v).$$

In a similar way, we find that $\varrho_{p(\cdot)}(|\nabla w|) \leqslant \varrho_{p(\cdot)}(|\nabla v|)$, and hence $\varrho_{1,p(\cdot)}(w) \leqslant \varrho_{1,p(\cdot)}(v)$. Since $v \in \widetilde{S}_{p(\cdot)}(E)$ was arbitrary, we obtain $C_{p(\cdot)}(E) \leqslant \widetilde{C}_{p(\cdot)}(E)$.

For the proof of the reverse inequality, assume that the variable exponent p satisfies the density condition. Let $E \subset \mathbb{R}^n$. Take $u \in S_{p(\cdot)}(E)$ and let $O \supset E$

be an open set such that $u \geqslant 1$ on O. By [11, Lemma 5.2], there exists a $p(\cdot)$-quasicontinuous function $\tilde{u}$ in $\mathbb{R}^n$ such that $u \geqslant 1$ a.e. in O. It follows from Lemma 2.1 (ii) that $\tilde{u} \geqslant 1$ $p(\cdot)$-q.e. in O. Hence $\tilde{u} \geqslant 1$ $p(\cdot)$-q.e. in E and thus $\tilde{u} \in \widetilde{S}_{p(\cdot)}(E)$. This yields $\widetilde{C}_{p(\cdot)}(E) \leqslant C_{p(\cdot)}(E)$, and finally combining this with (i) gives $C_{p(\cdot)}(E) = \widetilde{C}_{p(\cdot)}(E)$.

The following convergence result is a sharpening of [11, Lemma 5.1]; it corresponds to [17, Lemma 3.5] which is stated in metric measure spaces.

LEMMA 2.3. *Let* $1 < p^- \leqslant p^+ < \infty$. *Suppose that* $u_i \in W^{1,p(\cdot)}(\mathbb{R}^n)$ *are* $p(\cdot)$-*quasicontinuous functions for* $i = 1, 2, \ldots$ *such that* $u_i \to u$ *in* $W^{1,p(\cdot)}(\mathbb{R}^n)$. *Then* u *is* $p(\cdot)$-*quasicontinuous and there is a subsequence of* (u_i) *which converges pointwise to* u $p(\cdot)$-*quasieverywhere in* $\mathbb{R}^n$.

Proof. There is a subsequence of (u_i), denoted again by (u_i), such that

$$\sum_{i=1}^{\infty} 2^{ip^+} \|u_i - u_{i+1}\|_{1,p(\cdot)} < 1.$$

For $i = 1, 2 \ldots$, denote $E_i = \{x \in \mathbb{R}^n : |u_i(x) - u_{i+1}(x)| > 2^{-i}\}$ and $F_j = \bigcup_{i=j}^{\infty} E_i$. Clearly $2^i |u_i - u_{i+1}| \in \widetilde{S}_{p(\cdot)}(E_i)$ and hence using Theorem 2.2 (i) we obtain

$$\begin{aligned} C_{p(\cdot)}(E_i) &\leqslant \int_{\mathbb{R}^n} \left(|2^i(u_i - u_{i+1})|^{p(x)} + |\nabla(2^i(u_i - u_{i+1}))|^{p(x)} \right) dx \\ &\leqslant 2^{ip^+} \varrho_{1,p(\cdot)}(u_i - u_{i+1}). \end{aligned}$$

Using the subadditivity property (vii) of the Sobolev $p(\cdot)$-capacity and [19, (2.11)], we obtain

$$\begin{aligned} C_{p(\cdot)}(F_j) &\leqslant \sum_{i=j}^{\infty} C_{p(\cdot)}(E_i) \leqslant \sum_{i=j}^{\infty} 2^{ip^+} \varrho_{1,p(\cdot)}(u_i - u_{i+1}) \\ &\leqslant \sum_{i=j}^{\infty} 2^{ip^+} \|u_i - u_{i+1}\|_{1,p(\cdot)}. \end{aligned}$$

Since $\bigcap_{j=1}^{\infty} F_j \subset F_j$ for each j, the monotony property (ii) of the Sobolev $p(\cdot)$-capacity yields

$$C_{p(\cdot)}\left(\bigcap_{j=1}^{\infty} F_j \right) \leqslant \lim_{j \to \infty} C_{p(\cdot)}(F_j) = 0.$$

Moreover, $u_i \to u$ pointwise in $\mathbb{R}^n \setminus \bigcap_{j=1}^{\infty} F_j$, and so the convergence $p(\cdot)$-q.e. in $\mathbb{R}^n$ follows.

To prove the $p(\cdot)$-quasicontinuity of u, let $\varepsilon > 0$. By the first part of this proof, there is a set $F_j \subset \mathbb{R}^n$ such that $C_{p(\cdot)}(F_j) < \frac{\varepsilon}{2}$ and that $u_i \to u$ pointwise in $\mathbb{R}^n \setminus F_j$. Since every u_i is $p(\cdot)$-quasicontinuous in $\mathbb{R}^n$, we may choose open sets $G_i \subset \mathbb{R}^n$, $i = 1, 2, \ldots$, such that $C_{p(\cdot)}(G_i) < \frac{\varepsilon}{2^{i+1}}$ and $u_i|_{\mathbb{R}^n \setminus G_i}$ are continuous. Writing $G = \bigcup_i G_i$ we have

$$C_{p(\cdot)}(G) = C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} G_i\right) < \frac{\varepsilon}{2},$$

and the subaddivity property (vii) of the Sobolev $p(\cdot)$-capacity yields

$$C_{p(\cdot)}(F_j \cup G) \leqslant C_{p(\cdot)}(F_j) + C_{p(\cdot)}(G) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Moreover,

$$|u_l(x) - u_k(x)| \leqslant \sum_{i=l}^{k-1} |u_i(x) - u_{i+1}(x)| \leqslant \sum_{i=l}^{k-1} 2^{-i} < 2^{1-l}$$

for every $x \in \mathbb{R}^n \setminus (F_i \cup G)$ and every $k > l > i$. Therefore the convergence is uniform in $\mathbb{R}^n \setminus (F_i \cup G)$, and it follows that u is continuous in $\mathbb{R}^n \setminus (F_i \cup G)$. This completes the proof.

3. Variable exponent Sobolev spaces with zero boundary values

We assume throughout this section that $1 < p^- \leqslant p^+ < \infty$ in order to make sure that the Sobolev $p(\cdot)$-capacity is an outer measure and a Choquet capacity, [11, Corollary 3.3 and Corollary 3.4].

The variable exponent Sobolev spaces with zero boundary values are defined as in the metric measure spaces following [17]: Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote $u \in W_0^{1,p(\cdot)}(\Omega)$ and say that u belongs to the *variable exponent Sobolev space with zero boundary values* if there exists a $p(\cdot)$-quasicontinuous function $\tilde{u} \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $u = \tilde{u}$ almost everywhere in Ω and $\tilde{u} = 0$ $p(\cdot)$-quasieverywhere in $\mathbb{R}^n \setminus \Omega$. The set $W_0^{1,p(\cdot)}(\Omega)$ is endowed with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\tilde{u}\|_{W^{1,p(\cdot)}(\mathbb{R}^n)}.$$

A $p(\cdot)$-quasicontinuous function $\tilde{u} \in W^{1,p(\cdot)}(\mathbb{R}^n)$ is called a *canonical representative* of the function $u \in W_0^{1,p(\cdot)}(\Omega)$ if $u = \tilde{u}$ almost everywhere in Ω and $\tilde{u} = 0$ $p(\cdot)$-quasieverywhere in $\mathbb{R}^n \setminus \Omega$.

THEOREM 3.1. *If* $1 < p^- \leqslant p^+ < \infty$, *then* $W_0^{1,p(\cdot)}(\Omega)$ *is a Banach space.*

Proof. Suppose that (u_i) is a Cauchy sequence in $W_0^{1,p(\cdot)}(\Omega)$. Then there is a canonical representative $\tilde{u}_i$ of u_i for every $i = 1, 2, \ldots$ Since $W^{1,p(\cdot)}(\mathbb{R}^n)$ is a Banach space, there is $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $\tilde{u}_i \to u$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$ as $i \to \infty$. By Lemma 2.3, u is $p(\cdot)$-quasicontinuous and there is a subsequence of $(\tilde{u}_i)$ which converges to u $p(\cdot)$-quasieverywhere in $W^{1,p(\cdot)}(\mathbb{R}^n)$ as $i \to \infty$. This shows that $u = 0$ $p(\cdot)$-quasieverywhere in $\mathbb{R}^n \setminus \Omega$. Consequently $u \in W_0^{1,p(\cdot)}(\Omega)$ and the space $W_0^{1,p(\cdot)}(\Omega)$ is complete.

The definition of the space $W^{1,p(\cdot)}(\Omega)$ is analogous to that of $W^{1,p(\cdot)}(\mathbb{R}^n)$ in Section 2, one just changes every occurence of $\mathbb{R}^n$ by Ω, see [19] for details. By $H_0^{1,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in the space $W^{1,p(\cdot)}(\Omega)$. Note that $H_0^{1,p(\cdot)}(\Omega)$ is a Banach space.

COROLLARY 3.2. *If* $1 < p^- \leqslant p^+ < \infty$, *then* $H_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega) \subset W^{1,p(\cdot)}(\Omega)$.

Proof. Since $C_0^\infty(\Omega) \subset W_0^{1,p(\cdot)}(\Omega)$, the first inclusion follows from Theorem 3.1. The second inclusion follows directly from the definition of the space $W_0^{1,p(\cdot)}(\Omega)$.

The proof of the following result follows in part the arguments in Section 9.2 of [1].

THEOREM 3.3. *If p satisfies the density condition with* $1 < p^- \leqslant p^+ < \infty$, *then* $H_0^{1,p(\cdot)}(\Omega) = W_0^{1,p(\cdot)}(\Omega)$.

Proof. By Corollary 3.2 it suffices to show that $H_0^{1,p(\cdot)}(\Omega) \supset W_0^{1,p(\cdot)}(\Omega)$. Let $u \in W_0^{1,p(\cdot)}(\Omega)$ and let $\tilde{u}$ be its canonical representative. We need to show that there exist functions $\phi_i \in C_0^\infty(\Omega)$ which tend to $\tilde{u}$.

If we can construct such a sequence for $\tilde{u}_+(x) = \max\{\tilde{u}(x), 0\}$, then we can do it for $\tilde{u}_-$, as well, and combining these gives the result for $\tilde{u} = \tilde{u}_+ + \tilde{u}_-$. We therefore assume that $\tilde{u}$ is positive. Since we can approximate $\tilde{u}$ by $\tilde{u}_m(x) = \min\{\tilde{u}(x), m\}$, we see that it also suffices to consider only a bounded function $\tilde{u}$. Finally, applying progressively larger cut-off functions shows that we may assume that $\tilde{u}$ has compact support.

For $0 < \varepsilon < 1$ define $\tilde{u}_\varepsilon(x) = \max\{\tilde{u}(x) - \varepsilon, 0\}$. Let $\delta > 0$ and let G be an open set such that $\tilde{u}$ restricted to $\Omega \setminus G$ is continuous and $C_{p(\cdot)}(G) < \delta$. Let $\omega_\delta \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be such that $0 \leqslant \omega_\delta \leqslant 1$, $\omega_\delta|_G = 1$ and $\|\omega_\delta\|_{1,p(\cdot)} < \delta$. Since $\tilde{u}$ restricted to $\mathbb{R}^n \setminus G$ is continuous and equals zero almost everywhere in $\mathbb{R}^n \setminus \Omega$, we conclude that $\tilde{u}(x) = 0$ for every $x \in \mathbb{R}^n \setminus (\Omega \cup G)$. Thus there exists a set $F \subset \Omega \setminus G$ which is closed in $\mathbb{R}^n \setminus G$ (and hence in $\mathbb{R}^n$) such that $\tilde{u}_\varepsilon$ equals zero in $\mathbb{R}^n \setminus (G \cup F)$. It follows that $(1 - \omega_\delta)\tilde{u}_\varepsilon$ equals zero in $\mathbb{R}^n \setminus F$, and since F is closed, this means that the support of $(1 - \omega_\delta)\tilde{u}_\varepsilon$ is contained in F, so this function has compact support in Ω.

We also find that

$$\|\tilde{u} - (1-\omega_\delta)\tilde{u}_\varepsilon\|_{1,p(\cdot)} \leqslant \|\tilde{u} - \tilde{u}_\varepsilon\|_{1,p(\cdot)} + \|\omega_\delta \tilde{u}_\varepsilon\|_{1,p(\cdot)}.$$

We have

$$\|\tilde{u} - \tilde{u}_\varepsilon\|_{1,p(\cdot)} \leqslant \varepsilon\|\chi_{\operatorname{spt}\tilde{u}}\|_{p(\cdot)} + \|\chi_{\{0<\tilde{u}(x)\leqslant\varepsilon\}}\nabla\tilde{u}\|_{p(\cdot)},$$

and so we see that this term goes to zero with ε. We also find since $\tilde{u}$ is bounded that

$$\begin{aligned}\varrho_{1,p(\cdot)}(\omega_\delta\tilde{u}) &\leqslant \int_{\mathbb{R}^n} |\omega_\delta(x)\tilde{u}(x)|^{p(x)}\,dx + 2^{p^+}\int_{\mathbb{R}^n}\omega_\delta(x)^{p(x)}|\nabla\tilde{u}(x)|^{p(x)}\,dx \\ &\quad +2^{p^+}\int_{\mathbb{R}^n}|\nabla\omega_\delta(x)|^{p(x)}|\tilde{u}(x)|^{p(x)}\,dx \\ &\leqslant (2^{p^+}+1)\delta\sup_{x\in\mathbb{R}^n}\tilde{u}(x)^{p(x)} + 2^{p^+}\int_{\mathbb{R}^n}\omega_\delta(x)^{p(x)}|\nabla\tilde{u}(x)|^{p(x)}\,dx.\end{aligned}$$

Since $\omega_\delta \to 0$ in $L^{p(\cdot)}(\mathbb{R}^n)$, as $\delta \to 0$, we can choose a sequence ω_i which tends to 0 pointwise almost everywhere. Then $\int_{\mathbb{R}^n}\omega_i(x)^{p(x)}|\nabla\tilde{u}(x)|^{p(x)}\,dx \to 0$ by the dominated convergence theorem. Therefore $\varrho_{1,p(\cdot)}(\omega_\delta\tilde{u}) \to 0$ and so also $\|\omega_\delta\tilde{u}\|_{1,p(\cdot)} \to 0$ as $\delta \to 0$, [19, Theorem 2.4]. Thus we see that $(1-\omega_\delta)\tilde{u}_\varepsilon \to \tilde{u}$ as $\varepsilon, \delta \to 0$. In other words we have proven that every function in $W_0^{1,p(\cdot)}(\Omega)$ can be approximated by functions in $W^{1,p(\cdot)}(\Omega)$ with compact support in Ω.

Denote $w = (1-\omega_\delta)\tilde{u}_\varepsilon$. Let $\phi_i \in C^\infty(\mathbb{R}^n)$ be functions in $W^{1,p(\cdot)}(\mathbb{R}^n)$ which tend to w. Let $\psi \in C_0^\infty(\Omega)$ be a function which equals 1 in $\operatorname{spt} w$. Then

$$\begin{aligned}\varrho_{1,p(\cdot)}(w-\psi\phi_i) &= \int_{\operatorname{spt} w}|w(x)-\phi_i(x)|^{p(x)} + |\nabla(w(x)-\phi_i(x))|^{p(x)}\,dx \\ &\quad + \int_{\mathbb{R}^n\setminus\operatorname{spt} w}|\phi_i(x)\psi(x)|^{p(x)} + |\nabla(\phi_i(x)\psi(x))|^{p(x)}\,dx.\end{aligned}$$

Since $\phi_i \to w$, the first integral goes to zero. The second integral is less than

$$\text{const}\cdot\int_{\mathbb{R}^n\setminus\operatorname{spt} w}|\phi_i(x)|^{p(x)} + |\nabla\phi_i(x)|^{p(x)}\,dx,$$

which also tends to zero, since $\phi_i \to w$ and $w = 0$ in $\mathbb{R}^n \setminus \operatorname{spt} w$.

We have therefore constructed a sequence $(\psi\phi_i)$ which approaches w. But w can be chosen arbitrarily close to $\tilde{u}$, and so we get a sequence of $C_0^\infty(\Omega)$ functions tending to $\tilde{u}$.

THEOREM 3.4. *Let* $1 < q^-$, $p^+ < \infty$ *and* $p(x) \geqslant q(x)$ *for almost every* $x \in \mathbb{R}^n$. *Assume that* $\Omega \subset \mathbb{R}^n$ *is a bounded open set. Then*

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow W_0^{1,q(\cdot)}(\Omega).$$

Moreover, the norm of the embedding operator does not exceed $1 + |\Omega|$.

Proof. Let $u \in W_0^{1,p(\cdot)}(\Omega)$ and let $\tilde{u} \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be its canonical representative. By [19, Theorem 2.8], $\tilde{u} \in W^{1,q(\cdot)}(\mathbb{R}^n)$ and

$$\|\tilde{u}\|_{W^{1,q(\cdot)}(\mathbb{R}^n)} \leqslant (1 + |\Omega|)\|u\|_{W_0^{1,p(\cdot)}(\Omega)}.$$

We start by showing

$$C_{q(\cdot)}(\{\tilde{u} \neq 0\} \setminus \Omega) = 0.$$

We write $F = \{\tilde{u} \neq 0\} \setminus \Omega$. By the subadditivity of the capacity it is enough to show that

$$C_{q(\cdot)}(F \cap B(0,r)) = 0 \tag{3.1}$$

for every $r > 0$. Since $C_{p(\cdot)}(F \cap B(0,r)) = 0$ we can choose $v_i \in S_{p(\cdot)}(F \cap B(0,r))$ such that

$$\int_{\mathbb{R}^n} |v_i(x)|^{p(x)} + |\nabla v_i(x)|^{p(x)}\, dx \to 0$$

as $i \to \infty$. Let ϕ_r be a Lipschitz continuous cut-off function: $\phi_r = 1$ in $B(0,2r)$ and $\phi_r = 0$ outside $B(0,3r)$. Now by [19, Theorem 2.8] $\phi_r v_i \in S_{q(\cdot)}(F \cap B(0,r))$ and

$$\|\phi_r v_i\|_{W^{1,q(\cdot)}(\mathbb{R}^n)} \leqslant C(r)\|v_i\|_{W^{1,p(\cdot)}(\mathbb{R}^n)}.$$

By [19, (2.28)] we obtain (3.1).

To complete the proof we have to show that a $p(\cdot)$-quasicontinuous function $\tilde{u}$ is $q(\cdot)$-quasicontinuous. It is enough to verify this in every ball $B \subset \mathbb{R}^n$. Let $F_i \subset B$ be such that $\tilde{u}$ is continuous in $B \setminus F_i$ and $C_{p(\cdot)}(F_i) < \frac{1}{i}$. Let $v_i \in S_{p(\cdot)}(F_i)$ satisfy

$$\int_{\mathbb{R}^n} |v_i(x)|^{p(x)} + |\nabla v_i(x)|^{p(x)}\, dx < \frac{1}{i}.$$

Let ϕ be a cut-off function as before. Using ϕv_i as a test function we obtain $C_{q(\cdot)}(F_i) \to 0$ as $i \to \infty$. This completes the proof of Theorem 3.4.

Recall that the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and $W^{1,p(\cdot)}(\mathbb{R}^n)$ are reflexive if and only if the variable exponent $p : \mathbb{R}^n \to [1,\infty)$ satisfies $1 < p^- \leqslant p^+ < \infty$, [19, Corollary 2.7]. We prove this property for the variable exponent Sobolev spaces with zero boundary values; for more information on reflexive Banach spaces, see [24, Chapter 4].

THEOREM 3.5. *If* $1 < p^- \leqslant p^+ < \infty$, *then* $W_0^{1,p(\cdot)}(\Omega)$ *is reflexive.*

Proof. $W^{1,p(\cdot)}(\mathbb{R}^n)$ is a reflexive Banach space by [19, Theorem 3.1]. By Theorem 3.1, $W_0^{1,p(\cdot)}(\Omega)$ is closed, and the claim follows from [5, Theorem 23].

LEMMA 3.6. *Let* $1 < p^- \leqslant p^+ < \infty$. *Suppose that* $u \in W_0^{1,p(\cdot)}(\Omega)$ *and* $v \in W^{1,p(\cdot)}(\mathbb{R}^n)$ *are bounded functions. If* v *is* $p(\cdot)$*-quasicontinuous, then* $uv \in W_0^{1,p(\cdot)}(\Omega)$.

Proof. Let v be $p(\cdot)$-quasicontinuous. It is clear that $uv \in W^{1,p(\cdot)}(\Omega)$. Let $\tilde{u} \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be the canonical representative of u. Then $\tilde{u}v$ is $p(\cdot)$-quasicontinuous in $\mathbb{R}^n$ and it may be nonzero outside Ω only in a set $A \cup B$ where $A = \{x \in \mathbb{R}^n \setminus \Omega : \tilde{u}(x) \neq 0\}$ and $B = \{x \in \mathbb{R}^n \setminus \Omega : v(x) = \infty\}$. Both $C_{p(\cdot)}(A)$ and $C_{p(\cdot)}(B)$ vanish and hence property (iv) of the Sobolev $p(\cdot)$-capacity yields $C_{p(\cdot)}(A \cup B) = 0$. Therefore $\tilde{u}v = 0$ $p(\cdot)$-q.e. in $\mathbb{R}^n \setminus \Omega$. Since, in addition, $\tilde{u}v = uv$ a.e. in Ω, we have $uv \in W_0^{1,p(\cdot)}(\Omega)$.

Remark. To obtain the result of Lemma 3.6, we may relax the assumption that the function v is $p(\cdot)$-quasicontinuous. However, some additional assumption is needed to guarantee that v has a $p(\cdot)$-quasicontinuous representative in $\mathbb{R}^n$. One possibility is to suppose that p satisfies the density condition in $\mathbb{R}^n$, by [11, Theorem 5.2].

THEOREM 3.7. *Let* $1 < p^- \leqslant p^+ < \infty$*, and let* N *be a subset of* $\mathbb{R}^n$. *Then* $W_0^{1,p(\cdot)}(\Omega) = W_0^{1,p(\cdot)}(\Omega \setminus N)$ *if and only if* $C_{p(\cdot)}(N \cap \Omega) = 0$.

Proof. Suppose first that $C_{p(\cdot)}(N \cap \Omega) = 0$. It follows from [11, Lemma 4.1] that $|N \cap \Omega| = 0$ so that the notation $W_0^{1,p(\cdot)}(\Omega) = W_0^{1,p(\cdot)}(\Omega \setminus N)$ makes sense. It is clear that $W_0^{1,p(\cdot)}(\Omega \setminus N) \subset W_0^{1,p(\cdot)}(\Omega)$. Let $u \in W_0^{1,p(\cdot)}(\Omega)$ and $\tilde{u} \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be its canonical representative. We have $\tilde{u} = 0$ $p(\cdot)$-q.e. in $\mathbb{R}^n \setminus (\Omega \setminus N)$ as $C_{p(\cdot)}(N \cap \Omega) = 0$. Hence $u|_{\Omega \setminus N} \in W_0^{1,p(\cdot)}(\Omega \setminus N)$ because clearly $\tilde{u} = u$ a.e. in $\Omega \setminus N$. Moreover, we have

$$\|u|_{\Omega \setminus N}\|_{W_0^{1,p(\cdot)}(\Omega \setminus N)} = \|u\|_{W_0^{1,p(\cdot)}(\Omega)}.$$

The proof of the necessity goes along the same lines as the proof of [17, Theorem 4.8]. We may assume that $N \subset \Omega$. Let $x_0 \in \Omega$ and write

$$\Omega_i = B(x_0, i) \cap \{x \in \Omega : \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) > 1/i\}, \quad i = 1, 2, \ldots$$

Define $u_i : \mathbb{R}^n \to \mathbb{R}$ by $u_i(x) = \max(0, 1 - \operatorname{dist}(x, N \cap \Omega_i))$, $i = 1, 2, \ldots$ Then $u_i \in W^{1,p(\cdot)}(\mathbb{R}^n)$ is continuous, $0 \leqslant u_i \leqslant 1$ and $u = 1$ in $N \cap \Omega_i$. Define $v_i : \Omega_i \to \mathbb{R}$ as $v_i(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega_i)$, $i = 1, 2, \ldots$ Then v_i is continuous, hence $p(\cdot)$-quasicontinuous, and $v_i \in W_0^{1,p(\cdot)}(\Omega_i) \subset W_0^{1,p(\cdot)}(\Omega)$. Thus by Lemma 3.6 we have $u_i v_i \in W_0^{1,p(\cdot)}(\Omega) = W_0^{1,p(\cdot)}(\Omega \setminus N)$, $i = 1, 2, \ldots$ Fix i. If w is such a $p(\cdot)$-quasicontinuous function that $w = u_i v_i$ a.e. in $\Omega \setminus N$, then $w = u_i v_i$ a.e. in Ω since $|N| = 0$. Lemma 2.1 (i) implies that $w = u_i v_i$ $p(\cdot)$-q.e. in Ω. In particular, $w = u_i v_i > 0$ $p(\cdot)$-q.e. in $N \cap \Omega_i$. On the other hand, since $u_i v_i \in W_0^{1,p(\cdot)}(\Omega \setminus N)$, we may define $w = 0$ $p(\cdot)$-q.e. in $\mathbb{R}^n \setminus (\Omega \setminus N)$. In particular, we have $w = 0$ $p(\cdot)$-q.e. in $N \setminus \Omega_i$. This is possible only if

$C_{p(\cdot)}(N \setminus \Omega_i) = 0$ for every $i = 1, 2, \ldots$ and hence properties (ii) and (vii) of the Sobolev $p(\cdot)$-capacity yield

$$C_{p(\cdot)}(N) = C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty}(N \cap \Omega_i)\right) \leqslant \sum_{i=1}^{\infty} C_{p(\cdot)}(N \cap \Omega_i) = 0.$$

This completes the proof.

4. The $p(\cdot)$-Poincaré inequality

We write p_A^+ to denote the essential supremum of the function p in a set $A \cap \Omega$ and p_A^- to denote the essential infimum. If $p_\Omega^+ < \infty$ and if there exists $\delta > 0$ such that for every $x \in \Omega$ either

$$p^-_{B(x,\delta)} \geqslant n \tag{4.1}$$

or

$$p^+_{B(x,\delta)} \leqslant \frac{n \cdot p^-_{B(x,\delta)}}{n - p^-_{B(x,\delta)}} \tag{4.2}$$

holds, then the variable exponent p is said to satisfy the *jump condition* in Ω with constant δ. Roughly, the jump condition guarantees that p does not jump too much locally in Ω. We set

$$p^*_{B(x,\delta)} = \begin{cases} \frac{n \cdot p^-_{B(x,\delta)}}{n - p^-_{B(x,\delta)}}, & \text{if } p^-_{B(x,\delta)} < n, \\ p^+_{B(x,\delta)}, & \text{if } p^-_{B(x,\delta)} \geqslant n. \end{cases}$$

Note that if Ω is bounded and if p is continuous in $\overline{\Omega}$, then p satisfies the jump condition in Ω with some $\delta > 0$.

THEOREM 4.1. [The $p(\cdot)$-Poincaré inequality] *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that p satisfies the jump condition in Ω with $\delta > 0$. Then the inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leqslant C \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \tag{4.3}$$

holds for every $u \in W_0^{1,p(\cdot)}(\Omega)$. Here the constant C depends on the function p, $|\Omega|$, $\operatorname{diam}(\Omega)$, δ and the dimension n.

Proof. Since $\overline{\Omega}$ is compact, there exist $x_1, \ldots, x_j$ such that

$$D \subset \bigcup_{i=1}^{j} B(x_i, \delta).$$

We write $B_i = B(x_i, \delta)$ and denote by χ_i the characteristic function of B_i. Let $\tilde{u}$ be the canonical representative of u. By the triangle inequality and Theorem 3.4 we obtain

$$\begin{aligned}
\|u\|_{L^{p(\cdot)}(\Omega)} &= \|\tilde{u}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant \Big\|\tilde{u} \sum_i \chi_i\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant \sum_{i=1}^{j} \|\tilde{u}\chi_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&= \sum_{i=1}^{j} \|\tilde{u}\|_{L^{p(\cdot)}(B_i)} \leqslant (1+|\Omega|) \sum_{i=1}^{j} \|\tilde{u}\|_{L^{p^*_{B_i}}(B_i)} \\
&\leqslant (1+|\Omega|) \sum_{i=1}^{j} \Big(\|\tilde{u} - \tilde{u}_{B_i}\|_{L^{p^*_{B_i}}(B_i)} + |\tilde{u}_{B_i}|\, \|1\|_{L^{p^*_{B_i}}(B_i)} \Big)
\end{aligned}$$

The classical Sobolev-Poincaré inequality in the ball and [19, Theorem 2.8] imply that

$$\begin{aligned}
\|\tilde{u} - \tilde{u}_{B_i}\|_{L^{p^*_{B_i}}(B_i)} &\leqslant C(n, p^-_{B_i}, p^+_{B_i})(1+|B_i|)\|\nabla\tilde{u}\|_{L^{p^-_{B_i}}(B_i)} \\
&\leqslant C(n, p^-_{B_i}, p^+_{B_i})(1+|B_i|)^2\|\nabla\tilde{u}\|_{L^{p(\cdot)}(B_i)} \\
&\leqslant C(n, p^-_{B_i}, p^+_{B_i})(1+C(n)\delta^n)^2\|\nabla u\|_{L^{p(\cdot)}(\Omega)}
\end{aligned}$$

for every $i = 1, \dots j$. The classical Poincaré inequality implies that

$$\begin{aligned}
|\tilde{u}_{B_i}| &\leqslant \frac{C(n)}{\delta^n} \int_\Omega |u|\, dx \leqslant \frac{C(n)}{\delta^n} \operatorname{diam}(\Omega) \int_\Omega |\nabla u|\, dx \\
&\leqslant \frac{C(n)}{\delta^n} \operatorname{diam}(\Omega)(1+|\Omega|)\|\nabla u\|_{L^{p(\cdot)}(\Omega)}
\end{aligned}$$

again for every $i = 1, \dots j$. Since $\|1\|_{L^{p^*_{B_i}}(B_i)}$ depends only on $p^*_{B_i}$ and $|B_i|$, the previous inequalities imply the $p(\cdot)$-Poincaré inequality.

Remark. The condition $p^+_{B(x,\delta)} \leqslant \frac{n \cdot p^-_{B(x,\delta)}}{n - p^-_{B(x,\delta)}}$ for the exponent p is the best possible, see [12, Example 2.5].

Remark. When $1 < p^-_\Omega \leqslant p^+_\Omega < n$ and p is Lipschitz continuous, Edmunds and Rákosník have proven a Poincaré type inequality for functions in $W^{1,p(\cdot)}(\Omega)$ compactly supported in Ω, see [7, Lemma 3.1].

5. $p(\cdot)$-Dirichlet energy integral minimizers

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $w \in W^{1,p(\cdot)}(\Omega)$. The energy operator corresponding to the boundary value function w, acting on the space $W_0^{1,p(\cdot)}(\Omega)$ is defined by

$$I_{\Omega,w}^{p(\cdot)}(u) = \int_\Omega |\nabla u(x) + \nabla w(x)|^{p(x)}\,dx. \tag{5.1}$$

The general problem is to find a function that minimizes values of the operator $I_{\Omega,w}^{p(\cdot)}$ acting on the space $W_0^{1,p(\cdot)}(\Omega)$. It is clear that this problem is equivalent with the $p(\cdot)$-Dirichlet energy minimizing problem stated in the introduction. Here we use the same methods as in [26] to prove that a minimizer exists. The following is a well known lemma in functional analysis, see for example [18, Theorem 2.1].

LEMMA 5.1. *Let $\mathcal{B}$ be a reflexive Banach space. If $I : \mathcal{B} \to \mathbb{R}$ is a convex, lower semicontinuous and coercive operator, then there is an element in $\mathcal{B}$ that minimizes I.*

The operator I is said to be *convex* if for all $t \in [0, 1]$ and each pair $u, v \in \mathcal{B}$ the inequality $I(tu + (1-t)v) \leqslant tI(u) + (1-t)I(v)$ is satisfied. The operator I is *lower semicontinuous* if $I(u) \leqslant \liminf_{i\to\infty} I(u_i)$ whenever u_i is a sequence of elements in $\mathcal{B}$ converging to u, and *coercive* if $I(u_i) \to \infty$ whenever $\|u_i\|_{\mathcal{B}} \to \infty$.

THEOREM 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that p satisfies the jump condition in Ω and that $1 < p^- \leqslant p^+ < \infty$. Then there exists a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that*

$$I_{\Omega,w}^{p(\cdot)}(u) = \inf_{v \in W_0^{1,p(\cdot)}(\Omega)} I_{\Omega,w}^{p(\cdot)}(v). \tag{5.2}$$

Proof. By Theorems 3.1 and 3.5 we know that $W_0^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space. Since $x \mapsto x^p$ is convex for every fixed $1 < p < \infty$, we find that

$$(t|u(x)| + (1-t)|v(x)|)^{p(x)} \leqslant t|u(x)|^{p(x)} + (1-t)|v(x)|^{p(x)} \tag{5.3}$$

for every $0 < t < 1$, every $x \in \Omega$, and every $u, v \in W_0^{1,p(x)}(\Omega)$. Thus the operator $I_{\Omega,w}^{p(\cdot)}$ is convex.

Let (u_i) be a sequence of functions in $W_0^{1,p(\cdot)}(\Omega)$ which converge to $u \in W_0^{1,p(\cdot)}(\Omega)$. Then $\nabla(u_i + w)$ converges to $\nabla(u + w)$ in $L^{p(\cdot)}(\Omega)$. Since $p^+ < \infty$, we obtain by [19, Theorem 2.4] that

$$\varrho_{p(\cdot)}(\nabla(u_i + w) - \nabla(u + w)) \to 0$$

as $i \to \infty$. By [11, Lemma 2.6] this yields

$$\varrho_{p(\cdot)}(\nabla(u_i + w)) \to \varrho_{p(\cdot)}(\nabla(u + w)),$$

as $i \to \infty$. Hence the operator $I^{p(\cdot)}_{\Omega,w}$ is lower semicontinuous.

If $\|u_i\|_{W_0^{1,p(\cdot)}(\Omega)} \to \infty$, then inequality (4.3) implies that $\|\nabla u_i\|_{L^{p(\cdot)}(\Omega)} \to \infty$, which yields $\|\nabla u_i + \nabla w\|_{L^{p(\cdot)}(\Omega)} \to \infty$ as $i \to \infty$. Since $p^+ < \infty$, we obtain $I^{p(\cdot)}_{\Omega,w}(u_i) \to \infty$ as $i \to \infty$, so the operator $I^{p(\cdot)}_{\Omega,w}$ is coercive.

Now the theorem follows by Lemma 5.1.

THEOREM 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that p satisfies the jump condition in Ω and that $1 < p^- \leqslant p^+ < \infty$. The $p(\cdot)$-quasicontinuous representative $\tilde{u}$ of the minimizing function u in (5.2) is unique up to set of zero $p(\cdot)$-capacity.*

Proof. The following proof is a modification of the proof of [13, Theorem 5.27]. Assume that u_1 and u_2 are two minimizers of (5.2) with $|\{\nabla u_1 \neq \nabla u_2\}| > 0$. If $\nabla u_1(x) \neq \nabla u_2(x)$, we obtain as in (5.3) that

$$\Big(\frac{1}{2}|\nabla u_1(x)| + \frac{1}{2}|\nabla u_2(x)|\Big)^{p(x)} < \frac{1}{2}|\nabla u_1(x)|^{p(x)} + \frac{1}{2}|\nabla u_2(x)|^{p(x)}.$$

We set $v = \frac{1}{2}(u_1 + u_2)$. The previous inequality implies that

$$I^{p(\cdot)}_{\Omega,w}(v) < \frac{1}{2}I^{p(\cdot)}_{\Omega,w}(u_1) + \frac{1}{2}I^{p(\cdot)}_{\Omega,w}(u_2) = \inf_{u \in W_0^{1,p(\cdot)}(\Omega)} I^{p(\cdot)}_{\Omega,w}(u),$$

which is a contradiction. Therefore $|\{\nabla u_1 \neq \nabla u_2\}| = 0$. Since $u_1 - u_2 \in W_0^{1,p(\cdot)}(\Omega)$, we obtain by the Poincaré inequality (4.3) that

$$\|u_1 - u_2\|_{L^{p(\cdot)}(\Omega)} \leqslant C\|\nabla u_1 - \nabla u_2\|_{L^{p(\cdot)}(\Omega)} = 0,$$

and hence $u_1 = u_2$ for a.e. $x \in \Omega$. Let $\tilde{u}_1$ and $\tilde{u}_2$ be the $p(\cdot)$-quasicontinuous representatives of u_1 and u_2. Then $\tilde{u}_1 = \tilde{u}_2$ for almost every $x \in \Omega$ and Lemma 2.1 implies that $\tilde{u}_1 = \tilde{u}_2$ $p(\cdot)$-quasieverywhere in Ω.

THEOREM 5.4. *Let $1 < p^- \leqslant p^+ < \infty$ and $u \in W_0^{1,p(\cdot)}(\Omega)$. The following two conditions are equivalent:*

(i) *The function u minimizes the operator $I^{p(\cdot)}_{\Omega,w}$;*

(ii) *The function u is such that*

$$\int_\Omega p(x)|\nabla u(x) + \nabla w(x)|^{p(x)-2}(\nabla u(x) + \nabla w(x)) \cdot \nabla(v(x) - u(x))\, dx \geqslant 0$$

for every $v \in W_0^{1,p(\cdot)}(\Omega)$.

Proof. This proof is a modification of [13, Theorem 5.13]. First we prove that (i) implies (ii). We fix $v \in W_0^{1,p(\cdot)}(\Omega)$ and set $\phi = v - u$ and $f = u + w$. Let $0 < \varepsilon \leqslant 1$. Since $u + \varepsilon\phi \in W_0^{1,p(\cdot)}(\Omega)$, we obtain

$$I^{p(\cdot)}_{\Omega,w}(u) \leqslant I^{p(\cdot)}_{\Omega,w}(u + \varepsilon\phi),$$

and therefore

$$\int_\Omega \frac{|\varepsilon\nabla\phi(x) + \nabla f(x)|^{p(x)} - |\nabla f(x)|^{p(x)}}{\varepsilon}\, dx \geqslant 0. \tag{5.4}$$

Because

$$\lim_{\varepsilon\to 0} \frac{|\varepsilon\nabla\phi(x) + \nabla f(x)|^{p(x)} - |\nabla f(x)|^{p(x)}}{\varepsilon} = p(x)|\nabla f(x)|^{p(x)-2}\nabla f(x)\cdot\nabla\phi(x) \tag{5.5}$$

for almost every $x \in \Omega$, the condition (ii) follows from the Lebesgue dominated convergence theorem provided that we find a L^1-majorant independent of ε for the integrand in (5.4).

By the mean-value theorem there exists $\varepsilon' \in (0, \varepsilon)$ such that

$$\frac{|\varepsilon\nabla\phi(x) + \nabla f(x)|^{p(x)} - |\nabla f(x)|^{p(x)}}{\varepsilon} = p(x)|\varepsilon'\nabla\phi(x) + \nabla f(x)|^{p(x)-2}(\varepsilon'\nabla\phi(x) + \nabla f(x))\cdot\nabla\phi(x),$$

and thus

$$\left|\frac{|\varepsilon\nabla\phi(x) + \nabla f(x)|^{p(x)} - |\nabla f(x)|^{p(x)}}{\varepsilon}\right| \leqslant p^+2^{p^+}(|\nabla f(x)|^{p(x)-1}|\nabla\phi(x)| + |\nabla\phi(x)|^{p(x)}) = g(x).$$

Since $u, v, w \in W^{1,p(\cdot)}(\Omega)$, the Hölder inequality, [19, Theorem 2.1], implies that $g \in L^1(\Omega)$ is the desired majorant.

Then we prove that (ii) implies (i). Since

$$|\xi_2 + t(\xi_1 - \xi_2)|^p = |(1-t)\xi_2 + t\xi_1|^p \leqslant (1-t)|\xi_2|^p + t|\xi_1|^p$$

for $0 < t < 1$, we obtain by setting $\xi_1 = \nabla\phi + \nabla f$ and $\xi_2 = \nabla f$ that

$$\frac{|t\nabla\phi(x) + \nabla f(x)|^{p(x)} - |\nabla f(x)|^{p(x)}}{t} \leqslant |\nabla v(x) + \nabla w(x)|^{p(x)} - |\nabla u(x) + \nabla w(x)|^{p(x)}.$$

Letting $t \to 0$ this yields by (5.5) that

$$|\nabla v(x) + \nabla w(x)|^{p(x)} - |\nabla u(x) + \nabla w(x)|^{p(x)} \geqslant p(x)|\nabla f(x)|^{p(x)-2}\nabla f(x)\cdot\nabla\phi(x)$$

and hence $I^{p(\cdot)}_{\Omega,w}(u) \leqslant I^{p(\cdot)}_{\Omega,w}(v)$ for every $v \in W^{1,p(\cdot)}(\Omega)$.

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