DEVELOPMENT AND AWARENESS OF FUNCTION UNDERSTANDING IN FIRST YEAR UNIVERSITY STUDENTS

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This article presents results from a longitudinal study on the development of first year university students function concept, and of their awareness of this development. Our study indicates that students' function concepts do develop. However, there were some inconsistencies between students' actual development and their awareness of it.

Keywords: function, conceptual development, self-awareness, teacher education

INTRODUCTION

Too often our students complain that their mathematical training lacks relevance for their future career as teachers. Others value of the mathematical component of their study as a personal intellectual pursuit of "real mathematics" before returning to "school mathematics". We believe this is not good enough.

We see the mathematics studies as an opportunity to enjoy challenging yourself with mathematical problems; learn to value critical thinking and argumentation over dogma; and to observe your own development as well as that of your peers. In short, to develop mathematical habits of mind, as Cuoco, Goldenberg and Mark (1996) put it. We do not have space here to motivate our values, but some arguments are presented in the mentioned article and its references.

Goulding, Hatch and Rodd (2003) investigated what British students actually carry with them from their bachelor's degrees to the teacher training (PGCE) in a retrospective study. Of the seven emergent themes, only the two, "Understanding: Cognitive Demands" and "The Value and Nature of Mathematics" pertain to the issues mentioned in the previous paragraph; and within these categories many responses were actually negative, i.e., the students' views had changed in undesirable directions (e.g., "for exams only").

In 2010, we started a project aiming to investigate and address the situation at our university. Our focus on argumentation might suggest studying student teachers' developing understanding of proof (a fashionable topic of late, see, e.g., Stylianides & Ball, 2008). Proof is a topic where there is quite a drastic difference between school and university and it is not uncommon to find transition courses explicitly dealing with proof (Hanna & de Villiers, 2012). We chose a concept from the opposite end of the spectrum in which the development is subtle and slow, and may go unnoticed to the students. The concept of function satisfies this criterion (Carlson, 1998) and has been

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indicated as a one of the central topics for mathematics teacher education in the *standards* endeavor (Conference Board of the Mathematical Sciences, 2001).

We are developing a small course running alongside regular mathematics courses to help students reflect on the relevance of mathematics classes for their teaching careers. To support this work we started in the fall of 2011 a longitudinal study on students' development of the function concept and their self-awareness of said development. Here we report our findings from the first year of follow-up. In order to formulate our research questions we first present briefly some background material.

BACKGROUND AND RESEARCH QUESTIONS

Tall and Vinner (1981) introduced a theory distinguishing between the *concept image* and the *concept definition*. The concept image consists of all cognitive structures in one's mind that is associated with a given concept. Early on, this framework was applied also in the study of the function concept: it was found that a student's image of functions and his/her professed definition might be distinct and even partially contradictory (Vinner & Dreyfus, 1989). There are several ways to approach functions, each emphasizing different aspects. Therefore, one expects the function concept image to be rather complex and multifaceted. Understanding these approaches is important for mathematicians as well as mathematics subject teachers.

To understand the connections between different representations of functions and to move between them is a major challenge for students. Several studies indicate difficulties when describing mathematically identical concepts in different ways (see, e.g., Even, 1998; Bayazit, 2011). Students often lack the ability to present the same function using different representations. Also, the intuitive notion of the function often relies on an explicit algebraic formulation (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Tall & Vinner, 1981). Functions and related notions are often treated as if they were only symbols and representations instead of proper mathematical objects (Eisenberg, 1990). Although most students can visualize simple functions by drawing their graphs, it seems that they often lack ability to link the graph and the algebraic representations of the function (Vinner & Dreyfus, 1989).

Carlson (1998) found in a cross-sectional study that even A-level students take a long time to acquire solid understanding of functions: second year undergraduate students still showed many flawed conceptions. The complexity of the concept and its slow development suggest that it might be quite likely that some or even most students fail to notice their developing function concept during their mathematics studies. This is especially problematic for teachers since they then miss out on the opportunity to introspect on the difficulty of coming to terms with functions.

As far as we know, there is little research about students' self-awareness of long-term development of mathematical concepts. Hence, the next part is based on theories of reflection and metacognition in a more general context.

Generally, the concept of reflection can be understood as the conscious consideration of personal experiences; in our context, "reflection is characterized by distancing oneself from the action of doing mathematics" (Bjuland, 2004, p. 201). Bjuland (2004) points out that a central feature of reflection is the time frame involved. As we see it, both long and short time frames are necessary for self-awareness: immediate and short-term reflection is needed to notice the different aspects of the concept whereas reflection over an extended period of time is needed to compare these observations at different points in time and thus become aware of the development.

We have not found any studies in the context of learning mathematics in which this issue has been considered. For instance, Schoenfeld's (e.g., 1992) theory of metacognition for problem solving is centered in the here and now. The longest time frame is the "looking back" stage when one evaluates the steps taken in solving a given problem.

A general framework for addressing such issues was proposed by Schraw and Moshman (1995). According to them, a person's *metacognitive theory* integrates metacognitive knowledge and experiences, and can be used to explain and predict his/her cognitive behavior. They argue that one aspect in which individuals' metacognitive theories differ is the extent to which they are explicit (i.e., the extent to which one is aware of possessing such a theory).

Schraw and Moshman (1995, p. 360) point out that explicit knowledge about your own cognition makes it possible to reflect on your performance and to use this information to modify your future performance and thinking. They also seem to imply that individuals' metacognitive theories gradually develop via awareness of changes and reflection on them.

In contrast, tacit theories are developed without conscious reflection, based on personal experience or adaptations from others, and therefore it can be difficult to notice and report one's own development to others. Since an individual with a tacit metacognitive theory is not readily aware of either the theory itself or evidence that supports or refutes it, it can be very difficult to change the theory even when the theory is maladaptive and the individual is explicitly encouraged to do so. (Schraw & Moshman, 1995)

Further, the lack of conscious reflection might cause metacognitive knowledge and regulation to lack transferability between task-types. To come back to mathematics education, we think that an explicit metacognitive theory allows you to reflect on the consistency and connections between your concept definitions and concept images.

Even though we have presented reasons for the hypothesis that students might not be aware of the development of their function concept, we must of course also face the reality that some students may not notice any development since there simply is none to notice. Hence we divide our research question in two:

1. How do students' function concepts develop during the first year of study?
2. To what extent are students aware of these developments?

METHODOLOGY

Instruments

We used two questionnaires, both based mainly on Carlson's tests (1998). Our first test consisted of A1, A5, A6, A7, A14, B3 and a modification of A13 in which the linear graph was replaced by a piecewise linear one (the tasks can be found in Appendix A of Carlson's paper). The second test consisted of A2b, B2, A6, A7, and A8. The questions had been translated into Finnish and tested by P. Hästö and M. Leinonen for an earlier study (unpublished). The tests were scored following Carlson's (1998) rubrics with a few minor changes. For the modification of A13 we developed our own rubrics.

The second test also included six "self-awareness" claims which were answered on a five-point Likert scale and two other questions which are not analysed here. The claims were

- Q1. Examples of functions at the university are similar to those in high school.
- Q2. In high school it was clear to me what was meant by a function.
- Q3. It is (presently) clear to me what is meant by a function at the university.
- Q4. My understanding of functions has changed while at the university.
- Q5. I have pondered over my understanding of functions during my university studies.
- Q6. During my university studies there have appeared examples of functions contrary to my function concept.

Participants

The first test was administered in the second week of the first period in a class typically taken by first year students, both math majors and other students with a math component (mainly physics and chemistry majors). The second test was administered in the first week of the third (out of four) period in an analysis class typically taken by first year majors and second or later year minors. There were 98 participants in the first test and 64 in the second. Of these 38 participated in both tests. Two questions were repeated from the first to the second test; it should be noted that students got no feedback on the tests and solutions were also not distributed.

RESULTS AND ANALYSIS

Our first observation is that students in our follow-up became significantly better at the tasks. Their scores in the two repeated tasks improved, on average, over 2.5 points (the maximum score being 5 points on each task). To compare the other tasks we use Carlson's Group 2 (A-students from a 2nd year calculus course) as a reference. In the first test our students' scores were consistently (and sometimes considerably) below the reference, whereas in the second test exceeded the reference, except in one task.

This strongly suggests that there was on average much development in the function concept. One possible critique is that our students might have a different profile than Carlson's, so that using that as a baseline is a confounding factor. However, in general the relative difficult of a task in our tests and Carlson's test correlated strongly. Therefore this is unlikely to be a significant bias.

We tried to simplify and clarify the information through quantitative data reduction techniques. In particular, we looked at cross-correlations and did several test runs of principal component analyses. For instance, we looked for factors corresponding to graphical and analytic components of the function concept. Although there is a fairly clear division of the tasks into graphical and non-graphical ones, such components did not emerge from the data.

To get a view on individual improvement, we cross-tabulated the sum of scores from the two tasks included in both our tests, displayed in Table 1. From the table we see that all save three students improved their score, many by more than 5 points. The three exceptions in fact all dropped from 1 to 0 points. We have done a qualitative analysis of the results of four students whose results improved the most.

Table 1. Results from Tasks A6 and A7 in the two tests.

	Retake						
Initial	0	2	4–5	6–7	9	10	Total
0	1	2	2	2	0	4	11
1–3	3	0	1	2	2	11	19
5–7					1	4	5
10						3	3
Total	4	2	3	4	3	22	38

Based on the results in the two repeated tasks, we divided the students into three groups: LOW (less than 5 points in both tests), RISE (at least 5 point improvement between tests) and HIGH (at least 5 points in both tests). Three students belonged to none of these groups and are excluded from the next analysis.

Table 2. Means of opinion scores. The shaded cells are discussed in more detail below.

Group	*n*	Q1	Q2	Q3	Q4	Q5	Q6
LOW	8	.3	1.0	1.0	.3	1.0	–0.4
RISE	19	.3	.3	.8	1.0	.1	–0.3
HIGH	8	.0	.9	1.4	.8	.9	–1.3
Total	35	.2	.6	1.0	.8	.5	–0.5
All	64	.2	.6	1.1	.8	.5	–0.4

For these three groups we calculated the means of the answers to the six questions mentioned on page 4. The results appear in Table 2. The answers were coded so that "strongly agree" has value 2 and "strongly disagree" –2. From the table we see that there were no radical differences between the groups. In fact, in a one-way ANOVA even the biggest between-group difference, in Question 6, was only approaching statistical significance ($p = 0.085$).

Nevertheless, we combine indications from several answers to form a tentative conclusion, or, if you will, a hypothesis for further study. Q4 (change of understanding) is consistent with Q2 (high school) and Q3 (university) in that group RISE and HIGH now rate their understanding higher than in high school, and correspondingly rate it as having changed more that group LOW.

Group HIGH has consistently higher view of their function understanding than group RISE, which is consistent with test scores. However, group LOW find that they have had a fairly clear understanding of functions from high school, which has not changed. It appears that these students have an erroneous and overly optimistic view of the adequacy of their function concept.

Question 5 (pondering) is somewhat of a dilemma: the group with the least pondering (RISE) has improved most drastically. Finally, Q6 (new examples) is consistent with the groups in that the consistently high-scoring individuals found fewer examples which did not fit into their conception of function. Surprisingly, across all students there was a tendency to disagree with the statement that they had encountered new types of functions at the university. This may be due to the inclusion of the technical term "function concept" in the question.

Qualitative analysis

We identified four students as major improvers and did a more thorough data-driven analysis of their answers. Our first observation in the qualitative analysis of the two questionnaires was the obvious improvement in students' understanding of "the language of functions", as Carlson (1998) puts it.

Initially the scrutinized students demonstrated great difficulties in translating a verbal description of a function into the corresponding algebraic notation: e.g., when prompted to give an example of a function all of whose values are equal, one of them wrote "for example, $f(x):\ x=2$", and another one started to solve the equation $x-y=y-x$, and after a couple of steps concluded that $x=y$. Apparently these students understood neither what is meant by "the value of a function" nor the usual notation by which functions are defined. However, three of these students showed very little difficulty in algebraic manipulation while expressing the diameter of a circle as function of its area (the fourth one left it blank), but all of them neglected to sketch the graph. This is possibly due to the Finnish curriculum in which a considerable attention is paid to solving equations via algebraic manipulation.

It also seems that these students thought that all functions must be defined by a single algebraic formula. When asked to give an example of a function which assigns to every number different from 0 its square and to 0 assigns 1, two of the students simply wrote "x^2" while two left it blank. In the second part of the test only a few students (and none of these four) had difficulty in defining functions in a piece-wise manner.

Although there is little doubt about the students' improved writing in a formal mathematical style, there is some doubt about the depth of this improvement. For example, in the challenging task B2, one of the four students started the solution by defining $f(x) = x^2$ and then $f(y) = y^2$, as though the second did not follow from the first (a misconception also documented, e.g., by Sajka, 2003).

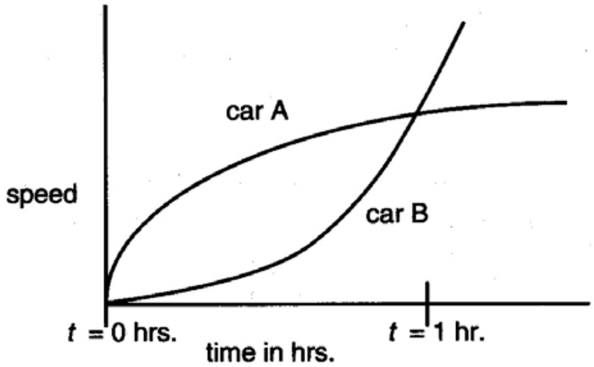

Figure 1. The graph from Task A8.

The difficulty in interpreting functional information from a given graph (rather than interpreting it literally) appears to be somewhat persistent. In Task A8 (second test) the students were given a graph illustrating speed of two cars, A and B, as a function of time (see Fig. 1). All the scrutinized students demonstrated very little difficulty in interpreting static graphical information or (relatively standard) dynamic information (comparing speed and acceleration of the two cars, respectively), yet all of them failed in Task A8d, in which they were asked to describe the relative position of the two cars over a time interval. Although they noticed that the car A was driving faster than B for the whole time, they still concluded that B was catching up with A because B had accelerated so much whereas A had driven with nearly constant speed. This is possibly due to the non-standard framing of the question.

DISCUSSION

Our results are consistent with Carlson's study (1998). In our first test students had difficulties very similar to those in Carlson's Group 1 (A-students from a college algebra course). For instance, they had problems understanding the language of functions, such as defining functions piecewise. In the second test, however, our students seemed to have the same kind of difficulties as Group 2 (A-students from a 2nd year calculus course), e.g., interpreting dynamic functional information over an interval. Interestingly, although many participants in our first test apparently thought that a function must be defined via a single algebraic formula, this was not the case in the second test; in contrast Carlson's Group 2 subjects still retained this misconception.

Our tests did not uncover a graphical and analytical component in the sense that there would have been greater of correlation within the groups of graphical and non-graphical tasks. This may be due to floor and ceiling effects in the task scores which led to skewed distributions; we found several nonlinear relationships between tasks where only students with 5/5 points on problem A got any point on problem B. This means that the difficulty range of the problems was not optimal. On the other hand, it seems that also in most other studies the graphical and analytical components are postulated rather than recovered from the data. Our observations of graphical components are limited to the qualitative analysis. Here our findings mirror Bayazit's (2011) results of students' limited abilities to shift from a graphical expression to algebraic expression, with most depending on algebraic expressions.

Functions in first year mathematics courses appear mainly through definitions and further properties (bijectivity, injectivity, etc.) whereas little explicit focus is placed on development of intuitions (and the construction of concept images). Nevertheless, the quantitative analysis showed that students made great progress in some type of function tasks. The qualitative analysis showed that this progress was tenuous in the sense that there was a tendency to relapse into old ideas when faced with difficulty. This is consistent with Carlson's (1998) finding that students do not use the newest tools in their conceptual arsenal proficiently. Unfortunately, we cannot say how much of the improved scores is due to conceptual change and how much is a consequence of adopting the formal language used in university courses.

Some answers indicate that the concept of variable is also problematic. For instance, one student introduced a function by defining it for several variables ($f(x) = x^2$ and $f(y) = y^2$) instead of just once. We do not know to what extent our students' problems are due to deficiencies in such prerequisite concepts as that of the variable.

About a quarter of the participants performed poorly and did not improve their performance between the first and second tests. Interestingly, these students on average had a high estimation of their understanding of functions — in fact, they estimated that the function concept had been clear in high school and had remained clear since then. If they do not come to terms with the need to develop their function concept, this might cause serious problems for their studies and teaching careers. From course grades and other feedback these students must know that their studies are not progressing as well as they could be. Therefore they seem to be misattributing this to some other factors than a weak function concept. This suggests that they do not have effective means of checking their metacognitive theories against reality, which indicates a persistent tacit metacognitive theory.

A second group, consisting of more than half of the participants, improved between the tests. Moreover, this group realistically estimated their function concept as having been average and then having improved. Unfortunately, we only asked participants to rate the clarity of the concept in the second test. Thus we have only a retrospective estimation of what the students thought after high school. Do they rate the conceptual clarity as low because they now understand it better? Or is it the case that they were unsatisfied with their understanding at the beginning of the studies, and were thus more open to new influences? This remains a question for further study.

If the retrospective rating is accurate, then it would allow us to design a diagnostic test for the beginning of studies which detects students at risk of belonging to the LOW category as those with low performance and high estimation of their function concept. Alcock and Simpson (2004) found that this kind of profile (low performance and high confidence) is typical for some graphically oriented students. Whether there is such a link in our case also remains to be determined. Another open question is whether this profile is context specific, or whether the students follow the same path in all areas of mathematics.

The results of the first year of our study in general agreed with our expectations. Students' concepts improved and most were said as much when prompted. Whether this group contains a subgroup of improvers who do not notice the change is a question for future study. Almost a quarter of the students did not improve, and showed low self-awareness of their situation. To probe the views of these groups, future versions of the study should also include an interview component.

REFERENCES

Alcock, L., & Simpson, S. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 57,* 1–32. doi: 10.1023/B:EDUC.0000047051.07646.92

Bayazit, I. (2011). Prospective teachers' inclination to single representation and their development of the function concept. *Educational Research and Reviews, 6*, 436–446.

Bjuland, R. (2004). Student teachers' reflections on their learning process through collaborative problem solving in geometry. *Educational Studies in Mathematics, 55,* 199–225. doi: 10.1023/B:EDUC.0000017690.90763.c1

Breidenbach, D., Dubinsky, E.D., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247–285. doi: 10.1007/BF02309532

Carlson, M.P. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A.H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education III* (114–162). Providence, RI, USA: American Mathematical Society.

Conference Board of the Mathematical Sciences (2001). The mathematical education of teachers. *Issues in Mathematics Education, 11*. Providence, RI, USA: American Mathematical Society.

Cuoco, A., Goldenberg, E.P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematical curricula. *Journal of Mathematical Behavior, 15,* 375–402. doi: 10.1016/S0732-3123(96)90023-1

Eisenberg, T. (1991). Function and associated learning difficulties. In D.O. Tall (Ed.), *Advanced mathematical thinking* (140–152). Dordrecht: Kluwer Academic Publishers.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17,* 105–121. doi: 10.1016/S0732-3123(99)80063-7

Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: Its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education, 6,* 361–393. doi: 10.1023/A:1026362813351

Hanna, G., & de Villiers, M. (Eds.) (2012). Proof and proving in mathematics education. *New ICMI Study Series, 15*. Heidelberg, Germany: Springer-Verlag. doi: 10.1007/978-94-007-2129-6.

Sajka, M. (2003). A secondary school student's understanding of the concept of function – A case study. *Educational Studies in Mathematics, 53,* 229–254. doi: 10.1023/A:1026033415747

Schoenfeld, A.H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (334–370). New York: MacMillan.

Schraw, G., & Moshman, D. (1995). Metacognitive theories. *Educational Psychology Review, 7,* 351–371. doi: 10.1007/BF02212307

Stylianides, A.J., & Ball, D.L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11,* 307–332. doi: 10.1007/s10857-008-9077-9

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12,* 151–169. doi: 10.1007/BF00305619

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20,* 356–366.