

GENERALIZED ORLICZ SPACES AND RELATED PDE

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ABSTRACT. We prove the boundedness of the maximal operator in generalized Orlicz spaces defined on subsets of \mathbb{R}^n . The proof is based on an extension result for Φ -functions. We study generalized Sobolev–Orlicz spaces and establish density of smooth functions and the Poincaré inequality. As applications we establish the existence of solutions of the φ -Laplace equation with zero and non-zero right-hand side. Further, we systematize assumptions for Φ -functions and prove several basic tools needed for the study of differential equations of generalized Orlicz growth.

1. INTRODUCTION

Generalized Orlicz spaces $L^{\varphi(\cdot)}$ have been studied since the 1940's. A major synthesis of functional analysis in these spaces is given in the 1983-monograph of Musielak [27] hence the alternative name Musielak–Orlicz spaces. These spaces are similar to Orlicz spaces, but defined by a more general function $\varphi(x, t)$ which may vary with the location in space: the norm is defined by means of the integral

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)|) dx,$$

whereas in an Orlicz spaces φ would be independent of x , $\varphi(|f(x)|)$. When $\varphi(t) = t^p$ we obtain classical Lebesgue spaces L^p .

The special case $\varphi(x, t) := t^{p(x)}$, so-called variable exponent spaces $L^{p(\cdot)}$, and corresponding differential equations with non-standard growth have been vigorously studied in recent years [10, 11, 17]. The reason that variable exponent spaces thrived while little was done in generalized Orlicz spaces was the belief that many classical results can be obtained in the former setting but not the latter. However, this belief has been challenged recently, based on new techniques that were developed and perfected in the context of variable exponent spaces [12, 15, 19, 20, 23, 24, 25, 28].

In addition to being a natural generalization which covers results from both variable exponent and Orlicz spaces, the study of generalized Orlicz spaces can be motivated by applications to image processing [2, 6, 16], fluid dynamics [33, 34] and differential equations.

Regarding regularity theory of differential equations, Giannetti and Passarelli di Napoli [13] and Ok [29] as well as Baroni, Colombo and Mingione [3, 4, 7, 8, 9] studied the minimization problems

$$\min_u \int |\nabla u|^{p(x)} \log(e + |\nabla u|) dx \quad \text{and} \quad \min_u \int |\nabla u|^p + a(x) |\nabla u|^q dx,$$

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respectively. They showed that the regularity of the minimizer depends on the regularity of the exponent p and the weight a . Both are special cases of generalized Orlicz growth.

In the function space setting the first steps from $L^{p(\cdot)}$ were similarly Φ -functions of type $t^{p(\cdot)} \log(e+t)^{q(\cdot)}$ like that of Giannetti, Passarelli di Napoli and Ok. Such Φ -functions were studied e.g. in [21, 26]. To our mind, a better approach is to develop stronger tools and to move directly to study general Φ -functions including, among others, those studied by Colombo and Mingione. Our assumptions (A0)–(A2) have been shown to correspond to known optimal conditions in these special cases [19].

In this article we systematize assumptions for Φ -functions and prove several basic tools needed for the study of differential equations of generalized Orlicz growth (Section 2–4). In order to use function spaces in subsets of \mathbb{R}^n , we study extensions of Φ -functions (Proposition 5.2). We consider generalized Sobolev–Orlicz spaces, including density of smooth functions (Theorem 6.6) and Poincaré inequality (Theorem 6.11). With these tools, we are able to establish the existence of solutions of the φ -Laplace equation with zero and non-zero right-hand side (Sections 7 and 8).

2. Φ -FUNCTIONS

The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq Cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. A function f is *almost increasing* if there exists a constant $L \geq 1$ such that $f(s) \leq Lf(t)$ for all $s \leq t$. *Almost decreasing* is defined similarly.

Let us start with looking at the foundations of Φ -functions. Different researchers have used different conditions, which we try to capture in the next definition.

Definition 2.1. We consider increasing functions $\varphi: [0, \infty) \rightarrow [0, \infty]$ with $\varphi(0) = \lim_{t \rightarrow 0^+} \varphi(t) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. Such φ is called a Φ -*prefunction*. Furthermore, φ is called

- (1) a *weak Φ -function* if, additionally, $t \mapsto \frac{\varphi(t)}{t}$ is almost increasing on $(0, \infty)$;
- (2) a Φ -*function* if, additionally, it is left-continuous and convex;
- (3) a *strong Φ -function* if, additionally, it is continuous in $\overline{\mathbb{R}}$ and convex.

While Φ -functions are most commonly used, the class of weak Φ -functions is good in the sense that it is invariant under equivalence. Strong Φ -functions, on the other hand, are nice when it comes to dealing with the inverse, see the end of this section.

If φ is a Φ -function we write $\varphi \in \Phi$. If φ is a convex Φ -prefunction and $s < t$, then $\varphi(s) = \varphi(\frac{s}{t}t) \leq \frac{s}{t}\varphi(t) + (1 - \frac{s}{t})\varphi(0) = \frac{s}{t}\varphi(t)$ so that $\frac{\varphi(t)}{t}$ is increasing. Hence we see that (3) \Rightarrow (2) \Rightarrow (1) in the previous definition. We will show that, up to equivalence, (1) \Leftrightarrow (3) in Definition 2.1.

Two functions φ and ψ are *equivalent*, $\varphi \simeq \psi$, if there exists $L \geq 1$ such that $\psi(\frac{t}{L}) \leq \varphi(t) \leq \psi(Lt)$ for all t . We shall see that equivalent Φ -functions give rise to the same space with comparable norms. If $\varphi, \psi \in \Phi$ and $\varphi \approx \psi$, then by convexity $\varphi \simeq \psi$. We say that φ is *doubling* if $\varphi(2t) \leq D\varphi(t)$ for every $t > 0$. For a doubling Φ -function \simeq and \approx are equivalent.

We introduce two conditions that will be used throughout the paper. Here $p \geq 1$.

- (Inc) $_p$ $s \mapsto s^{-p}\varphi(s)$ is increasing.
- (aInc) $_p$ $s \mapsto s^{-p}\varphi(s)$ is almost increasing.

Note that $(\text{aInc})_p$ is invariant under equivalence of Φ -functions, whereas $(\text{Inc})_p$ is not. For this reason, it is better to formulate results with the “almost” assumption when possible. The next results shows that $(\text{aInc})_p$ can be upgraded to $(\text{Inc})_p$.

Lemma 2.2. *If the Φ -prefunction φ satisfies $(\text{aInc})_p$, $p \geq 1$, then there exists $\psi \in \Phi$ equivalent to φ such that $\psi^{1/p}$ is convex. In particular, ψ satisfies $(\text{Inc})_p$.*

Proof. Assume first that φ satisfies $(\text{aInc})_1$. Let ψ be the greatest convex minorant of φ , and note that it is increasing. Since $0 \leq \psi \leq \varphi$, it follows that $\psi(0) = \lim_{t \rightarrow 0^+} \psi(t) = 0$.

Suppose that $\varphi(s) > 0$. Then $\varphi(t) \geq \beta \frac{t}{s} \varphi(s)$ for some $\beta \in (0, 1]$ and all $t > s$, since φ satisfies $(\text{aInc})_1$. Thus the function $t \mapsto \beta(\frac{t}{s} - 1)\varphi(s)$ is a convex minorant of φ on $[0, \infty)$ and since ψ is the greatest convex minorant we conclude that

$$\psi(t) \geq \beta\left(\frac{t}{s} - 1\right)\varphi(s).$$

It follows that $\lim_{t \rightarrow \infty} \psi(t) = \infty$. Furthermore, this inequality implies that $\psi(\frac{2}{\beta}s) \geq (2 - \beta)\varphi(s) \geq \varphi(s)$. Since also $\psi \leq \varphi$, we see that $\varphi \simeq \psi$.

Finally, since ψ is convex, it is continuous except at the (possible) left-most point t with $\psi(s) = \infty$ for $s > t$. We force ψ to be left-continuous by (re)defining $\psi(s) = \lim_{t \rightarrow s^-} \psi(t)$. The properties above still hold and $\psi \in \Phi$.

Assume then that φ satisfies $(\text{aInc})_p$, $p > 1$. Then $\varphi(t)^{1/p}$ satisfies $(\text{aInc})_1$. Hence by the first part of the proof there exists $\xi \in \Phi$ such that $\xi \simeq \varphi^{1/p}$. Set $\psi := \xi^p$. Since $p > 1$, $\psi \in \Phi$ and further $\psi \simeq \varphi$, as required.

Let $p \in [1, \infty)$. Since $\psi^{1/p}$ is convex, $\frac{\psi(t)^{1/p}}{t}$ is increasing. Hence $\frac{\psi(t)}{t^p}$ is increasing, and so $(\text{Inc})_p$ holds. \square

We next show how to “upgrade” a weak Φ -function to a strong Φ -function. Most of the work was done in the previous lemma; it remains to show that an equivalent function is continuous on $\overline{\mathbb{R}}$. Recall that a Φ -function can be represented as

$$\varphi(t) = \int_0^t \varphi'(s) ds,$$

where φ' is the right-continuous right-derivative of the convex function φ .

Proposition 2.3. *Every weak Φ -function is equivalent to a strong Φ -function.*

Proof. Let φ be a weak Φ -function. Since φ satisfies $(\text{aInc})_1$, it follows from Lemma 2.2 that there exists $\varphi_2 \in \Phi$ with $\varphi_2 \simeq \varphi$. The convexity implies that φ_2 is continuous on the set $\{\varphi_2 < \infty\}$. If this set equals $[0, \infty)$ we are done. Otherwise, denote $t_\infty := \inf\{t : \varphi_2(t) = \infty\} \in (0, \infty)$. Let us define φ_3 on $(\frac{1}{2}t_\infty, t_\infty]$ by setting

$$\varphi_3(t) := \varphi_2(t_0) + t_0 \varphi_2'(t_0) \xi\left(\frac{t - t_0}{t_0}\right),$$

where $t_0 := \frac{1}{2}t_\infty$ and $\xi(t) := \frac{t}{1-t}$. Elsewhere, we set $\varphi_3 := \varphi_2$. Then φ_3 is continuous in $[0, \infty]$ and convex since the right-derivative φ_3' is increasing. Further, $\varphi_3(\frac{t}{2}) \leq \varphi_2(t) \leq \varphi_3(2t)$, so $\varphi_3 \simeq \varphi_2 \simeq \varphi$. The conditions $\varphi_3(0) = \lim_{t \rightarrow 0^+} \varphi_3(t) = 0$, and $\lim_{t \rightarrow \infty} \varphi_3(t) = \infty$ are clear since they are invariant under equivalence of Φ -functions. Thus φ_3 is the required strong Φ -function. \square

The inverse of a Φ -function. By φ^{-1} we denote the left-continuous inverse of a weak Φ -function φ ,

$$\varphi^{-1}(\tau) := \inf\{t \geq 0 : \varphi(t) \geq \tau\}.$$

It follows directly from this definition that $\varphi^{-1}(\varphi(t)) \leq t$ and equality holds if φ is strictly increasing. When $\varphi \in \Phi$ we can be more precise: if $t_0 := \max\{t \mid \varphi(t) = 0\}$ and $t_\infty := \max\{t \mid \varphi(t) < \infty\}$, then

$$\varphi^{-1}(\varphi(t)) = \begin{cases} 0, & t \leq t_0, \\ t, & t_0 < t \leq t_\infty, \\ t_\infty & t \geq t_\infty. \end{cases}$$

In particular, $\varphi^{-1}(\varphi(t)) = t$ if $0 < \varphi(t) < \infty$. In the opposite order things work better: $\varphi(\varphi^{-1}(s)) \leq s$ and for continuous φ the equality holds; in particular, for strong Φ -functions, $\varphi(\varphi^{-1}(s)) = s$, which is the main reason for introducing this class.

Note that every Φ -function is strictly increasing on $\varphi^{-1}(0, \infty)$ and that $\varphi \simeq \psi$ if and only if $\varphi^{-1} \simeq \psi^{-1}$.

The conjugate Φ -function. The conjugate Φ -function φ^* is defined by the formula

$$\varphi^*(t) := \sup_{s \geq 0} (st - \varphi(s)).$$

We say that φ satisfies $(\text{aDec})_p$ if $s \mapsto s^{-p}\varphi(s)$ is almost decreasing. If $s \mapsto s^{-p}\varphi(s)$ is decreasing we say that φ satisfies $(\text{Dec})_p$. Note that $(\text{aDec})_p$ yields that φ is doubling:

$$L \frac{\varphi(t)}{t^p} \geq \frac{\varphi(2t)}{(2t)^p} \iff \varphi(2t) \leq L 2^p \varphi(t).$$

Lemma 2.4. *Let $p \in (1, \infty)$ and let p' be its Hölder conjugate exponent: $\frac{1}{p} + \frac{1}{p'} = 1$. If $\varphi \in \Phi$ is doubling and satisfies $(\text{aInc})_p$, then φ^* satisfies $(\text{aDec})_{p'}$.*

Proof. Assume that φ satisfies $(\text{aInc})_p$. By Lemma 2.2, there exists $\psi \in \Phi$ that satisfies $\psi \simeq \varphi$ and $(\text{Inc})_p$. By [15, Lemma 5.2], ψ^* satisfies $(\text{Dec})_{p'}$. By [11, Lemma 2.6.4], $\varphi^* \simeq \psi^*$ and thus φ^* satisfies $(\text{aDec})_{p'}$. \square

3. GENERALIZED Φ -FUNCTIONS

Throughout the paper we denote by $\Omega \subset \mathbb{R}^n$ an open set. By $L^0(\Omega)$ we denote the set of (Lebesgue) measurable functions on Ω . By $L^p(\Omega)$, $p \in [1, \infty]$ we denote the set of p -integrable functions on Ω . The (Hardy–Littlewood) maximal operator is defined for $f \in L^0(\mathbb{R}^n)$ by

$$Mf(x) := \sup_{r>0} \int_{B(x,r)} |f(y)| dy,$$

where $B(x, r)$ is the ball with center x and radius r , and \int denotes the integral average.

We recall some definitions pertaining to generalized Orlicz spaces.

Definition 3.1. The set $\Phi(\Omega)$ consists of those $\varphi: \Omega \times [0, \infty) \rightarrow [0, \infty]$ with

- (1) $\varphi(y, \cdot) \in \Phi$ for every $y \in \Omega$; and
- (2) $\varphi(\cdot, t) \in L^0(\Omega)$ for every $t \geq 0$.

Also the functions in $\Phi(\Omega)$ will be called Φ -functions. In sub- and superscripts the dependence on x will be emphasized by $\varphi(\cdot, \cdot): L^p$ (Orlicz) vs $L^{\varphi(\cdot)}$ (generalized Orlicz). Properties and definitions of Φ -functions carry over to generalized Φ -functions point-wise.

Some examples of generalized Φ -functions are:

$$\begin{aligned}\varphi_1(x, t) &= t^{p(x)} \log(1 + t), & \varphi_2(x, t) &= t^p + a(x)t^q, \quad q > p \\ \varphi_3(x, t) &= t^{p(x)}, & \varphi_4(x, t) &= \frac{1}{p(x)}t^{p(x)}, \\ \varphi_5(x, t) &= e^{p(x)t} - 1, & \varphi_6(x, t) &= \infty\chi_{(1, \infty)}(t).\end{aligned}$$

The first and second Φ -functions have been recently studied, as mentioned in the introduction. The second line contains variants of the $p(\cdot)$ -growth; the third line contains non-doubling Φ -functions corresponding to exponential growth and L^∞ . With appropriate assumptions, the maximal operator is bounded in all these cases [19, 20].

Definition 3.2. Let $\varphi \in \Phi(\Omega)$ and define the *modular* $\varrho_{\varphi(\cdot)}$ for $f \in L^0(\Omega)$ by

$$\varrho_{\varphi(\cdot)}(f) := \int_{\Omega} \varphi(x, |f(x)|) dx.$$

The *generalized Orlicz space*, also called Musielak–Orlicz space, is defined as the set

$$L^{\varphi(\cdot)}(\Omega) := \left\{ f \in L^0(\Omega) : \lim_{\lambda \rightarrow 0} \varrho_{\varphi(\cdot)}(\lambda f) = 0 \right\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{L^{\varphi(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \varrho_{\varphi(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

If the set is clear from the context we abbreviate $\|f\|_{L^{\varphi(\cdot)}(\Omega)}$ by $\|f\|_{\varphi(\cdot)}$.

The conditions $(\text{Inc})_p$ and $(\text{aInc})_p$ are applied to generalized Φ -functions *uniformly* point-wise: for example generalized Φ -function φ satisfies $(\text{aInc})_p$ if there exists $L \geq 1$ such that $s^{-p}\varphi(x, s) \leq Lt^{-p}\varphi(x, t)$ for every $0 < s < t$ and every $x \in \Omega$.

Lemma 3.3. *If $\varphi \in \Phi(\Omega)$ satisfies $(\text{aInc})_p$, $f \in L^0(\Omega)$ and $\int_{\Omega} \varphi(x, f) dx \geq 1$, then*

$$\|f\|_{L^{\varphi(\cdot)}(\Omega)}^p \lesssim \int_{\Omega} \varphi(x, f) dx.$$

Proof. Let us write $\lambda := \int_{\Omega} \varphi(x, f(x)) dx$. Let L be the almost increasing constant from $(\text{aInc})_p$. Since $\lambda, L \geq 1$, we obtain by $(\text{aInc})_p$ that

$$\frac{\varphi\left(x, \frac{f(x)}{(L\lambda)^{1/p}}\right)}{\left(\frac{f(x)}{(L\lambda)^{1/p}}\right)^p} \leq L \frac{\varphi(x, f(x))}{(f(x))^p}$$

and thus $\varphi\left(x, \frac{f(x)}{(L\lambda)^{1/p}}\right) \leq \frac{1}{\lambda}\varphi(x, f(x))$. This yields that

$$\int_{\Omega} \varphi\left(x, \frac{f(x)}{(L\lambda)^{1/p}}\right) dx \leq \frac{1}{\lambda} \int_{\Omega} \varphi(x, f(x)) dx = 1$$

so that $\|f\|_{L^{\varphi(\cdot)}(\Omega)} \leq (L\lambda)^{1/p}$ by the definition of the norm. \square

Assumption (A0). The following assumption means that we restrict our attention to the essentially “unweighted” case:

$$(A0) \quad \varphi^{-1}(x, 1) \approx 1.$$

This simple assumption already allows us to obtain several basic results.

Lemma 3.4. *Let $\varphi \in \Phi(\Omega)$. Then φ satisfies (A0) if and only if there exists $\beta \in (0, 1]$ such that $\varphi(x, \beta) \leq 1$ and $\varphi(x, 1/\beta) \geq 1$ for every $x \in \Omega$.*

Proof. Assume first that (A0) holds. Then there exists $\beta \in (0, 1]$ such that $\beta \leq \varphi^{-1}(x, 1) \leq \frac{1}{\beta}$ for all $x \in \Omega$. Applying φ to both inequalities, we obtain

$$\varphi\left(x, \frac{\beta}{2}\right) \leq \varphi(x, \beta) \leq \varphi(x, \varphi^{-1}(x, 1)) \leq 1,$$

and since $\varphi^{-1}(x, 1) \leq \frac{1}{\beta}$, we obtain by the definition of φ^{-1} that $\varphi(x, \frac{2}{\beta}) \geq 1$.

Assume then that the condition holds. By the definition of φ^{-1} , the inequality $\varphi(x, \frac{1}{\beta}) \geq 1$ yields $\varphi^{-1}(x, 1) \leq \frac{1}{\beta}$. By convexity $\varphi(x, \frac{\beta}{2}) \leq \frac{1}{2}\varphi(x, \beta) < 1$ and thus $\varphi^{-1}(x, 1) \geq \frac{\beta}{2}$. \square

4. FUNCTIONAL ANALYSIS IN GENERALIZED ORLICZ SPACES

In this section we have collected results on functional analytical properties of generalized Orlicz spaces. We proceed from properties which hold for all $\varphi \in \Phi(\Omega)$ to more restrictive results. The following theorem contains several nice convergence properties which hold without additional assumption.

Theorem 4.1 (Theorems 2.1.17, 2.2.8 and Lemma 2.3.16, [11]). *Let $\varphi \in \Phi(\Omega)$.*

- (1) *If $f_k \rightarrow f$ almost everywhere, then $\varrho_{\varphi(\cdot)}(f) \leq \liminf_{k \rightarrow \infty} \varrho_{\varphi(\cdot)}(f_k)$. (Fatou's lemma for the modular)*
- (2) *If $f_k \rightharpoonup f$ weakly in $L^{\varphi(\cdot)}$, then $\varrho_{\varphi(\cdot)}(f) \leq \liminf_{k \rightarrow \infty} \varrho_{\varphi(\cdot)}(f_k)$. (Lower semicontinuity)*
- (3) *If $|f_k| \nearrow |f|$ almost everywhere, then $\varrho_{\varphi(\cdot)}(f) = \lim_{k \rightarrow \infty} \varrho_{\varphi(\cdot)}(f_k)$. (Monotone convergence)*
- (4) *If $f_k \rightarrow f$ almost everywhere, $|f_k| \leq |g|$ almost everywhere, and $\varrho_{\varphi(\cdot)}(\lambda g) < \infty$ for every $\lambda > 0$, then $f_k \rightarrow f$ in $L^{\varphi(\cdot)}$. (Dominated convergence)*

For more advanced properties, we need some more assumptions on φ .

The space $E^{\varphi(\cdot)}$ of finite functions. The space $E^{\varphi(\cdot)}(\Omega)$ consists of those functions $f \in L^{\varphi(\cdot)}(\Omega)$ with $\varrho_{\varphi(\cdot)}(\lambda f) < \infty$ for every $\lambda > 0$. These functions are called *finite*. The equality $E^{\varphi(\cdot)}(\Omega) = L^{\varphi(\cdot)}(\Omega)$ holds if and only if $\varrho_{\varphi(\cdot)}(f) < \infty$ implies $\varrho_{\varphi(\cdot)}(2f) < \infty$ [11, p. 47]. If φ is doubling this surely holds, and in Orlicz spaces also the converse is true. However, in generalized Orlicz spaces there exist Φ -functions with $E^{\varphi(\cdot)} = L^{\varphi(\cdot)}$ which are not doubling, as the following example shows.

Example 4.2. Let $\lambda(x) := \frac{1}{2} \log \frac{1}{|x|}$, $|x| < 1$. Define $\varphi : B(0, 1) \times [0, \infty) \rightarrow [0, \infty)$ by

$$\varphi(x, t) := \begin{cases} e^t - 1, & \text{if } t \leq \lambda(x), \\ e^{\lambda(x)}(t - \lambda(x)) + e^{\lambda(x)} - 1, & \text{otherwise.} \end{cases}$$

Clearly φ is not doubling, since it has exponential growth near the origin. If $t > 3\lambda(x)$, then a calculation shows that $\varphi(x, 2t) \leq 3\varphi(x, t)$. On the other hand,

$$\varrho_{\varphi(\cdot)}(6\lambda) = \int_{B(0,1)} e^{\lambda(x)}(5\lambda(x) + 1) - 1 \, dx = \int_{B(0,1)} |x|^{-\frac{1}{2}} \left(\frac{5}{2} \log \frac{1}{|x|} + 1\right) - 1 \, dx < \infty.$$

Suppose that $\varrho_{\varphi(\cdot)}(f) < \infty$. Then

$$\varrho_{\varphi(\cdot)}(2f) \leq \varrho_{\varphi(\cdot)}(6\lambda) + \varrho_{\varphi(\cdot)}(2f\chi_{\{|f|>3\lambda\}}) \leq \varrho_{\varphi(\cdot)}(6\lambda) + 3\varrho_{\varphi(\cdot)}(f) < \infty.$$

Hence $E^{\varphi(\cdot)}(B(0, 1)) = L^{\varphi(\cdot)}(B(0, 1))$ even though φ is not doubling.

The space $E^{\varphi(\cdot)}$ has a number of nice properties. For instance, every $f \in E^{\varphi(\cdot)}$ has absolutely continuous norm [11, Remark 2.5.8]. A Φ -function is said to be *locally integrable* if $\varrho_{\varphi(\cdot)}(t\chi_E) < \infty$ for all $t > 0$ and sets $E \subset \Omega$ of finite measure. Note that every doubling φ satisfying (A0) is locally integrable.

Theorem 4.3. *Suppose that $\varphi \in \Phi(\Omega)$ is locally integrable. Then*

- (1) *Simple functions are dense in $E^{\varphi(\cdot)}$ [11, Theorem 2.5.9].*
- (2) *$E^{\varphi(\cdot)}$ is separable [11, Theorem 2.5.10].*

If φ is doubling, then these claims hold also for $L^{\varphi(\cdot)}$.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^n$ be bounded. If $\varphi \in \Phi(\Omega)$ satisfies (A0) and $(aInc)_p$, then $L^{\varphi(\cdot)}(\Omega) \hookrightarrow L^p(\Omega)$ and the embedding norm depends only on the (A0)-constant, the $(aInc)_p$ -constant and $|\Omega|$.*

Proof. By (A0) and Lemma 3.4, $\varphi(x, \frac{1}{\beta}) \geq 1$. By $(aInc)_p$,

$$\frac{\varphi(x, 1/\beta)}{(1/\beta)^p} \leq L \frac{\varphi(x, t)}{t^p}$$

for $t \geq 1/\beta$ and thus $t^p \leq L\beta^{-p}\varphi(x, t)$ for $t \geq 1/\beta$. Let $K := L\beta^{-p} + \beta^{-p}|\Omega| + 1$. Then

$$\left(\frac{t}{K}\right)^p \leq \frac{t^p}{K} \leq \varphi(x, t) + \frac{\chi_{\Omega}(x)}{|\Omega|},$$

where $\|\frac{\chi_{\Omega}(\cdot)}{|\Omega|}\|_{L^1(\Omega)} \leq 1$. Thus by [11, Theorem 2.8.1], $\|f\|_{L^p(\Omega)} \leq 2K\|f\|_{L^{\varphi(\cdot)}(\Omega)}$ for all $f \in L^{\varphi(\cdot)}(\Omega)$. \square

Theorem 4.5. *If $\varphi \in \Phi(\Omega)$ is doubling and satisfies (A0), then $C_0^\infty(\Omega)$ is dense in $L^{\varphi(\cdot)}(\Omega)$.*

Proof. Since simple functions are dense in $L^{\varphi(\cdot)}$ (Theorem 4.3), it suffices to show that every simple function can be approximated by $C_0^\infty(\Omega)$ -functions in $L^{\varphi(\cdot)}$.

If $t \in [0, \beta]$, where β is from Lemma 3.4, then by $(aInc)_1$

$$\varphi(x, t) = \varphi(x, \frac{t}{\beta}\beta) \lesssim \frac{t}{\beta}\varphi(x, \beta) \leq \frac{t}{\beta}.$$

Assume then that $t > \beta$. Set $p := \log_2 D$, where D is the doubling constant of φ . By iterating the doubling inequality we obtain that

$$\varphi(x, t) \leq c\left(\frac{t}{\beta}\right)^p \varphi(x, \beta) \leq \frac{c}{\beta^p} t^p$$

for some constant c depending only the doubling constant D . We conclude that

$$\varphi(x, t) \lesssim \max\{t^p, t\},$$

where the constant depends only on the doubling constant and the constants in (A0). Thus $L^1 \cap L^p \hookrightarrow L^{\varphi(\cdot)}$. Since every simple function belongs to $L^1 \cap L^p$, where it can be approximated by a sequence of C_0^∞ -functions, the claim follows. \square

Next, we relate the condition (A0) to previously studied conditions and results. A Φ -function $\varphi \in \Phi(\Omega)$ is called *proper* if φ and φ^* are locally integrable. If φ is proper and $E^{\varphi(\cdot)} = L^{\varphi(\cdot)}$, then $(L^{\varphi(\cdot)})^* = L^{\varphi^*(\cdot)}$ [11, Theorem 2.7.14]. Assumption (A0) implies that doubling Φ -functions are proper:

Proposition 4.6. *If $\varphi \in \Phi(\Omega)$ satisfies (A0) and φ and φ^* are doubling, then φ is proper, φ^* satisfies (A0) and $L^{\varphi(\cdot)}(\Omega)$ is reflexive.*

Proof. By [11, Corollary 2.7.9], φ is proper if and only if every simple function belongs to $L^{\varphi(\cdot)} \cap L^{\varphi^*(\cdot)}$. Since both φ and φ^* are doubling, it suffices to show that $\varrho_{\varphi(\cdot)}(\beta\chi_E) < \infty$ and $\varrho_{\varphi^*(\cdot)}(\beta\chi_E) < \infty$ for some β and every $E \subset \Omega$ of finite measure. Since $\varphi(x, \beta) \leq 1$ (Lemma 3.4), $\varrho_{\varphi(\cdot)}(\beta\chi_E) \leq |E|$ so the first condition holds.

By [11, Corollary 2.7.18], $L^{\varphi(\cdot)}(\Omega)$ is reflexive, because φ is proper.

It remains to show that also φ^* satisfies (A0). For this we note by Lemma 3.4 that

$$\varphi^*(x, \beta) = \sup_{s>0}(\beta s - \varphi(x, s)) = \sup_{s \in [0, \frac{1}{\beta}]}(\beta s - \varphi(x, s)) \leq \sup_{s \in [0, \frac{1}{\beta}]} \beta s = 1;$$

the second equality follows since $\varphi(x, \frac{1}{\beta}) \geq 1$, so for $s \geq \frac{1}{\beta}$ the expression is negative. On the other hand $\varphi^*(x, \frac{2}{\beta}) \geq \frac{2}{\beta}\beta - \varphi(x, \beta) \geq 1$. Therefore also φ^* satisfies (A0) by Lemma 3.4. \square

For more advanced properties, we need to control the variability of φ between neighboring points. For this we have two further assumptions from [15].

(A1) There exists $\beta \in (0, 1]$ such that $\beta\varphi^{-1}(x, t) \leq \varphi^{-1}(y, t)$ for every $t \in [1, \frac{1}{|\beta|}]$, every $x, y \in B \cap \Omega$ and every ball B with $|B| \leq 1$.

(A2) $L^{\varphi(\cdot)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) = L^{\varphi_\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $\varphi_\infty(t) := \limsup_{|x| \rightarrow \infty} \varphi(x, t)$.

Note that conditions (A0), (A1) and (A2) are invariant with respect of equivalence of functions: if $\varphi \simeq \psi$, then φ satisfies (Ai) if and only if ψ satisfies (Ai), $i=0, 1, 2$. Also $(\text{aInc})_p$ is invariant with respect of equivalence of functions.

Theorem 4.7. *Let φ be a weak Φ -function on \mathbb{R}^n and satisfy assumptions (A0)–(A2) and $(\text{aInc})_p$ for some $p > 1$. Then*

$$M : L^{\varphi(\cdot)}(\mathbb{R}^n) \rightarrow L^{\varphi(\cdot)}(\mathbb{R}^n)$$

is bounded.

Proof. By Lemma 2.2 there exists $\psi \in \Phi(\mathbb{R}^n)$ with $\psi \simeq \varphi$ which ψ satisfies $(\text{Inc})_p$. Since (A0)–(A2) are invariant under the equivalence of Φ -functions, ψ satisfies (A0)–(A2). Thus the claim follows by [15, Corollary 3.7]. \square

5. ASSUMPTION (A1) IN DOMAINS

Of the assumptions (A0)–(A2) in the previous section, (A0) generalizes easily to the case of a subdomain of \mathbb{R}^n . Assumption (A2) is only relevant to unbounded domains. However, generalizing (A1) to the domain-case is not entirely straight-forward. We propose the following assumption, and spend the rest of the section showing that it is indeed a natural and correct counterpart to (A1) in Ω .

(A1) $_\Omega$ There exist constants $\alpha, \beta \in (0, 1]$ such that $\alpha\beta^{t^{1/n}|x-y|}\varphi^{-1}(y, t) \leq \varphi^{-1}(x, t)$ for all $x, y \in \Omega$ and $t \geq 1$.

A domain $\Omega \subset \mathbb{R}^n$ is *quasi-convex*, if there exists a constant $K \geq 1$ such that every pair $x, y \in \Omega$ can be connect by a rectifiable path $\gamma \subset \Omega$ from x to y such that $\ell(\gamma) \leq K|x - y|$. By $\ell(\gamma)$ we denote the length of the path γ .

Lemma 5.1. *If $\Omega \subset \mathbb{R}^n$ is quasi-convex, then $\varphi \in \Phi(\Omega)$ satisfies (A1) if and only if it satisfies (A1) $_\Omega$.*

Proof. It is clear that (A1) $_\Omega$ implies (A1). We proceed to prove the converse.

Let $x, y \in \Omega$ and $t \geq 1$. Let $\gamma \subset \Omega$ be a path connecting x and y of length at most $K|x - y|$. Let $x_0 = x$ and $c(n)$ be a constant such that $|B(x, r)| = c(n)r^n$. For

$j = 1, 2, \dots, k-1$ choose points $x_j \in \gamma$ such that $\ell(\gamma(x, x_j)) < \ell(\gamma(x, x_{j+1}))$ and $|x_{j-1} - x_j| = \frac{1}{(2c(n)t)^{1/n}}$ if possible and finally set $x_k = y$. Then $|x_{k-1} - x_k| \leq \frac{1}{(2c(n)t)^{1/n}}$. Let B be an open ball such that $x_j, x_{j+1} \in B$ and $\text{diam}(B) = 2|x_j - x_{j+1}|$. Then $\frac{1}{|B|} = \frac{1}{c(n)|x_j - x_{j+1}|^n} \geq 2t$ so that $\frac{1}{|B|} > t$. Thus t is in the allowed range and we obtain by (A1) that $\beta\varphi^{-1}(x_{j+1}, t) \leq \varphi^{-1}(x_j, t)$. With this chain of inequalities, we obtain that $\beta^k\varphi^{-1}(y, t) \leq \varphi^{-1}(x, t)$. On the other hand, at most

$$k = \frac{\ell(\gamma(x, x_{k-1}))}{\text{diam}(B)} + 1 \leq \frac{K|x-y|}{\text{diam}(B)} + 1 \leq \frac{K|x-y|}{2/(2c(n)t)^{1/n}} + 1 = c'Kt^{1/n}|x-y| + 1$$

points x_j are needed, so that $\beta^{c'Kt^{1/n}|x-y|+1}\varphi^{-1}(y, t) \leq \varphi^{-1}(x, t)$ for all $x, y \in \Omega$ and $t \geq 1$. \square

We say that $\psi \in \Phi(\mathbb{R}^n)$ is an *extension* of $\varphi \in \Phi(\Omega)$ if $\psi|_{\Omega} \simeq \varphi$.

Proposition 5.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded and $\varphi \in \Phi(\Omega)$ satisfies (A0).*

Then there exists an extension $\psi \in \Phi(\mathbb{R}^n)$ of φ which satisfies (A0)–(A2) if and only if φ satisfies (A1) $_{\Omega}$.

If φ satisfies (aInc) $_p$, then the extension can be taken to satisfy it as well.

Proof. Let $t \geq 1$. Suppose first that there exists an extension $\psi \in \Phi(\mathbb{R}^n)$ which satisfies (A0)–(A2). To compare $\varphi^{-1}(x, t)$ and $\varphi^{-1}(y, t)$, $x, y \in \Omega$, it suffices to use $\alpha\beta^{t^{1/n}|x-y|}\psi^{-1}(y, t) \leq \psi^{-1}(x, t)$ which follows from (A1) $_{\mathbb{R}^n}$ for ψ by Lemma 5.1 since \mathbb{R}^n is quasi-convex. So we can move on to the converse implication.

Define $\psi : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty]$ by

$$\psi^{-1}(x, t) := \sup_{y \in \Omega} \beta^{t^{1/n}|x-y|}\varphi^{-1}(y, t).$$

By (A1) $_{\Omega}$, we have $\psi^{-1} \leq \frac{1}{\alpha}\varphi^{-1}$ in Ω . Choosing $y = x$ in the supremum, we see that $\psi^{-1} \geq \varphi^{-1}$. Thus $\psi^{-1} \approx \varphi^{-1}$ and $\psi \simeq \varphi$ in Ω .

Let $B \subset \mathbb{R}^n$ be an open ball with $|B| \leq 1$. Let $t \in [1, \frac{1}{|B|}]$ and $x, y \in B$. Then $t^{1/n}|x-y| \leq c(n)$. Choose $z \in \Omega$ such that

$$\psi^{-1}(x, t) \leq 2\beta^{t^{1/n}|x-z|}\varphi^{-1}(z, t).$$

By definition of ψ , $\beta \in (0, 1]$, the triangle inequality and $t^{1/n}|x-y| \leq c(n)$,

$$\psi^{-1}(y, t) \geq \beta^{t^{1/n}|y-z|}\varphi^{-1}(z, t) \geq \beta^{t^{1/n}(|x-y|+|x-z|)}\varphi^{-1}(z, t) \geq \frac{1}{2}\beta^{c(n)}\psi^{-1}(x, t).$$

Hence ψ satisfies condition (A1) with constant $\frac{1}{2}\beta^{c(n)}$.

Let then B be a ball containing Ω and let $\eta \in C_0^\infty(\mathbb{R}^n)$ equal 1 in B and 0 in $(2B)^c$. Choose $x_0 \in \Omega$ and define $\psi_2 : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty]$ by

$$\psi_2^{-1}(x, t) := \eta(x)\psi^{-1}(x, t) + (1 - \eta(x))\psi^{-1}(x_0, t).$$

Clearly $\psi_2 \simeq \varphi$ in Ω . By definition $\psi_2 = \psi(x_0, \cdot)$ in $2B^c$ is independent of x , so (A2) is clear. Since ψ and $\psi(x_0, \cdot)$ satisfy (A1), the condition also follows for ψ_2 .

Let us consider condition (A0). If $x \in \mathbb{R}^n \setminus 2B$, then $\psi_2(x, t) = \psi(x_0, t) \simeq \varphi(x_0, t)$ and thus for those x condition (A0) holds. Let r be the diameter of $2B$. If $x \in 2B$, then by (A1)

$$\psi_2^{-1}(x, 1) \geq \beta^r\varphi^{-1}(x_0, 1) \gtrsim \beta^r.$$

On the other hand, $\psi_2^{-1}(x, 1) \leq \sup_{y \in \Omega} \varphi^{-1}(y, 1) \lesssim 1$. Thus (A0) holds for ψ_2 .

With $s = \varphi^{-1}(x, t)$, we have

$$s^{-p}\varphi(x, s) = (\varphi^{-1}(x, t))^{-p}t = \left(\varphi^{-1}(x, t)t^{-\frac{1}{p}}\right)^{-p}.$$

Since φ satisfies $(\text{aInc})_p$, $t \mapsto t^{-\frac{1}{p}}\varphi^{-1}(x, t)$ is almost decreasing. Then for every y

$$t \mapsto t^{-\frac{1}{p}}\beta^{t^{1/n}|x-y|}\varphi^{-1}(y, t)$$

is almost decreasing (since $\beta < 1$). Therefore the same holds for ψ^{-1} since the supremum of almost decreasing functions is almost decreasing. Thus ψ_2 satisfies $(\text{aInc})_p$.

Conditions (A0) and $(\text{aInc})_p$ imply that $\lim_{t \rightarrow \infty} \psi_2(x, t) = \infty$ and $\lim_{t \rightarrow 0^+} \psi_2(x, t) = 0 = \psi_2(x, 0)$. Thus $\psi_2(x, \cdot)$ is a weak Φ -function for all $y \in \mathbb{R}^n$. By Lemma 2.2 there exists $\psi_3 \in \Phi(\mathbb{R}^n)$ such that $\psi_3 \simeq \psi_2$, and thus ψ_3 satisfies (A0)–(A2) and $(\text{aInc})_p$. \square

In view of the extension result the boundedness of the maximal operator is obtained also in subdomains $\Omega \subset \mathbb{R}^n$.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $\varphi \in \Phi(\Omega)$ satisfy assumptions (A0), (A1) $_{\Omega}$ and $(\text{aInc})_p$ for some $p > 1$. Then*

$$M : L^{\varphi(\cdot)}(\Omega) \rightarrow L^{\varphi(\cdot)}(\Omega)$$

is bounded.

Proof. Let $f \in L^{\varphi(\cdot)}(\Omega)$. By Proposition 5.2, there exists an extension $\psi \in \Phi(\mathbb{R}^n)$ of φ which satisfies (A0)–(A2) and $(\text{aInc})_p$. Then f extended by 0 to $\mathbb{R}^n \setminus \Omega$ belongs to $L^{\psi(\cdot)}(\mathbb{R}^n)$. By Theorem 4.7,

$$\|Mf\|_{L^{\varphi(\cdot)}(\Omega)} \approx \|Mf\|_{L^{\psi(\cdot)}(\Omega)} \leq \|Mf\|_{L^{\psi(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\psi(\cdot)}(\Omega)} \approx \|f\|_{L^{\varphi(\cdot)}(\Omega)}. \quad \square$$

6. SOBOLEV–ORLICZ SPACES

Definition 6.1. A function $u \in L^{\varphi(\cdot)}(\Omega)$ belongs to the *Sobolev–Orlicz space* $W^{1, \varphi(\cdot)}(\Omega)$ if its weak partial derivatives $\partial_1 u, \dots, \partial_n u$ exist and belong to $L^{\varphi(\cdot)}(\Omega)$.

We define a modular on $W^{1, \varphi(\cdot)}(\Omega)$ by

$$\begin{aligned} \varrho_{W^{1, \varphi(\cdot)}(\Omega)}(u) &:= \varrho_{\varphi(\cdot)}(u) + \sum_{i=1}^n \varrho_{\varphi(\cdot)}(\partial_i u) \\ &= \int_{\Omega} \varphi(x, |u(x)|) dx + \sum_{i=1}^n \int_{\Omega} \varphi(x, |\partial_i u(x)|) dx \end{aligned}$$

which induces a norm by

$$\|u\|_{W^{1, \varphi(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \varrho_{W^{1, \varphi(\cdot)}(\Omega)}(u/\lambda) \leq 1 \}.$$

Let $\nabla u = (\partial_1 u, \dots, \partial_n u)$. Note that for $\varphi \in \Phi(\Omega)$

$$\sum_{i=1}^n \varphi(x, |\partial_i u(x)|) \leq n\varphi(x, |\nabla u(x)|) \leq \varphi(x, n|\nabla u(x)|)$$

and

$$\varphi(x, \frac{1}{n}|\nabla u(x)|) \leq \varphi\left(x, \frac{1}{n} \sum_{i=1}^n |\partial_i u(x)|\right) \leq \frac{1}{n} \sum_{i=1}^n \varphi(x, |\partial_i u(x)|),$$

where the convexity has been used in both inequalities. Thus $\int_{\Omega} \varphi(x, |u(x)|) dx + \int_{\Omega} \varphi(x, |\nabla u(x)|) dx$ gives an equivalent modular in the case $\varphi \in \Phi(\Omega)$.

Next we show that $\|u\|_{W^{1,\varphi(\cdot)}(\Omega)} \approx \|u\|_{L^{\varphi(\cdot)}(\Omega)} + \|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}$ where $\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}$ is an abbreviation of $\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}$. Clearly $\|u\|_{L^{\varphi(\cdot)}(\Omega)} + \|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)} \leq 2\|u\|_{W^{1,\varphi(\cdot)}(\Omega)}$. Let $\lambda > \|u\|_{L^{\varphi(\cdot)}(\Omega)}$ and $\mu > \|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}$. Then by convexity

$$\begin{aligned} & \int_{\Omega} \varphi\left(x, \frac{u(x)}{2(\lambda + \mu)}\right) + \varphi\left(u, \frac{|\nabla u(x)|}{2(\lambda + \mu)}\right) dx \\ & \leq \frac{1}{2} \int_{\Omega} \varphi\left(x, \frac{u(x)}{\lambda}\right) + \varphi\left(u, \frac{|\nabla u(x)|}{\mu}\right) dx \leq 1 \end{aligned}$$

and thus $\|u\|_{W^{1,\varphi(\cdot)}(\Omega)} \leq 2\|u\|_{L^{\varphi(\cdot)}(\Omega)} + 2\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}$.

Lemma 4.4 gives the following results.

Lemma 6.2. *If $\varphi \in \Phi(\Omega)$ satisfies (A0) and $(aInc)_p$, then $W^{1,\varphi(\cdot)}(\Omega) \subset W_{loc}^{1,p}(\Omega)$.*

The previous lemma yields that $W^{1,\varphi(\cdot)}(\Omega)$ is a *lattice*: if $u, v \in W^{1,\varphi(\cdot)}(\Omega)$, then $\max\{u, v\}, \min\{u, v\}, |u| \in W^{1,\varphi(\cdot)}(\Omega)$. The proof is as in the variable exponent case, see [11, Proposition 8.1.9].

The Sobolev–Orlicz space $W^{1,\varphi(\cdot)}$ inherits most properties of the generalized Orlicz space $L^{\varphi(\cdot)}$. Since a closed subspace of a reflexive/separable Banach spaces is also reflexive/separable, we obtain the following result by Theorem 4.3 and Proposition 4.6.

Theorem 6.3. *If φ and φ^* are doubling and (A0) holds, then $W^{1,\varphi(\cdot)}(\Omega)$ is a reflexive and separable Banach space.*

Theorem 6.4. *Let $\varphi \in \Phi(\mathbb{R}^n)$ be such that $\varrho_{\varphi(\cdot)}(f_k) \rightarrow 0$ implies $\varrho_{\varphi(\cdot)}(2f_k) \rightarrow 0$. Then Sobolev functions with compact support in \mathbb{R}^n are dense in $W^{1,\varphi(\cdot)}(\mathbb{R}^n)$.*

Proof. Let us denote $B_t := B(0, t)$, $t \geq 1$. Let $\psi_r \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function with $\psi_r = 1$ on B_r , $\psi_r = 0$ on $\mathbb{R}^n \setminus B_{r+1}$, $0 \leq \psi_r(x) \leq 1$ and $|\nabla \psi_r| \leq 2$. We show that $u\psi_r \rightarrow u$ in $W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ as $r \rightarrow \infty$. Note first that

$$\|u - u\psi_r\|_{W^{1,\varphi(\cdot)}(\mathbb{R}^n)} \leq \|u\|_{W^{1,\varphi(\cdot)}(\mathbb{R}^n \setminus B_{r+1})} + \|u - u\psi_r\|_{W^{1,\varphi(\cdot)}(B_{r+1} \setminus B_r)}.$$

Since $E^{\varphi(\cdot)}(\mathbb{R}^n) = L^{\varphi(\cdot)}(\mathbb{R}^n)$ (by $\varrho_{\varphi(\cdot)}(f_k) \rightarrow 0 \Rightarrow \varrho_{\varphi(\cdot)}(2f_k) \rightarrow 0$), the absolute continuity of the norm implies that $\|u\|_{W^{1,\varphi(\cdot)}(\mathbb{R}^n \setminus B_{r+1})} \rightarrow 0$ as $r \rightarrow \infty$. To handle the second term in the above inequality we observe that

$$|\nabla u - \nabla(u\psi_r)| \leq (1 - \psi_r)|\nabla u| + |\nabla \psi_r||u| \leq |\nabla u| + 2|u|.$$

Thus $\|u - u\psi_r\|_{W^{1,\varphi(\cdot)}(B_{r+1} \setminus B_r)} \leq \| |\nabla u| + 2|u| \|_{W^{1,\varphi(\cdot)}(\mathbb{R}^n \setminus B_r)}$. By absolute continuity of the norm, this converges to 0 as $r \rightarrow \infty$. \square

Theorem 6.5. *Let $\varphi \in \Phi(\mathbb{R}^n)$ be doubling and satisfy (A0). Moreover assume that $M : L^{\varphi(\cdot)}(\mathbb{R}^n) \rightarrow L^{\varphi(\cdot)}(\mathbb{R}^n)$ is bounded. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,\varphi(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $u \in W^{1,\varphi(\cdot)}(\mathbb{R}^n)$ and let $\varepsilon > 0$ be arbitrary. By Theorem 6.4 we may assume that u has compact support in \mathbb{R}^n . Let ψ_ε be the standard mollifier. Then $u * \psi_\varepsilon$ belongs to $C_0^\infty(\mathbb{R}^n)$ and

$$\nabla(u * \psi_\varepsilon) - \nabla u = (\nabla u) * \psi_\varepsilon - \nabla u.$$

The claim follows once we show that $\|f * \psi_\varepsilon - f\|_{\varphi(\cdot)} \rightarrow 0$ as $\varepsilon \rightarrow 0$ for every $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$.

Let $\delta > 0$. Note that by doubling and (A0), φ is locally integrable. Then by the density of simple functions in $L^{\varphi(\cdot)}(\mathbb{R}^n)$, Theorem 4.3, we can find a simple function

g with $\|f - g\|_{\varphi(\cdot)} \leq \delta$. This implies that

$$\|f * \psi_\varepsilon - f\|_{\varphi(\cdot)} \leq \|g * \psi_\varepsilon - g\|_{\varphi(\cdot)} + \|(f - g) * \psi_\varepsilon - (f - g)\|_{\varphi(\cdot)} =: (I) + (II).$$

Since g is a simple function, $g \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, where p is as in Theorem 4.5. The classical theorem on mollification implies that $g * \psi_\varepsilon \rightarrow g$ in $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Thus $g * \psi_\varepsilon \rightarrow g$ in $L^{\varphi(\cdot)}(\mathbb{R}^n)$ as in Theorem 4.5. This proves (I) $\rightarrow 0$ for $\varepsilon \rightarrow 0$.

Let us study the term (II). By Lemma 4.4 with $p = 1$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ so that $|f * \psi_\varepsilon| \leq 2Mf$ by [11, Lemma 4.6.3]. We obtain

$$\begin{aligned} (II) &= \|(f - g) * \psi_\varepsilon - (f - g)\|_{\varphi(\cdot)} \leq \|(f - g) * \psi_\varepsilon\|_{\varphi(\cdot)} + \|f - g\|_{\varphi(\cdot)} \\ &\leq 2\|M(f - g)\|_{\varphi(\cdot)} + \|f - g\|_{\varphi(\cdot)} \leq c\|f - g\|_{\varphi(\cdot)} \leq c\delta \end{aligned}$$

where the boundedness of M has been used in the second to last inequality. This implies

$$\limsup_{\varepsilon \rightarrow 0} \|f * \psi_\varepsilon - f\|_{\varphi(\cdot)} \leq c\delta.$$

Since $\delta > 0$ was arbitrary, this yields $\|f * \psi_\varepsilon - f\|_{\varphi(\cdot)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. \square

Theorem 6.6. *Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $\varphi \in \Phi(\Omega)$ be doubling and satisfy (A0), (A1) $_\Omega$ and (aInc) $_p$ for some $p > 1$. Then $C^\infty(\Omega) \cap W^{1,\varphi(\cdot)}(\Omega)$ is dense in $W^{1,\varphi(\cdot)}(\Omega)$.*

Proof. By Proposition 5.2, we extend φ to \mathbb{R}^n . Let $u \in W^{1,\varphi(\cdot)}(\Omega)$. Fix $\varepsilon > 0$ and define $\Omega_0 := \emptyset$,

$$\Omega_m = \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{1}{m} \right\} \quad \text{and} \quad U_m := \Omega_{m+1} \setminus \overline{\Omega}_{m-1}$$

for $m = 1, 2, \dots$. Let (ξ_m) be a partition of unity subordinate to the covering (U_m) , i.e. $\xi_m \in C_0^\infty(U_m)$ and $\sum_{m=1}^\infty \xi_m(x) = 1$ for every $x \in \Omega$. Let ψ_δ be the standard mollifier. For every m there exists δ_m such that

$$\text{spt}((\xi_m u) * \psi_{\delta_m}) \subset U_m \subset \Omega$$

and as in the proof of Theorem 6.5 we conclude by choosing a smaller δ_m if necessary, that

$$\|(\xi_m u) - (\xi_m u) * \psi_{\delta_m}\|_{W^{1,\varphi(\cdot)}(\Omega)} \leq \varepsilon 2^{-m}.$$

We define

$$u_\varepsilon := \sum_{m=1}^\infty (\xi_m u) * \psi_{\delta_m}.$$

Every point $x \in \Omega$ has a neighborhood such that the above sum has only finitely many non-zero terms and thus $u_\varepsilon \in C^\infty(\Omega)$. Furthermore, this is an approximating sequence, since

$$\|u - u_\varepsilon\|_{W^{1,\varphi(\cdot)}(\Omega)} \leq \sum_{m=1}^\infty \|\xi_m u - (\xi_m u) * \psi_{\delta_m}\|_{W^{1,\varphi(\cdot)}(\Omega)} \leq \varepsilon. \quad \square$$

Definition 6.7. $W_0^{1,\varphi(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W^{1,\varphi(\cdot)}(\Omega)$.

Since $W_0^{1,\varphi(\cdot)}(\Omega)$ is a closed subspace of $W^{1,\varphi(\cdot)}(\Omega)$, we obtain the following result.

Theorem 6.8. *If φ and φ^* are doubling and (A0) holds, then $W_0^{1,\varphi(\cdot)}(\Omega)$ is a reflexive and separable Banach space.*

Lemma 6.9. *Let Ω be a bounded open set. Let $\varphi \in \Phi(\Omega)$ satisfy (A0) and (aInc) $_p$. Then $W_0^{1,\varphi(\cdot)}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ and the embedding norm is bounded by a constant depending on the (A0)-constant, the (aInc) $_p$ -constant and $|\Omega|$.*

Proof. By Lemma 4.4, we obtain

$$\begin{aligned} \|u\|_{W^{1,p}(\Omega)} &\lesssim \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \\ &\lesssim \|u\|_{L^{\varphi(\cdot)}(\Omega)} + \|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)} \lesssim \|u\|_{W^{1,\varphi(\cdot)}(\Omega)} \end{aligned}$$

for all $u \in W^{1,\varphi(\cdot)}(\Omega)$.

Let $u \in W_0^{1,\varphi(\cdot)}(\Omega)$. Let (ξ_j) be a sequence of $C_0^\infty(\Omega)$ -functions converging to u in $W^{1,\varphi(\cdot)}(\Omega)$. Then $\xi_j \rightarrow u$ also in $W^{1,p}(\Omega)$ and the claim follows. \square

The proof of the following theorem is as in the variable exponent case, see [11, Proposition 8.1.14].

Theorem 6.10. *Let $\varphi \in \Phi(\mathbb{R}^n)$ and $u \in W_0^{1,\varphi(\cdot)}(\Omega)$. Then u extend by zero to $\mathbb{R}^n \setminus \Omega$ belongs to $W^{1,\varphi(\cdot)}(\mathbb{R}^n)$.*

Let $\Omega \subset \mathbb{R}^n$ be bounded. If $u \in W_0^{1,1}(\Omega)$, then

$$|u(x)| \lesssim \text{diam}(\Omega)M|\nabla u|(x) \quad \text{a.e.}$$

by [5, Chap. 6]. In the same source it is also shown that in a John domain, we have

$$|u(x) - u_\Omega| \lesssim \text{diam}(\Omega)M|\nabla u|(x) \quad \text{a.e.}$$

From this and the boundedness of M , we directly get:

Theorem 6.11 (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^n$ be bounded. Suppose $\varphi \in \Phi(\Omega)$ satisfies $(A0)$, $(A1)_\Omega$ and $(aInc)_p$ for some $p > 1$. Then*

$$\|u\|_{L^{\varphi(\cdot)}(\Omega)} \lesssim \text{diam}(\Omega)\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)},$$

for every $u \in W_0^{1,\varphi(\cdot)}(\Omega)$. If $\Omega \subset \mathbb{R}^n$ is a John domain, then

$$\|u - u_\Omega\|_{L^{\varphi(\cdot)}(\Omega)} \lesssim \text{diam}(\Omega)\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)},$$

for every $u \in W^{1,\varphi(\cdot)}(\Omega)$.

Note that a bounded convex domain is a John domain.

7. DIRICHLET ENERGY INTEGRAL MINIMIZERS

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $w \in W^{1,\varphi(\cdot)}(\Omega)$. We consider the minimization of the energy

$$(7.1) \quad I_{\varphi(\cdot)}(u) := \varrho_{\varphi(\cdot)}(\nabla u) = \int_{\Omega} \varphi(x, \nabla u(x)) \, dx$$

on the affine set $w + W_0^{1,\varphi(\cdot)}(\Omega) := \{w + u : u \in W_0^{1,\varphi(\cdot)}(\Omega)\}$. Here we use the same methods as in [32] to prove that a minimizer exists.

Let V be a reflexive Banach space and $I : V \rightarrow \mathbb{R}$. The operator I is said to be *convex* if $I(tv + (1-t)u) \leq tI(v) + (1-t)I(u)$ for all $t \in [0, 1]$ and $u, v \in V$. It is *lower semicontinuous* if $I(u) \leq \liminf_{i \rightarrow \infty} I(u_i)$ whenever $u_i \rightarrow u$ in V , and *coercive* if $I(u_i) \rightarrow \infty$ whenever $\|u_i\|_V \rightarrow \infty$. The following is a well known lemma in functional analysis, see for example [22, Theorem 2.1].

Lemma 7.2. *Let V be a reflexive Banach space. If $I : V \rightarrow \mathbb{R}$ is a convex, lower semicontinuous and coercive operator, then there is an element in V that minimizes I .*

Theorem 7.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $\varphi \in \Phi(\Omega)$ is doubling and satisfies (A0), (A1) $_{\Omega}$ and (aInc) $_p$ for some $p > 1$. Let $w \in W^{1,\varphi(\cdot)}(\Omega)$. Then there exists a function $u \in w + W_0^{1,\varphi(\cdot)}(\Omega)$ such that*

$$(7.4) \quad I_{\varphi(\cdot)}(u) = \inf_{v \in w + W_0^{1,\varphi(\cdot)}(\Omega)} I_{\varphi(\cdot)}(v).$$

Proof. Let $J(u) := I_{\varphi(\cdot)}(u + w)$, where $u \in W_0^{1,\varphi(\cdot)}(\Omega)$. By Lemma 2.4, (aInc) $_p$ yields that φ^* satisfies (aDec) $_{p'}$ and thus φ^* is doubling. Hence by Theorem 6.8, the generalized Orlicz–Sobolev space $W_0^{1,\varphi(\cdot)}(\Omega)$ is a reflexive Banach space. We show that J is convex, lower semicontinuous and coercive. Then the assertion follows from Lemma 7.2.

The operator J is convex, since φ is convex. By Theorem 4.1, J is lower semicontinuous. Let (u_i) be a sequence of functions in $W_0^{1,\varphi(\cdot)}(\Omega)$. If $\|u_i\|_{W^{1,\varphi(\cdot)}(\Omega)} \rightarrow \infty$, then by the Poincaré inequality (Theorem 6.11) we have $\|\nabla u_i\|_{W^{1,\varphi(\cdot)}(\Omega)} \rightarrow \infty$. Thus $J(u_i) = \varrho_{\varphi}(\nabla u_i + \nabla w) \rightarrow \infty$ as $i \rightarrow \infty$ and the operator J is coercive. \square

The operator I is strictly convex if $I(tu + (1-t)v) < tI(u) + (1-t)I(v)$ when $u \neq v$ and $t \in (0, 1)$.

Theorem 7.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $\varphi \in \Phi(\Omega)$ is strictly convex and satisfies (A0). Then the possible minimizing function u is unique up to set of measure zero.*

Proof. Assume that u_1 and u_2 are two minimizers of (7.4) with $|\{\nabla u_1 \neq \nabla u_2\}| > 0$. When $\nabla u_1(x) \neq \nabla u_2(x)$, we obtain by strict convexity that

$$\varphi\left(x, \frac{1}{2}\nabla u_1(x) + \frac{1}{2}\nabla u_2(x)\right) < \frac{1}{2}\varphi(x, \nabla u_1(x)) + \frac{1}{2}\varphi(x, \nabla u_2(x)).$$

We set $v = \frac{1}{2}(u_1 + u_2)$. The previous inequality implies that

$$I_{\varphi(\cdot)}(v) < \frac{1}{2}I_{\varphi(\cdot)}(u_1) + \frac{1}{2}I_{\varphi(\cdot)}(u_2) = \inf_{u \in w + W_0^{1,\varphi(\cdot)}(\Omega)} I_{\varphi(\cdot)}(u),$$

which is a contradiction. Therefore $\nabla u_1 = \nabla u_2$ almost everywhere.

Since $u_1 - u_2 \in W_0^{1,\varphi(\cdot)}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$ by Lemma 6.9, we obtain by the Poincaré inequality in $W_0^{1,1}(\Omega)$ [14, 7.44, p. 164] that

$$\|u_1 - u_2\|_{L^1(\Omega)} \lesssim \|\nabla u_1 - \nabla u_2\|_{L^1(\Omega)} = 0,$$

and hence $u_1 = u_2$ for a.e. $x \in \Omega$. \square

If φ is a generalized Φ -function, we abbreviate $\varphi \in C^1(\mathbb{R}^+)$ if $t \mapsto \varphi(x, t)$ is continuously differentiable in $\mathbb{R}^+ := [0, \infty)$ for every $x \in \Omega$.

Theorem 7.6. *Let $\varphi \in \Phi(\Omega) \cap C^1(\mathbb{R}^+)$ be doubling and $w \in W^{1,\varphi(\cdot)}(\Omega)$. The following three conditions are equivalent:*

- (i) *The function $u \in w + W_0^{1,\varphi(\cdot)}(\Omega)$ minimizes $I_{\varphi(\cdot)}$.*
- (ii) *The function $u \in w + W_0^{1,\varphi(\cdot)}(\Omega)$ satisfies*

$$\int_{\Omega} \frac{\varphi'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx \geq 0$$

for every $v \in W_0^{1,p(\cdot)}(\Omega)$.

(iii) The function $u \in w + W_0^{1,\varphi(\cdot)}(\Omega)$ satisfies

$$\int_{\Omega} \frac{\varphi'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx = 0$$

for every $v \in W_0^{1,p(\cdot)}(\Omega)$.

Proof. This proof is a modification of [18, Theorem 5.13]. It is clear that (iii) implies (ii). By considering in (ii) the functions v and $-v$, we see that it implies (iii). So it remains to prove the equivalence of (i) and (ii).

First we prove that (i) implies (ii). We fix $v \in w + W_0^{1,\varphi(\cdot)}(\Omega)$ and set $\lambda = v - u$. Then $\lambda \in W_0^{1,\varphi(\cdot)}(\Omega)$. Let $0 < \varepsilon \leq 1$. Since $u + \varepsilon\lambda \in w + W_0^{1,\varphi(\cdot)}(\Omega)$, we obtain

$$I_{\varphi(\cdot)}(u) \leq I_{\varphi(\cdot)}(u + \varepsilon\lambda),$$

and therefore

$$(7.7) \quad \int_{\Omega} \frac{1}{\varepsilon} (\varphi(x, |\nabla u + \varepsilon \nabla \lambda|) - \varphi(x, |\nabla u|)) \, dx \geq 0.$$

By l'Hôpital's rule,

$$\frac{\varphi(x, |\nabla u + \varepsilon \nabla \lambda|) - \varphi(x, |\nabla u|)}{\varepsilon} \rightarrow \varphi'(x, |\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \nabla \lambda$$

as $\varepsilon \rightarrow 0^+$ for almost every $x \in \Omega$. Then condition (ii) follows by dominated convergence provided that we find an L^1 -majorant independent of ε for the integrand in (7.7).

By the mean value theorem there exists $\varepsilon' \in (0, \varepsilon)$ such that

$$\frac{\varphi(x, |\nabla u + \varepsilon \nabla \lambda|) - \varphi(x, |\nabla u|)}{\varepsilon} = \varphi'(x, |\nabla u + \varepsilon' \nabla \lambda|) \frac{\nabla u + \varepsilon' \nabla \lambda}{|\nabla u + \varepsilon' \nabla \lambda|} \cdot \nabla \lambda.$$

Furthermore,

$$\left| \varphi'(x, |\nabla u + \varepsilon' \nabla \lambda|) \frac{\nabla u + \varepsilon' \nabla \lambda}{|\nabla u + \varepsilon' \nabla \lambda|} \cdot \nabla \lambda \right| \leq \varphi'(x, |\nabla u + \varepsilon' \nabla \lambda|) |\nabla \lambda|.$$

Since φ is convex we obtain $\varphi(2s) \geq \varphi(s) + \varphi'(s)(2s - s)$ and by doubling $s\varphi'(s) \lesssim \varphi(s)$. Thus

$$\varphi'(x, |\nabla u + \varepsilon' \nabla \lambda|) |\nabla \lambda| \leq \varphi'(x, |\nabla u| + |\nabla \lambda|) (|\nabla u| + |\nabla \lambda|) \lesssim \varphi(x, |\nabla u| + |\nabla \lambda|).$$

The upper bound is integrable and independent of ε , so it provides the required majorant.

Then we prove that (ii) implies (i). By convexity,

$$\varphi(x, \xi_2 + \varepsilon(\xi_1 - \xi_2)) = \varphi(x, (1 - \varepsilon)\xi_2 + \varepsilon\xi_1) \leq (1 - \varepsilon)\varphi(x, \xi_2) + \varepsilon\varphi(x, \xi_1)$$

for $0 < \varepsilon < 1$. Then we obtain by setting $\xi = \xi_1 - \xi_2$ that

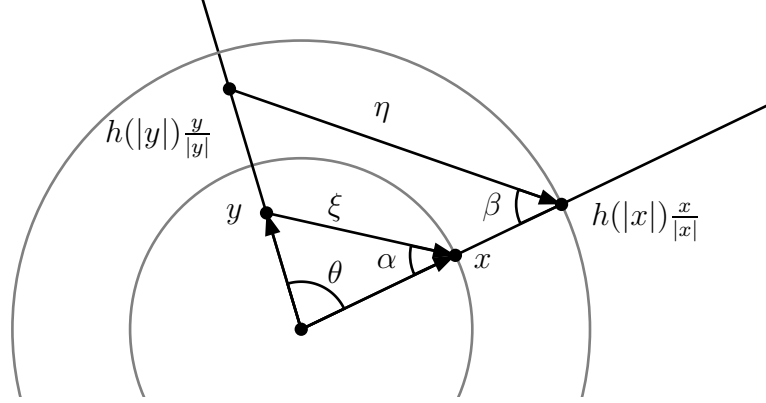
$$\varphi(x, \xi_2 + \varepsilon\xi) - \varphi(x, \xi_2) \leq \varepsilon(\varphi(x, \xi + \xi_2) - \varphi(x, \xi_2)).$$

With the choice $\xi = \nabla v$ and $\xi_2 = \nabla u$, we find that

$$\frac{\varphi(x, \nabla u + \varepsilon \nabla v) - \varphi(x, \nabla u)}{\varepsilon} \leq \varphi(x, \nabla u + \nabla v) - \varphi(x, \nabla u).$$

Letting $\varepsilon \rightarrow 0^+$, we find by l'Hôpital's rule that

$$\varphi'(x, |\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \nabla v \leq \varphi(x, \nabla u + \nabla v) - \varphi(x, \nabla u)$$

FIGURE 1. Variables η and ξ in the proof of Lemma 8.2.

and hence $I_{\varphi(\cdot)}(u) \leq I_{\varphi(\cdot)}(u + v)$ for every $v \in W_0^{1,\varphi(\cdot)}(\Omega)$. \square

8. NON-ZERO RIGHT HAND SIDE

We continue the study of the energy $I_{\varphi(\cdot)}$ defined in (7.1). In Theorem 7.6 we saw that the minimizing problem corresponds to a weak solution of the equation $A(u) = 0$, where the operator $A : W_0^{1,\varphi(\cdot)}(\Omega) \rightarrow (W_0^{1,\varphi(\cdot)})^*(\Omega)$ is defined by

$$\langle A(u), v \rangle := \int_{\Omega} \frac{\varphi'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx \quad \text{for all } v \in W_0^{1,\varphi(\cdot)},$$

and $\langle \cdot, \cdot \rangle$ denotes the inner product $\langle f, g \rangle = \int_{\Omega} f(x)g(x) \, dx$. In this section we use a different approach to study existence of weak solutions of the equation $A(u) = f$.

We recall some definitions (cf., e.g., [31, Definition 2.3]).

Definition 8.1. Let $A : V \rightarrow V^*$ be an operator on a separable, reflexive Banach space V . The operator is:

- (1) *monotone*, if $\langle A(u) - A(v), u - v \rangle \geq 0$ for all $u, v \in V$.
- (2) *hemicontinuous*, if the mapping $t \mapsto \langle A(u + tv), w \rangle$ is continuous for all fixed $u, v, w \in V$.
- (3) *radially continuous*, if the mapping $t \mapsto \langle A(u + tv), v \rangle$ is continuous for all fixed $u, v \in V$.
- (4) *coercive*, if

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle A(u), u \rangle}{\|u\|} = \infty.$$

Clearly a hemicontinuous operator is radially continuous. By the Minty–Browder Theorem, any radially continuous, monotone and coercive operator A is surjective; i.e. for any $f \in V^*$, there is at least one solution to the equation $A(u) = f$, see, e.g., [31, Theorem 2.18].

Lemma 8.2. *If $h : [0, \infty) \rightarrow [0, \infty)$ is increasing, then $h(|x|) \frac{x}{|x|}$ is monotone on \mathbb{R}^n .*

Proof. Let $x, y \in \mathbb{R}^n$, with angle $\theta \in [0, \pi]$ between them at the origin and denote $\xi := x - y$ and $\eta := h(|x|) \frac{x}{|x|} - h(|y|) \frac{y}{|y|}$, see Figure 1. The claim is that $\xi \cdot \eta \geq 0$.

Let l be the line through the origin and x . Without loss of generality, we assume that $|y| \leq |x|$; then $h(|y|) \leq h(|x|)$. Hence the angle α between ξ and l lies in $[0, \frac{\pi-\theta}{2}]$, as does the angle β between η and l . Consequently, the angle between ξ and η lies between $\pm \frac{\pi-\theta}{2}$. Hence $\xi \cdot \eta \geq 0$. \square

Corollary 8.3. *If $\varphi \in \Phi(\Omega)$, then the operator A is monotone.*

Proof. Since φ is convex, φ' is increasing and so Lemma 8.2 implies that

$$\langle A(u) - A(v), u - v \rangle = \int_{\mathbb{R}^n} \left(\varphi'(x, |\nabla u|) \frac{\nabla u}{|\nabla u|} - \varphi'(x, |\nabla v|) \frac{\nabla v}{|\nabla v|} \right) \cdot (\nabla u - \nabla v) dx \geq 0. \quad \square$$

Proposition 8.4. *If $\varphi \in \Phi(\Omega) \cap C^1(\mathbb{R}^+)$ is doubling and $\varphi'(x, 0) \equiv 0$, then the operator A is hemicontinuous.*

Proof. Since φ is convex and doubling we obtain, for $s \in (t, 2t]$, that

$$\frac{\varphi(x, s) - \varphi(x, t)}{s - t} \leq \frac{\varphi(x, 2t) - \varphi(x, t)}{2t - t} \leq (K - 1) \frac{\varphi(x, t)}{t}$$

and thus $\varphi'(x, t) \leq (K - 1) \frac{\varphi(x, t)}{t}$. Hence

$$\left| \frac{\varphi'(x, \xi)}{|\xi|} \xi \cdot \zeta \right| \leq \varphi'(x, \xi) |\zeta| \lesssim \frac{\varphi(x, \xi)}{|\xi|} |\zeta|.$$

Furthermore, we recall that $\frac{\varphi(x, t)}{t}$ is increasing.

Fix $u, v, w \in W^{1, \varphi(\cdot)}(\mathbb{R}^n)$. Let $t_0 > 0$ and $g := (1 + t_0) \max\{|\nabla u|, |\nabla v|, |\nabla w|\}$. Then, for $t \in [0, t_0]$, $|\nabla(u + tv)|, |\nabla w| \leq g$ and it follows that

$$\left| \frac{\varphi'(x, |\nabla(u + tv)|)}{|\nabla(u + tv)|} \nabla(u + tv) \cdot \nabla w \right| \lesssim \frac{\varphi(x, |\nabla(u + tv)|)}{|\nabla(u + tv)|} |\nabla w| \leq \frac{\varphi(x, g)}{g} g = \varphi(x, g).$$

By the definition of g and the doubling condition, $\varphi(x, g) \lesssim \varphi(x, \nabla u) + \varphi(x, \nabla v) + \varphi(x, \nabla w)$. Therefore $\varphi(x, g)$ is integrable. Also, if $t \rightarrow t_1 \in [0, t_0]$, then

$$\left| \frac{\varphi'(x, |\nabla(u + tv)|)}{|\nabla(u + tv)|} \nabla(u + tv) \cdot \nabla w \right| \rightarrow \left| \frac{\varphi'(x, |\nabla(u + t_1v)|)}{|\nabla(u + t_1v)|} \nabla(u + t_1v) \cdot \nabla w \right|;$$

here the assumption $\varphi'(x, 0) \equiv 0$ is used to ensure the continuity of $\varphi'(\xi) \frac{\xi}{|\xi|}$ at the origin. Since $\varphi(x, g)$ is a majorant of the expression for all $t \in [0, t_0]$, the continuity of $t \mapsto \langle A(u + tv), w \rangle$ in $[0, t_0]$ follows by dominated convergence. Since t_0 is arbitrary, the continuity holds in all of \mathbb{R}^+ . \square

Theorem 8.5. *Let Ω be a bounded domain. If $\varphi \in \Phi(\Omega) \cap C^1(\mathbb{R}^+)$ is doubling and satisfies (A0), (A1) $_{\Omega}$, (aInc) $_p$ for some $p > 1$ and $f \in (W_0^{1, \varphi(\cdot)})^*(\Omega)$, then there exists a solution $u \in W_0^{1, \varphi(\cdot)}(\Omega)$ of $A(u) = f$.*

Proof. By Corollary 8.3, A is monotone. By (aInc) $_p$, $p > 1$, $\frac{\varphi(x, t)}{t} \rightarrow 0$ as $t \rightarrow 0$. Therefore $\varphi'(x, 0) \equiv 0$ and so A is radially continuous by Proposition 8.4. Furthermore, if A is coercive, then the claim follows by the Minty–Browder Theorem [31, Theorem 2.18].

So we show that A is coercive. By convexity, $\varphi'(x, t) \geq \frac{\varphi(x, t)}{t}$, so that

$$\langle A(u), u \rangle = \int_{\Omega} \frac{\varphi'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla u dx = \int_{\Omega} \varphi'(x, |\nabla u|) |\nabla u| dx \geq \int_{\Omega} \varphi(x, |\nabla u|) dx.$$

By the Poincaré inequality, Theorem 6.11,

$$\|u\|_{W_0^{1, \varphi(\cdot)}(\Omega)} \lesssim \|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}.$$

Thus, by Lemma 3.3,

$$\frac{\langle A(u), u \rangle}{\|u\|_{W_0^{1,\varphi(\cdot)}(\Omega)}} \gtrsim \frac{\int_{\Omega} \varphi(x, |\nabla u|) dx}{\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}} \gtrsim \frac{\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}^p}{\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}} \rightarrow \infty$$

as $\|u\|_{W_0^{1,\varphi(\cdot)}(\Omega)} \rightarrow \infty$ since $p > 1$. □

Remark 8.6. Since $W_0^{1,\varphi(\cdot)} \subset L_0^{\varphi(\cdot)}$, it follows that $L^{\varphi^*(\cdot)} \subset (L_0^{\varphi(\cdot)})^* \subset (W_0^{1,\varphi(\cdot)})^*$, so the previous theorem hold in particular when $f \in L^{\varphi^*(\cdot)}(\Omega)$. With the Sobolev embedding (cf. [15]) we may conclude that the larger generalized Orlicz space $L^{(\varphi^\#)^*(\cdot)}$ belongs to $(W_0^{1,\varphi(\cdot)})^*$ which leads to the existence of solutions in this case also.

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