

# HOMOGENEOUS VARIABLE EXPONENT BESOV AND TRIEBEL–LIZORKIN SPACES

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ABSTRACT. We introduce homogeneous Besov and Triebel–Lizorkin spaces with variable indexes. We show that their study reduces to the study of inhomogeneous variable exponent spaces and homogeneous constant exponent spaces. Corollaries include trace space characterizations and Sobolev embeddings.

## 1. INTRODUCTION

In this paper we deal with homogeneous Besov and Triebel–Lizorkin spaces with variable indexes. To contextualize the study we start with a review of previous results.

Spaces of variable integrability, also known as variable exponent function spaces, can be traced back to 1931 and Orlicz [44], but the modern development started with the paper [35] of Kováčik and Rákosník in 1991. Corresponding PDE with non-standard growth have been studied since the same time. For an overview we refer to the monographs [15, 17] and the survey [27]. Apart from interesting theoretical considerations, the motivation to study such function spaces comes from applications to fluid dynamics [46], image processing [13, 26, 38], PDE and the calculus of variations [1, 2, 8, 9, 12].

Some ten years ago, Sobolev spaces were extended to variable exponent Bessel potential spaces  $\mathcal{L}^{\alpha,p(\cdot)}$ ,  $\alpha \in \mathbb{R}$  constant, by Almeida and Samko [7] and Gurka, Harjulehto and Nekvinda [25]. As in the classical case, this space coincides with the Sobolev space for integer  $\alpha \geq 0$ . Then Xu [52, 53], considered Besov  $B_{p(\cdot),q}^\alpha$  and Triebel–Lizorkin  $F_{p(\cdot),q}^\alpha$  spaces with variable  $p$ , but fixed  $q$  and  $\alpha$ .

Along a different line of inquiry, Leopold [36, 37] considered a generalization of Besov spaces with smoothness index determined by certain symbols of hypoelliptic pseudo-differential operators. In particular, for symbols of the form  $(1 + |\xi|^2)^{m(x)/2}$  the related spaces coincide with  $B_{p,p}^{m(\cdot)}$ . Function spaces of variable smoothness have been studied by Besov [11]: he generalized Leopold's work by considering both Triebel–Lizorkin spaces  $F_{p,q}^{\alpha(\cdot)}$  and Besov spaces  $B_{p,q}^{\alpha(\cdot)}$  in  $\mathbb{R}^n$ . By way of application, Schneider, Reichmann and Schwab [48] used  $B_{2,2}^{m(\cdot)}(\mathbb{R})$  in the analysis of certain Black–Scholes equations. More recently, Tyulenev [50] showed that such spaces appear as traces of (classical) Sobolev spaces with Muckenhoupt weights.

Integrating the above mentioned spaces into the full Besov and Triebel–Lizorkin scales simplifies the theory as one result then applies in all special cases; however, this requires all

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the indexes to be variable. This three-index generalization was done by Diening, Hästö and Roudenko [19] for Triebel–Lizorkin spaces  $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ , and by Almeida and Hästö [5] for Besov spaces  $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ . Additional results, including Sobolev embeddings and atomic decompositions, have subsequently been proven by Vybíral and Kempka [30, 31, 32, 33, 51], and others [3, 4, 20, 21, 23, 29, 42, 55]; see also the papers [6, 24, 40] for the study of traces.

Recently, results from variable exponent spaces have been derived in the more general Musielak–Orlicz setting, see, e.g., [14, 28, 39, 43]. Yang, Yuan and Zhou [54] considered versions of Besov and Triebel–Lizorkin spaces in this setting.

All the papers mentioned so far dealt only with inhomogeneous spaces. Homogeneous variable exponent spaces have been studied in only a few papers: Hardy spaces by Nakai and Sawano [41, 47] and by Cruz-Uribe and Wang [16], and interpolation between BMO and  $L^{p(\cdot)}$  by Kopalani [34].

Our main result shows that the study of homogeneous Besov and Triebel–Lizorkin spaces reduces to the study of homogeneous spaces with constant exponent and inhomogeneous spaces with variable exponent when the smoothness does not vary greatly. Therefore, we obtain as immediate corollaries for instance Sobolev embeddings and trace theorems in Sections 5 and 6.

We use a Fourier analytical approach to the Besov and Triebel–Lizorkin spaces. For this we need some general definitions, well-known from the constant exponent case. For the notation see Section 2.

**Definition 1.1.** Let  $\varphi$  be a function in  $\mathcal{S}$  satisfying

- $\text{supp } \hat{\varphi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$  and  $|\hat{\varphi}(\xi)| \geq c > 0$  when  $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$ ,
- $\sum_{\nu \in \mathbb{Z}} \hat{\varphi}_\nu(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$ ,

where  $\varphi_\nu(x) := 2^{\nu n} \varphi(2^\nu x)$  for  $\nu \in \mathbb{Z}$ . The system  $\{\varphi_\nu\}$  is called *admissible*. Also we define  $\Phi_- \in \mathcal{S}$  by

$$\hat{\Phi}_-(\xi) := \sum_{\nu=-\infty}^0 \hat{\varphi}(2^{-\nu}\xi) \text{ if } \xi \neq 0, \quad \text{and} \quad \hat{\Phi}_-(0) := 1.$$

**Definition 1.2.** Let  $\varphi_\nu$  be as in Definition 1.1. For  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $p, q \in \mathcal{P}_0$ , the *homogeneous Besov space*  $\dot{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$  and the *homogeneous Triebel–Lizorkin space*  $\dot{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$  consist respectively of all distributions  $f \in \mathcal{S}'$  such that

$$\begin{aligned} \|f\|_{\dot{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^\varphi &:= \|(2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu\|_{\ell_{\mathbb{Z}}^{q(\cdot)}(L^{p(\cdot)})} < \infty, \\ \|f\|_{\dot{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^\varphi &:= \|(2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu\|_{L^{p(\cdot)}(\ell_{\mathbb{Z}}^{q(\cdot)})} < \infty. \end{aligned}$$

The main result of this paper is that the previous “norms” (cf. Remark 1.5) are equivalent to the sum of a variable exponent part with positive  $\nu$  which is identical to the previously considered inhomogeneous spaces’ (quasi)norms [5, 19], and to a constant index part with negative  $\nu$ . See also Theorem 4.4 for the equality of norms without the independence from the basis functions under more general assumptions.

**Theorem 1.3.** For  $f \in \mathcal{S}'$  we set  $f^l := f * \Phi_-$  and  $f^h = f - f^l$ . Let  $p, q \in \mathcal{P}_0^{\text{log}}$  and  $\alpha \in L^\infty(\mathbb{R}^n)$  be log-Hölder continuous. Suppose that  $p \in L^\infty(\mathbb{R}^n)$ , and for the Triebel–Lizorkin case also  $q \in L^\infty(\mathbb{R}^n)$ . Assume that  $\alpha^+ - \alpha^- < \frac{n}{p^+}$ . Then the B- and F-spaces

are well-defined, i.e. they are independent of the choice  $(\varphi_\nu)$ , and

$$\begin{aligned}\|f\|_{\dot{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &\approx \|f^h\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} + \|f^l\|_{\dot{B}_{p_\infty, q_\infty}^{\alpha_\infty}}, \\ \|f\|_{\dot{F}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &\approx \|f^h\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} + \|f^l\|_{\dot{F}_{p_\infty, q_\infty}^{\alpha_\infty}}.\end{aligned}$$

In particular, if  $\alpha$  is constant we have the following consequence.

**Corollary 1.4.** *For  $f \in \mathcal{S}'$  we set  $f^l := f * \Phi_-$  and  $f^h = f - f^l$ . Let  $p, q \in \mathcal{P}_0^{\log}$  and  $\alpha \in \mathbb{R}$ . Suppose that  $p \in L^\infty(\mathbb{R}^n)$ , and for the Triebel–Lizorkin case also  $q \in L^\infty(\mathbb{R}^n)$ . Then the  $B$ - and  $F$ -spaces are well-defined and*

$$\begin{aligned}\|f\|_{\dot{B}_{p(\cdot), q(\cdot)}^\alpha} &\approx \|f^h\|_{B_{p(\cdot), q(\cdot)}^\alpha} + \|f^l\|_{\dot{B}_{p_\infty, q_\infty}^\alpha}, \\ \|f\|_{\dot{F}_{p(\cdot), q(\cdot)}^\alpha} &\approx \|f^h\|_{F_{p(\cdot), q(\cdot)}^\alpha} + \|f^l\|_{\dot{F}_{p_\infty, q_\infty}^\alpha}.\end{aligned}$$

The results state that there are no essential differences between the homogeneous and inhomogeneous spaces when we deal with high frequencies. On the other hand, the low frequency part behaves like in the constant exponent case. Note that any  $f \in \mathcal{S}'$  can be decomposed as

$$f = \Phi_- * f + \sum_{\nu=1}^{\infty} \varphi_\nu * f \quad (\text{convergence in } \mathcal{S}').$$

*Remark 1.5.* For  $f \in \mathcal{S}'$ , we have  $\|f\|_{\dot{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = 0 = \|f\|_{\dot{F}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}$  if and only if  $\text{supp } \hat{f} = \{0\}$ , i.e. if and only if  $f$  is a polynomial. This means that  $\|\cdot\|_{\dot{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}$  and  $\|\cdot\|_{\dot{F}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}$  are not exactly (quasi)norms. The way of remedying this would be considering the  $\dot{B}$  and  $\dot{F}$  spaces as subspaces of  $\mathcal{S}'/P$ , where  $P$  stands for the set of all polynomials in  $\mathbb{R}^n$ . We refer to the monographs [10], [45] and [49] for further details.

*Remark 1.6.* We can show by example that the result does not necessarily hold, even when  $p = q = 2$ , in the case  $\alpha^+ - \alpha^- > n$ . It is unclear what happens when  $\alpha^+ - \alpha^- \in [\frac{n}{p}, n]$ .

The results also raise the question of whether this definition of the homogeneous space is good when the variability in  $\alpha$  is large. Perhaps  $2^{\nu\alpha(x)}$  should be replaced by  $2^{\nu\alpha_{B(x, 2^\nu)}}$  for some average  $\alpha_{B(x, 2^\nu)}$  at the appropriate scale, which would guarantee that the low frequencies are affected only by  $\alpha_\infty$ . This question is left for future research.

## 2. BACKGROUND MATERIAL

As usual, we denote by  $\mathbb{R}^n$  the  $n$ -dimensional real Euclidean space,  $\mathbb{N}$  the set of all natural numbers and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . We write  $B(x, r)$  for the open ball in  $\mathbb{R}^n$  centered at  $x \in \mathbb{R}^n$  with radius  $r > 0$ . We use  $c$  as a generic positive constant, i.e. a constant whose value may change from appearance to appearance. The expression  $f \lesssim g$  means that  $f \leq cg$  for some suitably independent constant  $c$ , and  $f \approx g$  means  $f \lesssim g \lesssim f$ .

The notation  $X \hookrightarrow Y$  stands for a continuous embedding from  $X$  to  $Y$ , where  $X$  and  $Y$  are quasi-normed spaces. If  $E \subset \mathbb{R}^n$  is a measurable set, then  $|E|$  stands for its (Lebesgue) measure and  $\chi_E$  denotes its characteristic function. By  $\text{supp } f$  we denote the support of the function  $f$ , i.e. the closure of its zero set.

The symbol  $\mathcal{S}$  denotes the Schwartz class of infinitely differentiable rapidly decreasing complex-valued functions and  $\mathcal{S}'$  denotes the dual space of tempered distributions. The Fourier transform of a tempered distribution  $f$  is denoted by  $\hat{f}$  or  $\mathcal{F}f$ .

We denote the sequence spaces over  $\mathbb{Z}$ ,  $\mathbb{N}_0$  and  $-\mathbb{N}_0$  by  $\ell^q$ ,  $\ell_+^q$  and  $\ell_-^q$ , respectively.

**2.1. Variable exponents.** We present some basic background on variable exponent spaces. For more information and proofs see, e.g., [17]. We denote by  $\mathcal{P}_0$  the set of measurable functions  $p : \mathbb{R}^n \rightarrow (0, \infty]$  (called *variable exponents*) which are bounded away from zero. The subset of variable exponents with range  $[1, \infty]$  is denoted by  $\mathcal{P}$ . For  $A \subset \mathbb{R}^n$  and  $p \in \mathcal{P}_0$  we denote  $p_A^+ = \text{ess sup}_A p(x)$  and  $p_A^- = \text{ess inf}_A p(x)$ ; we abbreviate  $p^+ = p_{\mathbb{R}^n}^+$  and  $p^- = p_{\mathbb{R}^n}^-$ .

Let

$$\varphi_p(t) = \begin{cases} t^p & \text{if } p \in (0, \infty), \\ 0 & \text{if } p = \infty \text{ and } t \leq 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1. \end{cases}$$

The convention  $1^\infty = 0$  is adopted in order that  $\varphi_p$  be left-continuous. In what follows we write  $t^p$  instead of  $\varphi_p(t)$ , with this convention implied. The variable exponent modular is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \varphi_{p(x)}(|f(x)|) dx.$$

The *variable exponent Lebesgue space*  $L^{p(\cdot)}$  is the class of all measurable functions  $f$  on  $\mathbb{R}^n$  such that  $\varrho_{p(\cdot)}(\lambda f) < \infty$  for some  $\lambda > 0$ . This is a quasi-Banach space when equipped with the quasi-norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)}\left(\frac{1}{\lambda}f\right) \leq 1 \right\}.$$

If  $p(x) \equiv p$  is constant, then  $L^{p(\cdot)} = L^p$  is the classical Lebesgue space.

Let  $U \subset \mathbb{R}^n$ . We say that  $g : U \rightarrow \mathbb{R}$  satisfies the *local log-Hölder continuity* condition if

$$|g(x) - g(y)| \leq \frac{c}{\log(e + 1/|x - y|)}$$

for all  $x, y \in U$ . If

$$|g(x) - g_\infty| \leq \frac{c'}{\log(e + |x|)}$$

for some  $g_\infty \in \mathbb{R}$ ,  $c' > 0$  and all  $x \in U$ , then we say  $g$  satisfies the *log-Hölder decay condition* (at infinity). If both conditions are satisfied, we simply speak of *log-Hölder continuity*. By the *log-Hölder constant* we mean  $c_{\log}(g) := \max\{c, c'\}$ . The subset of  $\mathcal{P}_0$  and  $\mathcal{P}$  with  $\frac{1}{p}$  log-Hölder continuous is denoted by  $\mathcal{P}_0^{\log}$  and  $\mathcal{P}^{\log}$ , respectively.

We say that a function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies the *log log-Hölder decay condition* if

$$|g(x) - g_\infty| \leq \frac{c}{\log(e + \log(e + |x|))}$$

for all  $x \in \mathbb{R}^n$ . A sequence  $(a_\nu)_\nu$  satisfies the log-Hölder decay condition if

$$|a_\nu - a_\infty| \leq \frac{c}{\log(e + \nu)}$$

for all  $\nu \in \mathbb{N}$ .

As in [19], we define a class of  *$\eta$ -functions* on  $\mathbb{R}^n$  by

$$\eta_{\nu, m}(x) := \frac{2^{n\nu}}{(1 + 2^\nu|x|)^m}$$

with  $\nu \in \mathbb{Z}$  and  $m > 0$ . Note that  $\eta_{\nu, m} \in L^1$  when  $m > n$  and that  $\|\eta_{\nu, m}\|_1 = c_m$  is independent of  $\nu$ .

The next lemma often allows us to deal with exponents which are smaller than 1. Although the statement in the reference is only for  $\nu \geq 0$ , the same proof works for negative  $\nu$  as well.

**Lemma 2.1** (“The  $r$ -trick”, [19]). *Let  $r > 0$ ,  $\nu \in \mathbb{Z}$  and  $m > n$ . Then*

$$|g(x)| \lesssim (\eta_{\nu,m} * |g|^r(x))^{1/r}, \quad x \in \mathbb{R}^n$$

for all  $g \in \mathcal{S}'$  with  $\text{supp } \hat{g} \subset \{\xi : |\xi| \leq 2^{\nu+1}\}$ . The implicit constant is independent of  $\nu$ .

**2.2. Mixed spaces.** The *mixed Lebesgue-sequence space*  $L^{p(\cdot)}(\ell^{q(\cdot)})$  is easy to define: for each point  $x$  we just use the constant exponent sequence space  $\ell^{q(x)}$ . Thus

$$\|(f_\nu)_\nu\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} := \left\| \left\| (f_\nu)_\nu \right\|_{\ell^{q(x)}} \right\|_{L^{p(\cdot)}}.$$

This space was studied in [19]. The opposite case is more complicated; the following definition is from [5].

**Definition 2.2.** Let  $p, q \in \mathcal{P}_0$ . The *mixed sequence-Lebesgue space*  $\ell^{q(\cdot)}(L^{p(\cdot)})$  is defined on sequences of  $L^{p(\cdot)}$ -functions by the semimodular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_\nu)_\nu) := \sum_{\nu \in \mathbb{Z}} \inf \left\{ \lambda_\nu > 0 \mid \varrho_{p(\cdot)}\left(f_\nu / \lambda_\nu^{\frac{1}{q(\cdot)}}\right) \leq 1 \right\}.$$

Here we use the convention  $\lambda^{1/\infty} = 1$ . The (quasi)norm is defined from this as usual:

$$\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf \left\{ \mu > 0 \mid \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_\nu)_\nu\right) \leq 1 \right\}.$$

Note that  $\inf \left\{ \lambda > 0 \mid \varrho_{p(\cdot)}\left(f / \lambda^{\frac{1}{q(\cdot)}}\right) \leq 1 \right\} = \| |f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}}$  when  $q^+ < \infty$ .

### 2.3. Inhomogeneous Besov and Triebel–Lizorkin spaces with variable indexes.

Besov and Triebel–Lizorkin spaces ( $B$  and  $F$  spaces for short) with variable smoothness and integrability were introduced in [5] and [19], respectively. Let  $\varphi_\nu$  be as in Definition 1.1 for  $\nu \geq 1$  and set for this paragraph  $\varphi_0(x) := \Phi_-(x)$ . For measurable  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $p, q \in \mathcal{P}_0$ , the (inhomogeneous) Besov space  $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$  is defined as the class of all distributions  $f \in \mathcal{S}'$  such that

$$\|f\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}^\varphi := \left\| (2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu \right\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The (inhomogeneous) Triebel–Lizorkin space  $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$  consists of all  $f \in \mathcal{S}'$  such that

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}^\varphi := \left\| (2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu \right\|_{L^{p(\cdot)}(\ell_+^{q(\cdot)})} < \infty.$$

As in the constant exponent case, they are quasi-normed spaces and they coincide when  $p = q$ , i.e.,  $B_{p(\cdot), p(\cdot)}^{\alpha(\cdot)} = F_{p(\cdot), p(\cdot)}^{\alpha(\cdot)}$  (with  $\alpha$  and  $p$  bounded). These spaces are in most cases well-defined in the sense that different admissible systems  $(\varphi_\nu)$  produce the same spaces (up to equivalence of quasinorms). For Besov spaces this is the case at least when  $p, q \in \mathcal{P}_0^{\text{log}}$  and  $\alpha$  is locally log-Hölder continuous [5, Theorem 5.5]. For Triebel–Lizorkin spaces it holds when  $p, q \in \mathcal{P}_0^{\text{log}}$  are bounded and  $\alpha$  has a limit at infinity [19, Theorem 3.10].

As regards the independence of the spaces from the admissible system, we also refer to the recent paper [3, Section 3] for further details on this topic, even in the more general setting of 2-microlocal spaces with variable integrability.

The scale  $F_{p(\cdot),q(\cdot)}^\alpha$  includes Bessel potential spaces and Sobolev spaces with variable integrability as special cases: For  $\alpha \in [0, \infty)$  and  $p \in \mathcal{P}^{\log}$  with  $1 < p^- \leq p^+ < \infty$ , it was proved in [18, Theorem 4.5] that  $F_{p(\cdot),2}^\alpha = \mathcal{L}^{\alpha,p(\cdot)}$  are the variable exponent Bessel potential spaces introduced in [7, 25]. In particular,  $\mathcal{L}^{k,p(\cdot)} = W^{k,p(\cdot)}$  are Sobolev spaces for  $k \in \mathbb{N}_0$ , under the same assumptions on  $p$  [7, Corollary 6.2].

The Besov scale includes variable order Hölder–Zygmund spaces as special cases [5, Theorem 7.2]. For  $\alpha$  locally log-Hölder continuous with  $\alpha^- > 0$ , we have

$$B_{\infty,\infty}^{\alpha(\cdot)} = \mathcal{C}^{\alpha(\cdot)} \quad (\alpha^+ \leq 1) \quad \text{and} \quad B_{\infty,\infty}^{\alpha(\cdot)} = C^{\alpha(\cdot)} \quad (\alpha^+ < 1).$$

### 3. EQUIVALENCE OF (QUASI)NORMS

We now start constructing the proof of the main theorem. First we deal with just the iterated norms of sequences of functions.

**Lemma 3.1.** *Let  $q_1, q_2 \in (0, \infty]$ . If  $|a_\nu| \leq \beta^{-\nu}$ ,  $\beta > 1$ ,  $\nu \in \mathbb{N}_0$ , then*

$$\|(a_\nu)\|_{\ell_+^{q_1}} \lesssim \log \left( e + \frac{1}{\|(a_\nu)\|_{\ell_+^{q_2}}} \right)^{|\frac{1}{q_1} - \frac{1}{q_2}|} \|(a_\nu)\|_{\ell_+^{q_2}}.$$

*Proof.* If  $q_1 \geq q_2$ , then  $\ell_+^{q_2} \hookrightarrow \ell_+^{q_1}$ , so the claim is trivial; thus we assume that  $q_1 < q_2$ . Let  $\nu_0 \in \mathbb{N}$  and  $z := \|(a_\nu)\|_{\ell_+^{q_2}}$ . We assume that  $z > 0$ , since  $z = 0$  is also trivial. Let  $s > 0$  be such that  $1/q_1 = 1/q_2 + 1/s$ . By Hölder's inequality

$$\|(a_1, \dots, a_{\nu_0}, 0, 0, \dots)\|_{\ell_+^{q_1}} \leq \|(1, \dots, 1, 0, 0, \dots)\|_{\ell_+^s} \|(a_1, \dots, a_{\nu_0}, 0, 0, \dots)\|_{\ell_+^{q_2}} \leq \nu_0^{\frac{1}{s}} z.$$

For the terms  $a_{\nu_0+1}, a_{\nu_0+2}, \dots$  we use the estimate  $|a_\nu| \leq \beta^{-\nu}$ . Thus

$$(3.2) \quad \frac{\|(a_\nu)\|_{\ell_+^{q_1}}}{\|(a_\nu)\|_{\ell_+^{q_2}}} \leq \frac{1}{z} \left( z \nu_0^{\frac{1}{s}} + \left( \sum_{\nu > \nu_0} \beta^{-\nu} \right)^{\frac{1}{q_1}} \right) = \nu_0^{\frac{1}{s}} + \frac{1}{z} \left( \frac{\beta^{-\nu_0}}{\beta - 1} \right)^{\frac{1}{q_1}}$$

With the choice  $\nu_0 := q_0 \lceil \frac{\log(e+1/z)}{\log \beta} \rceil$  we find that

$$\frac{1}{z} \beta^{-\frac{\nu_0}{q_0}} \leq \frac{1}{z(e+1/z)} = \frac{1}{1+ez} \leq 1.$$

On the other hand,  $\nu_0^{\frac{1}{s}} \geq 1$ , so that

$$\nu_0^{\frac{1}{s}} + \frac{1}{z} \left( \frac{\beta^{-\nu_0}}{\beta - 1} \right)^{\frac{1}{q_1}} \leq \nu_0^{\frac{1}{s}} + (\beta - 1)^{-\frac{1}{q_1}} \leq (1 + (\beta - 1)^{-\frac{1}{q_1}}) \nu_0^{\frac{1}{s}},$$

and so the claim follows from (3.2).  $\square$

**Lemma 3.3.** *Let  $p \in \mathcal{P}_0$ . If  $\frac{1}{q}$  satisfies the log log-Hölder decay condition and  $|f_\nu| \lesssim \beta^{-\nu} \|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q_\infty})}$ , then*

$$\|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q(\cdot)})} \approx \|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q_\infty})}.$$

*Proof.* We prove that  $\|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q(\cdot)})} \lesssim \|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q_\infty})}$ ; the other inequality is analogous. We assume first that  $\|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q_\infty})} = 1$ . If  $\|(f_\nu(x))\|_{\ell_+^{q_\infty}} > \eta_{0,m}(x)$ , for some fixed

$m > 2n/p^-$ , then by Lemma 3.1,

$$\begin{aligned} \|(f_\nu(x))\|_{\ell_+^{q(x)}} &\lesssim \log \left( e + \frac{1}{\|(f_\nu(x))\|_{\ell_+^{q_\infty}}} \right)^{|\frac{1}{q(x)} - \frac{1}{q_\infty}|} \|(f_\nu(x))\|_{\ell_+^{q_\infty}} \\ &\lesssim \log \left( e + \frac{1}{\eta_{0,m}(x)} \right)^{|\frac{1}{q(x)} - \frac{1}{q_\infty}|} \|(f_\nu(x))\|_{\ell_+^{q_\infty}} \\ &\lesssim \|(f_\nu(x))\|_{\ell_+^{q_\infty}}, \end{aligned}$$

where the last inequality follows from the decay condition on  $q$ .

There exists a constant such that

$$\log \left( e + \frac{1}{s} \right)^{\frac{1}{q^-}} s \leq c \sqrt{t}$$

for  $0 \leq s \leq t \leq 1$ . Now if  $\|(f_\nu(x))\|_{\ell_+^{q_\infty}} \leq \eta_{0,m}(x)$ , then

$$\begin{aligned} \|(f_\nu(x))\|_{\ell_+^{q(x)}} &\lesssim \log \left( e + \frac{1}{\|(f_\nu(x))\|_{\ell_+^{q_\infty}}} \right)^{|\frac{1}{q(x)} - \frac{1}{q_\infty}|} \|(f_\nu(x))\|_{\ell_+^{q_\infty}} \\ &\leq \log \left( e + \frac{1}{\|(f_\nu(x))\|_{\ell_+^{q_\infty}}} \right)^{\frac{1}{q^-}} \|(f_\nu(x))\|_{\ell_+^{q_\infty}} \\ &\lesssim \eta_{0,m}(x)^{1/2}. \end{aligned}$$

Combining these two estimates, we conclude that

$$\|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q(\cdot)})} \lesssim \left\| \|(f_\nu)\|_{\ell_+^{q_\infty}} + \eta_{0,m}^{1/2} \right\|_{L^{p(\cdot)}} \leq \|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q_\infty})} + \|\eta_{0,m}^{1/2}\|_{L^{p(\cdot)}} \leq c.$$

Now we proceed from this by a *scaling argument*: for a sequence of functions  $g_\nu$  satisfying the assumptions of the lemma we consider  $f_\nu := g_\nu / \|(g_\nu)\|_{L^{p(\cdot)}(\ell_+^{q_\infty})}$ . Then  $f_\nu$  satisfies the assumption of the proof, so that

$$\frac{1}{\|(g_\nu)\|_{L^{p(\cdot)}(\ell_+^{q_\infty})}} \|(g_\nu)\|_{L^{p(\cdot)}(\ell_+^{q(\cdot)})} = \|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q(\cdot)})} \lesssim 1,$$

from which the claim follows.  $\square$

*Remark 3.4.* The proof also works with the scaling  $|f_\nu| \leq \beta^{-\nu} \|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q(\cdot)})}$ , i.e. with  $q(\cdot)$  in place of  $q_\infty$ . In this case we first consider the normalization  $\|(f_\nu)\|_{L^{p(\cdot)}(\ell_+^{q(\cdot)})} = 1$ .

We now move on to the mixed spaces used in Besov spaces. Note that  $\ell^{(q_\nu)}$  is the variable exponent sequence space defined on the set  $\mathbb{N}$  with the counting measure, cf. [17, p. 82].

**Lemma 3.5.** *Let  $p, q \in \mathcal{P}_0$  and  $\beta > 1$ . Suppose that  $|f_\nu| \lesssim \beta^{-\nu} \|(f_\nu)\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})}$ . Then there exists  $\varepsilon > 0$  such that*

$$\|(f_\nu)\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|(f_\nu)\|_{\ell_+^{(q_\nu^-)}(L^{p(\cdot)})} \quad \text{and} \quad \|(f_\nu)\|_{\ell_+^{(q_\nu^+)}(L^{p(\cdot)})} \lesssim \|(f_\nu)\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})}$$

for  $B_\nu := B(0, \varepsilon 2^{\varepsilon \nu})$  and  $q_\nu^\pm := q_{\mathbb{R}^n \setminus B_\nu}^\pm$ .

*Proof.* We start with the second claim. By the triangle inequality

$$\|(f_\nu)\|_{\ell_+^{(q_\nu^+)}(L^{p(\cdot)})} \lesssim \|(f_\nu \chi_{B_\nu})\|_{\ell_+^{(q_\nu^+)}(L^{p(\cdot)})} + \|(f_\nu \chi_{\mathbb{R}^n \setminus B_\nu})\|_{\ell_+^{(q_\nu^+)}(L^{p(\cdot)})}.$$

In  $\mathbb{R}^n \setminus B_\nu$ ,  $q \leq q_\nu^+$ , and so

$$\|(f_\nu \chi_{\mathbb{R}^n \setminus B_\nu})\|_{\ell_+^{(q_\nu^+)}(L^{p(\cdot)})} \leq \|(f_\nu \chi_{\mathbb{R}^n \setminus B_\nu})\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} \leq \|f_\nu\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})}.$$

For the other part, we use the estimate on  $|f_\nu|$ :

$$\|(f_\nu \chi_{B_\nu})\|_{\ell_+^{(q_\nu^+)}(L^{p(\cdot)})} \lesssim \|f_\nu\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} \|(\beta^{-\nu} \chi_{B_\nu})\|_{\ell_+^{(q_\nu^+)}(L^{p(\cdot)})}.$$

All that remains is to bound the latter norm by a constant. To this end we estimate the corresponding semimodular:

$$\varrho_{\ell_+^{(q_\nu^+)}(L^{p(\cdot)})}((\beta^{-\nu} \chi_{B_\nu})) \lesssim \varrho_{\ell_+^{q^-}(L^{p^-})}((\beta^{-\nu} \chi_{B_\nu})) = \sum_{\nu \geq 0} (\beta^{-\nu p^-} |B_\nu|)^{q^-} < \infty$$

provided  $\beta^{-p^-} 2^{\varepsilon n} < 1$  (usual modifications when  $p^- = \infty$  and/or  $q^- = \infty$ ).

By a similar reasoning in the first claim, we conclude that

$$\|f_\nu\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} \leq A(\|f_\nu\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} \|(\beta^{-\nu} \chi_{B_\nu})\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} + \|f_\nu \chi_{\mathbb{R}^n \setminus B_\nu}\|_{\ell_+^{(q_\nu^-)}(L^{p(\cdot)})}),$$

where  $A$  is the constant from the quasi-triangle inequality. If  $\varepsilon$  is chosen so small that  $\|(\beta^{-\nu} \chi_{B_\nu})\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} \leq \frac{1}{2A}$ , then the first term on the right hand side can be absorbed into the left hand side, and so we get the claim.  $\square$

**Lemma 3.6.** *Let  $p \in \mathcal{P}_0$  and let  $\frac{1}{q}$  satisfy the log log-Hölder decay condition. If  $|f_\nu| \lesssim \beta^{-\nu} \|f_\nu\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})}$ , then*

$$\|f_\nu\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} \approx \|f_\nu\|_{\ell_+^{q_\infty}(L^{p(\cdot)})}.$$

*Proof.* Let  $B_\nu$  and  $q_\nu^\pm$  be as in Lemma 3.5. Then

$$\left| \frac{1}{q_\nu^+} - \frac{1}{q_\nu^-} \right| \leq \frac{c}{\log(e + \log(e + \gamma^\nu))} \approx \frac{c}{\log(e + \nu)},$$

so the sequences  $(q_\nu^\pm)_\nu$  satisfy the decay condition. Hence  $\ell_+^{(q_\nu^+)} \simeq \ell_+^{(q_\nu^-)} \simeq \ell_+^{q_\infty}$  according to [17, Corollary 4.1.9]. By this and Lemma 3.5 we obtain that

$$\|f\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|f\|_{\ell_+^{(q_\nu^-)}(L^{p(\cdot)})} \approx \|f\|_{\ell_+^{(q_\nu^+)}(L^{p(\cdot)})} \lesssim \|f\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})}. \quad \square$$

#### 4. PROOF OF THEOREM 1.3

We first consider moving the smoothness parameter inside a convolution. This will incur an error term, which we later have to control.

**Lemma 4.1.** *Suppose that  $p \in \mathcal{P}^{\log}$ ,  $\alpha$  is log-Hölder continuous and  $\|f\|_{p(\cdot)} \leq 2^{-\nu\alpha^+}$  or  $\|f\|_{p_\infty} \leq 2^{-\nu\alpha^+}$ . For arbitrarily small  $\varepsilon > 0$  and all  $\nu \leq 0$ , we have*

$$2^{\nu\alpha(x)} (\eta_{\nu,m} * f)(x) \lesssim \eta_{\nu,m} * (2^{\nu\alpha_\infty} |f|)(x) + 2^{(\alpha^- - \alpha^+)\nu} 2^{\frac{n\nu}{p^\pm}} \chi_{B_\nu}(x),$$

$$2^{\nu\alpha_\infty} (\eta_{\nu,m} * f)(x) \lesssim \eta_{\nu,m} * (2^{\nu\alpha(\cdot)} |f|)(x) + 2^{(\alpha^- - \alpha^+)\nu} 2^{\frac{n\nu}{p^\pm}} \chi_{B_\nu}(x),$$

where  $B_\nu := B(0, 2^{-\varepsilon\nu})$ .

*Proof.* Let  $\varepsilon \in (0, 1)$ . Outside  $B_\nu$ ,  $2^{\nu\alpha(x)} \approx 2^{\nu\alpha_\infty}$  by the decay property of  $\alpha$ . Hence

$$2^{\nu\alpha(x)} (\eta_{\nu,m} * f)(x) \lesssim \eta_{\nu,m} * (2^{\nu\alpha_\infty} f)(x) \chi_{\mathbb{R}^n \setminus B_\nu} + 2^{\nu\alpha^-} (\eta_{\nu,m} * f)(x) \chi_{B_\nu}.$$

The other inequality holds with the same error term by the same token. We need to control the error term. If  $\|f\|_{p(\cdot)} \leq 2^{-\nu\alpha^+}$ , we find by Hölder's inequality that

(4.2)

$$|\eta_{\nu,m} * f(x)| \leq 2 \|\eta_{\nu,m}\|_{L^{p'(\cdot)}} \|f\|_{L^{p(\cdot)}} \lesssim 2^{-\nu\alpha^+} \|\eta_{\nu,m}\|_{L^{(p^+)'}} = 2^{(\frac{n}{p^+} - \alpha^+)\nu} \|\eta_{0,m}\|_{L^{(p^+)'}} \lesssim 2^{(\frac{n}{p^+} - \alpha^+)\nu};$$

here we used that  $L^{p'(\cdot)} \subset L^{(p^-)'} \cup L^{(p^+)}'$  to estimate the norm of  $\eta_{\nu,m}$ . The case  $\|f\|_{p_\infty} \leq 2^{-\nu\alpha^+}$  is handled in the same way.  $\square$

**Lemma 4.3.** *Suppose that  $\|f_\nu\|_{p(\cdot)} \leq 2^{-\nu\alpha^+}$  for every  $\nu$  (alternatively,  $\|f_\nu\|_{p_\infty} \leq 2^{-\nu\alpha^+}$ ),  $\alpha$  is log-Hölder continuous,  $q \in \mathcal{P}$  and that  $p \in \mathcal{P}^{\log}$  is bounded. If  $\alpha^+ - \alpha^- < \frac{n}{p^+}$ , then*

$$\begin{aligned} \left\| (2^{\nu\alpha(\cdot)} \eta_{\nu,m} * f_\nu)_\nu \right\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})} &\lesssim \left\| (\eta_{\nu,m} * (2^{\nu\alpha_\infty} f_\nu))_\nu \right\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})} + 1, \\ \left\| (2^{\nu\alpha(\cdot)} \eta_{\nu,m} * f_\nu)_\nu \right\|_{L^{p(\cdot)}(\ell_-^{q(\cdot)})} &\lesssim \left\| (\eta_{\nu,m} * (2^{\nu\alpha_\infty} f_\nu))_\nu \right\|_{L^{p(\cdot)}(\ell_-^{q(\cdot)})} + 1. \end{aligned}$$

*Proof.* Let us denote  $\delta := -\alpha^+ + \alpha^-$ . We consider first the  $\ell_-^{q(\cdot)}(L^{p(\cdot)})$  spaces. Taking the the iterated norm of the first inequality in Lemma 4.1, we get

$$\left\| (2^{\nu\alpha(\cdot)} \eta_{\nu,m} * f_\nu)_\nu \right\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})} \lesssim \left\| (\eta_{\nu,m} * (2^{\nu\alpha_\infty} f_\nu))_\nu \right\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})} + \left\| \left( 2^{(\delta + \frac{n}{p^+})\nu} \chi_{B_\nu} \right)_\nu \right\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})}$$

with  $B_\nu$  defined as in Lemma 4.1. (A similar result holds for the second inequality in Lemma 4.1.) For the latter term we then use the inequality  $\|(a_\nu)_\nu\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})} \leq \|(a_\nu)_\nu\|_{\ell_-^1(L^{p(\cdot)})}$ . Since  $\|\chi_{B_\nu}\|_{L^{p(\cdot)}} \lesssim 2^{-\frac{\varepsilon n \nu}{p_\infty}}$  [17, Corollary 4.5.9] and  $\delta + \frac{n}{p^+} > 0$ , we obtain for sufficiently small  $\varepsilon$  that

$$\left\| \left( 2^{(\delta + \frac{n}{p^+})\nu} \chi_{B_\nu} \right)_\nu \right\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})} \lesssim \left\| 2^{(\delta + \frac{n}{p^+} - \frac{\varepsilon n}{p_\infty})\nu} \right\|_{\ell_-^1} \lesssim 1.$$

Combining the estimates, we conclude the proof of the first inequality.

For the  $L^{p(\cdot)}(\ell_-^{q(\cdot)})$ -space we again start with the point-wise estimate of Lemma 4.1 and take the mixed norm. This time we need to control

$$\left\| \left( 2^{(\delta + \frac{n}{p^+})\nu} \chi_{B_\nu} \right)_\nu \right\|_{L^{p(\cdot)}(\ell_-^1)}$$

For this we calculate

$$\left\| \left( 2^{(\delta + \frac{n}{p^+})\nu} \chi_{B_\nu} \right)_\nu \right\|_{\ell_-^1} = \sum_{\nu=-\infty}^{\min\{0, \frac{1}{\varepsilon} \log_2 \frac{1}{|x|}\}} 2^{(\delta + \frac{n}{p^+})\nu} \approx \min\{1, |x|^{-\frac{1}{\varepsilon}(\delta + \frac{n}{p^+})}\} \in L^{p(\cdot)}$$

for small enough  $\varepsilon$ . Therefore the error term can be bounded by a constant also in the this case, so we are done with the proof of the second inequality.  $\square$

**Theorem 4.4.** *For  $f \in S'$  we set  $f^l := f * \Phi_-$  and  $f^h := f - f^l$ . Suppose that  $q \in \mathcal{P}_0$  with  $\frac{1}{q} \log \log$ -Hölder continuous,  $\alpha \in L^\infty$  is log-Hölder continuous, and  $p \in \mathcal{P}_0^{\log}$  is bounded. If  $\alpha^+ - \alpha^- < \frac{n}{p^+}$ , then*

$$\begin{aligned} \|f\|_{\dot{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &\approx \|f^h\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} + \|f^l\|_{\dot{B}_{p_\infty, q_\infty}^{\alpha_\infty}}, \\ \|f\|_{\dot{F}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &\approx \|f^h\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} + \|f^l\|_{\dot{F}_{p_\infty, q_\infty}^{\alpha_\infty}}. \end{aligned}$$

*Remark 4.5.* The previous statement should be interpreted in the following sense: if the (quasi)norm on the left is defined by a certain admissible system  $\{\varphi_\nu\}$ , then the sum of the (quasi)norms on the right are equivalent when calculated with the same system.

*Proof.* We give the proof for Besov spaces. The proof for the spaces  $\dot{F}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$  is similar: we need only use Lemma 3.3 instead of Lemma 3.6.

We first observe that

$$\|f\|_{\dot{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \|(2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})} + \|(2^{\nu\alpha_\infty}\varphi_\nu * f)_\nu\|_{\ell_+^{q(\cdot)}(L^{p(\cdot)})}.$$

Therefore it suffices to show that

$$\|(2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})} \approx \|(2^{\nu\alpha_\infty}\varphi_\nu * f)_\nu\|_{\ell_-^{q_\infty}(L^{p_\infty})}.$$

Observe that  $\|2^{\nu\alpha(\cdot)}\varphi_\nu * f\|_{p_\infty} \leq \|f^l\|_{\dot{B}_{p_\infty, \infty}^{\alpha_\infty}} \leq \|f^l\|_{\dot{B}_{p_\infty, q_\infty}^{\alpha_\infty}}$  so that  $\|\varphi_\nu * f\|_{p_\infty} \leq 2^{-\nu\alpha^+} \|f^l\|_{\dot{B}_{p_\infty, q_\infty}^{\alpha_\infty}}$ .

Let  $m > n$ . By the  $r$ -trick (Lemma 2.1) we obtain that

$$|\varphi_\nu * f| \lesssim \eta_{\nu, m} * (|\varphi_\nu * f|).$$

It follows as in (4.2) that

$$|\varphi_\nu * f| \lesssim |\eta_{\nu, m} * (|\varphi_\nu * f|)| \leq \|\eta_{\nu, m}\|_{p'_\infty} \|\varphi_\nu * f\|_{p_\infty} \lesssim 2^{\frac{n\nu}{p_\infty}} 2^{-\nu\alpha^+}.$$

Hence  $2^{\nu\alpha(\cdot)}|\varphi_\nu * f|$  satisfies the decay condition, and so it follows from Lemma 3.6 that

$$\|(2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu\|_{\ell_-^{q(\cdot)}(L^{p(\cdot)})} \approx \|(2^{\nu\alpha(\cdot)}|\varphi_\nu * f|)_\nu\|_{\ell_-^{q_\infty}(L^{p_\infty})} \approx \|(2^{\nu\alpha(\cdot)}|\varphi_\nu * f|)_\nu\|_{\ell_-^{q_\infty}(L^{p_\infty})}$$

where we used in the second step that  $L^{p(\cdot)} \cap L^\infty = L^{p_\infty} \cap L^\infty$  with equivalent norms [17, Lemma 3.3.12] and the uniform bound on the function.

It remains to show that

$$\|(2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu\|_{\ell_-^q(L^p)} \approx \|(2^{\nu\alpha_\infty}\varphi_\nu * f)_\nu\|_{\ell_-^q(L^p)}$$

for constant  $p \in (0, \infty)$  and  $q \in (0, \infty]$ . We assume without loss of generality that  $\|f^l\|_{\dot{B}_{p, q}^{\alpha_\infty}} = 1$ . Let  $r < p$ . By the  $r$ -trick (Lemma 2.1) we obtain that

$$|\varphi_\nu * f|^r \lesssim \eta_{\nu, m} * (|\varphi_\nu * f|^r).$$

Thus we obtain by Lemma 4.3 and the boundedness of convolution on the constant exponent iterated space that

$$\begin{aligned} \|(2^{\nu\alpha(\cdot)}\varphi_\nu * f)_\nu\|_{\ell_-^q(L^p)} &= \|(2^{\nu\alpha(\cdot)r}|\varphi_\nu * f|^r)_\nu\|_{\ell_-^{\frac{q}{r}}(L^{\frac{p}{r}})}^{1/r} \\ &\lesssim \|(2^{\nu\alpha(\cdot)r}\eta_{\nu, m} * |\varphi_\nu * f|^r)_\nu\|_{\ell_-^{\frac{q}{r}}(L^{\frac{p}{r}})}^{1/r} \\ &\lesssim \|(\eta_{\nu, m} * (2^{\nu\alpha_\infty r}|\varphi_\nu * f|^r))_\nu\|_{\ell_-^{\frac{q}{r}}(L^{\frac{p}{r}})}^{1/r} + 1 \\ &\lesssim \|(2^{\nu\alpha_\infty r}|\varphi_\nu * f|^r)_\nu\|_{\ell_-^{\frac{q}{r}}(L^{\frac{p}{r}})(\mathbb{R}^n)}^{1/r} + 1 \\ &= \|(2^{\nu\alpha_\infty}\varphi_\nu * f)_\nu\|_{\ell_-^q(L^p)} + 1 = 2. \end{aligned}$$

The general inequality follows from this by a scaling argument. The opposite inequality can be proved with exactly the same steps, with the second inequality of Lemma 4.1.  $\square$

Suppose that  $p, q$  and  $\alpha$  are such that the inhomogeneous spaces are independent of the basis functions (recall the related results given in [5, Theorem 5.5], [19, Theorem 3.10] and in [3, Section 3]). For the constant exponent spaces this requires  $p_\infty < \infty$  for the  $F$ -case. Then the previous theorem can be used to derive independence of the the variable index homogeneous spaces. Collecting all of these informations, we arrive at Theorem 1.3.

## 5. EMBEDDINGS

The following elementary embeddings can be proved as in the inhomogeneous case, cf. [5, Theorem 6.1].

**Proposition 5.1.** *Let  $\alpha \in L^\infty$  and  $p, q_0, q_1 \in \mathcal{P}_0$ .*

(i) *If  $q_0 \leq q_1$ , then*

$$\dot{B}_{p(\cdot), q_0(\cdot)}^{\alpha(\cdot)} \hookrightarrow \dot{B}_{p(\cdot), q_1(\cdot)}^{\alpha(\cdot)} \quad \text{and} \quad \dot{F}_{p(\cdot), q_0(\cdot)}^{\alpha(\cdot)} \hookrightarrow \dot{F}_{p(\cdot), q_1(\cdot)}^{\alpha(\cdot)}.$$

(ii) *If  $p^+ < \infty$ , then*

$$\dot{B}_{p(\cdot), \min\{p(\cdot), q(\cdot)\}}^{\alpha(\cdot)} \hookrightarrow \dot{F}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} \hookrightarrow \dot{B}_{p(\cdot), \max\{p(\cdot), q(\cdot)\}}^{\alpha(\cdot)}.$$

As for constant exponents, homogeneous spaces are not monotone with respect to  $\alpha$ .

It is known that Sobolev and Jawerth embeddings are valid for variable index inhomogeneous spaces ([5, Theorem 6.4] and [51, Theorems 3.4, 3.7]):

**Proposition 5.2.** *Let  $p_0, p_1, q \in \mathcal{P}_0$ ,  $\alpha_0, \alpha_1 \in L^\infty$  with  $\alpha_0 \geq \alpha_1$ , and*

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}.$$

(i) *If  $1/q$  and  $\alpha_0 - n/p_0 = \alpha_1 - n/p_1$  are locally log-Hölder continuous, then*

$$B_{p_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow B_{p_1(\cdot), q(\cdot)}^{\alpha_1(\cdot)}.$$

(ii) *Let  $p_0, p_1, q, r \in \mathcal{P}_0^{\log}$  be bounded and let  $\alpha_0, \alpha_1$  be locally log-Hölder continuous with limit at infinity. Then*

$$F_{p_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow F_{p_1(\cdot), q(\cdot)}^{\alpha_1(\cdot)}.$$

*If, in addition,  $(\alpha_0 - \alpha_1)^- = n(1/p_0 - 1/p_1)^- > 0$ , then*

$$F_{p_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow F_{p_1(\cdot), r(\cdot)}^{\alpha_1(\cdot)}.$$

Note that the results in (ii) contain various interesting special cases, since the  $F$  scale includes Lebesgue, Sobolev and Bessel potential spaces (cf. Section 2.3).

It is also known that similar embeddings hold for homogeneous spaces with constant exponents (cf. [10, Section 6.5]). Using this fact together with Theorem 1.3 and Proposition 5.2, we can extend the embedding results to the variable exponent homogeneous spaces.

**Theorem 5.3** (Sobolev–Jawerth embedding). *Let  $p_0, p_1, q \in \mathcal{P}_0^{\log}$  with  $p_0, p_1 \in L^\infty$ . Let  $\alpha_0, \alpha_1 \in L^\infty$  be log-Hölder continuous with  $\alpha_0 \geq \alpha_1$ , and*

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}.$$

*Assume that  $\alpha_j^+ - \alpha_j^- < \frac{n}{p_j^+}$  for  $j \in \{0, 1\}$ .*

(i) *Then  $\dot{B}_{p_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \dot{B}_{p_1(\cdot), q(\cdot)}^{\alpha_1(\cdot)}$ .*

- (ii) If  $q \in L^\infty$ , then  $\dot{F}_{p_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \dot{F}_{p_1(\cdot), q(\cdot)}^{\alpha_1(\cdot)}$ .  
 (1) If, additionally,  $r \in L^\infty$  and  $(\alpha_0 - \alpha_1)^- = n(1/p_0 - 1/p_1)^- > 0$ , then

$$\dot{F}_{p_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \dot{F}_{p_1(\cdot), r(\cdot)}^{\alpha_1(\cdot)}.$$

The following embedding is an immediate consequence of Theorem 5.3 and Proposition 5.1.

**Corollary 5.4.** *Let  $p_0, p_1 \in \mathcal{P}_0^{\log} \cap L^\infty$ , and  $\alpha_0, \alpha_1 \in L^\infty$  be log-Hölder continuous with  $\alpha_0 \geq \alpha_1$  and*

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}.$$

*Assume that  $\alpha_j^+ - \alpha_j^- < \frac{n}{p_j^+}$  for  $j \in \{0, 1\}$ . If  $q \leq p_0$  and  $(\alpha_0 - \alpha_1)^- = n(1/p_0 - 1/p_1)^- > 0$ , then*

$$\dot{B}_{p_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \dot{B}_{p_1(\cdot), r(\cdot)}^{\alpha_1(\cdot)}.$$

## 6. TRACES

Trace properties have been studied for inhomogeneous variable exponent spaces (see [6, Theorem 5.2], [19, Theorem 3.13], and also the recent papers [24], [40]). If  $p \in \mathcal{P}_0^{\log} \cap L^\infty$ ,  $q \in (0, \infty]$ ,  $\alpha$  is log-Hölder continuous, and

$$(6.1) \quad \left( \alpha - \frac{1}{p} - (n-1) \max \left\{ 0, \frac{1}{p} - 1 \right\} \right)^- > 0,$$

then

$$\mathrm{Tr} B_{p(\cdot), q}^{\alpha(\cdot)}(\mathbb{R}^n) = B_{p(\cdot), q}^{\alpha(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1})$$

and

$$\mathrm{Tr} F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = F_{p(\cdot), p(\cdot)}^{\alpha(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1}).$$

Note that  $q$  is assumed to be constant in the Besov space (the reason is that the result was obtained by interpolation from the corresponding result for  $F$  spaces proved in [19]).

From the trace property of the inhomogeneous spaces and Theorem 1.3, we can get the same property for variable exponent homogeneous spaces. We refer to [22] for some references on the traces in classical homogeneous spaces.

**Theorem 6.2.** *Let  $n > 1$ ,  $p, q \in \mathcal{P}_0^{\log} \cap L^\infty$  and  $\alpha \in L^\infty$  be log-Hölder continuous. Assume that  $\alpha^+ - \alpha^- < \frac{n}{p^+}$ . If (6.1) holds, then*

$$\mathrm{Tr} \dot{F}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = \dot{F}_{p(\cdot), p(\cdot)}^{\alpha(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1}).$$

Moreover, for constant  $q$  we also have

$$\mathrm{Tr} \dot{B}_{p(\cdot), q}^{\alpha(\cdot)}(\mathbb{R}^n) = \dot{B}_{p(\cdot), q}^{\alpha(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1}).$$

Combining Proposition 6.2 with Theorem 5.3, we can obtain an embedding for traces.

**Corollary 6.3.** *Let  $p \in \mathcal{P}^{\log}$  with  $p^+ < \infty$ . If  $\alpha$  is log-Hölder continuous with  $1 < (\alpha p)^- \leq (\alpha p)^+ < n$  and  $\alpha^+ - \alpha^- < \frac{n}{p^+}$ , then*

$$\mathrm{Tr} \dot{F}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{\frac{(n-1)p(\cdot)}{n-\alpha(\cdot)p(\cdot)}}(\mathbb{R}^{n-1}).$$

*Proof.* We have

$$\mathrm{Tr} \dot{F}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = \dot{F}_{p(\cdot), p(\cdot)}^{\alpha(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1}) \hookrightarrow \dot{F}_{p_*(\cdot), 2}^0(\mathbb{R}^{n-1}) = L^{p_*(\cdot)}(\mathbb{R}^{n-1}),$$

where the exponent  $p_*$  is given by

$$\alpha(x) - \frac{1}{p(x)} - \frac{n-1}{p(x)} = -\frac{n-1}{p_*(x)}.$$

We used a Littlewood-Paley characterization on the right hand side of the chain above, see [41, Lemma 3.1 and Theorem 5.7].  $\square$

A counterpart of the trace embedding for inhomogeneous Sobolev spaces was obtained in [51].

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