

# ISOMETRIES OF SOME HYPERBOLIC-TYPE PATH METRICS, AND THE HYPERBOLIC MEDIAL AXIS

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## 1. INTRODUCTION

A conformal path metric is a special kind of Finslerian metric, in which the density depends only on the location, not on the direction. If  $D$  is a connected subset of  $\mathbb{R}^n$  and  $p$  is a non-negative real valued function defined on  $D$ , then we can define such a metric by

$$d_p(x, y) = \inf_{\gamma} \int_{\gamma} p(z) |dz|,$$

where  $|dz|$  represents integration with respect to path-length, and the infimum is taken over all paths  $\gamma$  joining  $x, y \in D$ . If  $p$  is a  $C^2$  function, then we are in the standard Riemannian setting, but there is nothing preventing us from considering also a more general  $p$ .

In fact, choosing  $p(z) = \delta(z)^{-1}$ , where  $\delta$  is the distance-to-the-boundary function, gives us the well-known quasihyperbolic metric. Despite the prominence of this metric, there have been almost no investigations of its geometry (some exceptions are [17, 18]). Part of the reasons for this lack of geometrical investigations is probably that the density of the quasihyperbolic metric is not differentiable in the entire domain, which places the metric outside the standard framework of Riemannian metrics.

At least two modifications of the quasihyperbolic metric have been proposed which go some way to alleviate this problem. J. Ferrand [6] suggested replacing the density  $\delta(z)^{-1}$  by

$$\sigma_D(x) = \sup_{a, b \in \partial D} \frac{|a - b|}{|a - x| |b - x|}.$$

Note that  $\delta(x)^{-1} \leq \sigma_D(x) \leq 2\delta(x)^{-1}$ , so the Ferrand metric and the quasihyperbolic metric are bilipschitz equivalent. Moreover, the Ferrand metric is Möbius invariant, whereas the quasihyperbolic metric is only Möbius quasi-invariant. A second variant was proposed more recently by R. Kulkarni and U. Pinkall [16], see also [15]. The K–P metric is defined by the density

$$\mu_D(x) = \inf \left\{ \lambda_B(x) : x \in B \subset D, B \text{ is a disc or a half-plane} \right\}.$$

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Here  $\lambda_B$  is the density of the hyperbolic metric in  $B$ . Recall that if  $B = B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ , then

$$\lambda_B(x) = \frac{2r}{r^2 - |x - x_0|^2}.$$

This density satisfies the same estimates as Ferrand's density, i.e.  $\delta(x)^{-1} \leq \mu_D(x) \leq 2\delta(x)^{-1}$ , and the K–P metric is also Möbius invariant. Although the Ferrand and the K–P metrics are in some sense better behaved than the quasihyperbolic metric, they suffer from the short-coming that it is very difficult to get a grip of the density, even in simple domains.

Despite this, D. Herron, Z. Ibragimov and D. Minda [14] recently managed to solve the isometry problem of the K–P metric in most cases. By the isometry problem of the metric  $d$  we mean characterizing mappings  $f: D \rightarrow \mathbb{R}^2$  with

$$d_D(x, y) = d_{f(D)}(f(x), f(y))$$

for all  $x, y \in D$ . Notice that in some sense we are here dealing with two different metrics, due to the dependence on the domain. Hence the usual way of approaching the isometry problem is by looking at some intrinsic features of the metric which are then preserved under the isometry. Since irregularities (e.g. cusps) in the domain often lead to more distinctive features, this implies that the problem is often easier for more complicated domains. The work by Herron, Ibragimov and Minda [14] bears out this heuristic – they were able to show that all isometries of the K–P metric are Möbius mappings except possibly in simply and doubly connected domains. Their proof is based on studying the circular geodesics of the K–P metric.

There are three steps in characterizing isometries of a conformal metric like the quasihyperbolic metric:

- (1) to show that they are conformal or anti-conformal;
- (2) to show that they are Möbius; and
- (3) to show that they are similarities.

Note that step (2) is trivial in dimensions 3 and higher, and that step (3) is not relevant for Möbius invariant metrics like the K–P metric and Ferrand's metric. The first step has been carried out by Martin and Osgood [18, Theorem 2.6] for arbitrary domains assuming only that the density is continuous, so there is no more work to do there.

In Section 2 the work of Hästö [9] on steps (2) and (3) for the quasihyperbolic metric is described. For this metric, formulae for the curvature were worked out in [18] (see Section 2, below), and were used in that paper to prove that all the isometries of the disc are similarity mappings. The proofs in [9] are based on both the curvature and its gradient and work for domains with  $C^3$  boundary. In Section 3 we relate how Herron, Ibragimov and Minda [14] used circular geodesics and the curvature of the metric to take care of the second step for the K–P metric for most domains. Finally, in Section 4 we show how the isometry problem can be solved for the K–P metric in all doubly connected domains using a new concept which we call the hyperbolic medial axis, and also give some minor new results for the quasihyperbolic metric and Ferrand's metric.

Besides the aforementioned works, the proofs in this paper were inspired by our work on the isometries of other, non-length metric [10, 11, 13].

**Notation.** If  $D \subset \mathbb{R}^n$ , we denote by  $\partial D$  and  $\overline{D}$  its boundary and closure, respectively. For  $x \in D \subsetneq \mathbb{R}^n$  we denote  $\delta(x) = d(x, \partial D) = \min\{|x - z| : z \in \partial D\}$ . We tacitly identify  $\mathbb{R}^2$  with  $\mathbb{C}$ , and speak about real and imaginary axes, etc. By  $B(x, r)$  we denote a disc with center  $x$  and radius  $r$ , and by  $[x, y]$ ,  $(x, y]$  the closed and half-open segment between  $x$  and  $y$ , respectively.

We denote by  $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$  the one point compactification of  $\mathbb{R}^n$ . The cross-ratio  $|a, b, c, d|$  is defined by

$$|a, b, c, d| = \frac{|a - c| |b - d|}{|a - b| |c - d|}$$

for distinct points  $a, b, c, d \in \overline{\mathbb{R}^2}$ , with the understanding that  $|\infty - x|/|\infty - y| = 1$  for all  $x, y \in \mathbb{R}^2$ . A homeomorphism  $f: \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$  is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|$$

for every quadruple of distinct points  $a, b, c, d \in \overline{\mathbb{R}^n}$ . A mapping of a subdomain of  $\overline{\mathbb{R}^n}$  is Möbius, if it is a restriction of a Möbius mapping defined on  $\overline{\mathbb{R}^n}$ . For more information on Möbius mappings, see e.g. [1, Section 3]. Note that a Möbius mapping can always be decomposed as  $i \circ s$ , where  $i$  is an inversion or the identity and  $s$  is a similarity.

## 2. ISOMETRIES OF $k_D$

By  $f \in C^k$  we mean that  $f$  is a  $k$  times continuously differentiable function. By a  $C^k$  domain we mean a domain whose boundary can be locally represented as the graph of a  $C^k$  function. To carry out step (2) of the isometry program above, the following result was proved in [9, Proposition 2.2].

**Proposition 2.1.** *Let  $D \subsetneq \mathbb{R}^2$  be a  $C^1$  domain, and let  $f: D \rightarrow \mathbb{R}^2$  be a quasihyperbolic isometry which is also Möbius. If  $D$  is not a half-plane, then  $f$  is a similarity.*

Note that if we do not assume  $C^1$  boundary, then there are some domains with non-similarity isometries: punctured planes  $\mathbb{R}^2 \setminus \{a\}$  and sector domains (i.e. domains whose boundary consists of two rays). In both cases inversions centered at the distinguished boundary point ( $a$  or the vertex of the sector) are also isometries. The previous proposition strongly suggests that these are all the examples of domains with non-similarity isometries.

An immediate consequence is the solution of the isometry problem in higher dimensions for  $C^1$  domains [9, Corollary 2.3]:

**Corollary 2.2.** *Let  $D$  be a  $C^1$  domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , which is not a half-space. Then every quasihyperbolic isometry is a similarity mapping.*

The *medial axis* of  $D$  is the set of centers of maximal balls (with respect to the inclusion order) in  $D$ . The medial axis is denoted by  $\text{MA}(D)$ . For some mathematical

investigations of the medial axis, see [4, 5], and for an application to the quasiworld see [3].

By  $R_\zeta$  we denote the reciprocal of the curvature of  $\partial D$  at the boundary point  $\zeta$ . The principal tool in [9] for attacking the main problem in the isometry program, namely step (3), is the following curvature formula which is based on the estimates of Martin and Osgood [18].

**Proposition 2.3** (Proposition 3.2, [9]). *Let  $D \subsetneq \mathbb{R}^2$  be a  $C^2$  domain and  $z \in D \setminus \text{MA}(D)$  have closest boundary point  $\zeta \in \partial D$ . Then*

$$\kappa_D(z) = -\frac{R_\zeta}{R_\zeta - \delta(z)} = -\frac{1}{1 - \delta(z)/R_\zeta}.$$

*If  $z$  lies on the medial axis, then  $\kappa_D(z) = -\infty$ .*

Using this proposition, the following theorem was proven in [9, Theorem 4.3].

**Theorem 2.4.** *Let  $D \subsetneq \mathbb{R}^2$  be a  $C^3$  domain, which is not a half-plane. Then every isometry  $f: D \rightarrow \mathbb{R}^2$  of the quasihyperbolic metric is a similarity mapping.*

The idea of the proof is the following: Let  $z \in D \setminus \text{MA}(D)$  and let  $\zeta_z$  be its unique nearest boundary point. Then the half-open segment  $[z, \zeta_z)$  is a geodesic half-line with respect to the quasihyperbolic metric. The proof of the theorem is based on showing that this type of geodesic is somehow special, and thus mapped to another geodesic half-line of the same type. There are a couple of different cases based on the curvature  $R_\zeta$  at the nearest boundary point, but essentially this part of the proof is based on Proposition 2.3. We have then shown that an isometry maps a segment to a segment, which implies that it is a Möbius mapping. The proof is concluded by applying Proposition 2.1, which says that the Möbius isometry is a similarity.

In fact, the smoothness assumption on the boundary of the domain can be dropped to  $C^2$ , except in two special cases, namely, when the domain is strictly convex or strictly concave! Corollary 4.7, proved below, takes care of the concave case, so only the convex case remains.

### 3. ISOMETRIES OF $\mu_D$

*Extremal discs* and *circular geodesics* are important objects when discussing the isometries of the K–P metric. Consider a domain  $D$  in  $\mathbb{C}$  with  $\text{card}(\partial D) \geq 2$ . A disc or a half-plane  $B \subset D$  with  $\text{card}(\partial B \cap \partial D) \geq 2$  is called an extremal disc. We call  $\Gamma$  a circular geodesic in  $D$  if there exists an extremal disc  $B \subset D$  such that  $\Gamma$  is a hyperbolic geodesic line in  $B$  with endpoints in  $\partial B \cap \partial D$ . While the definition may not suggest the importance of extremal discs, it is the existence of a unique extremal disc associated to each point in the domain that plays a crucial role in the study of the K–P metric.

More precisely, given a domain  $D \subset \mathbb{C}$  with  $\text{card}(\partial D) \geq 2$  and a point  $z \in D$ , let  $i_z$  be the inversion in a circle centered at  $z$  with radius 1. Then the complement of  $i_z(D)$  is a compact set in  $\mathbb{C}$  and hence by Jung’s Theorem (see [2, 11.5.8, p. 357]) there exists a unique disc  $B$  of smallest radius whose closure contains the set. In particular,  $\text{card}(\partial B \cap \partial D) \geq 2$  and  $z \in i_z(\overline{\mathbb{C}} \setminus \overline{B}) \subset D$ . Hence the set  $i_z(\overline{\mathbb{C}} \setminus \overline{B})$  is an extremal

disc, which is properly called the extremal disc at  $z$  and is denoted by  $B_z$ . An observant reader will notice that  $B_z$  is also the extremal disc for each point of a circular geodesic contained in  $B_z$ . Using more delicate arguments it is proved that  $B_z$  is the extremal disc for each point of  $\hat{K}_z$  and only for these points, where  $\hat{K}_z$  is the hyperbolic convex hull of the set  $\partial B_z \cap \partial D$  in  $B_z$  (see [14, Proposition 2.5]). In particular, for each pair of points  $z, w \in D$ , we have either  $\hat{K}_z = \hat{K}_w$  or  $\hat{K}_z \cap \hat{K}_w = \emptyset$ .

Another important property of the extremal discs is that  $\mu_D(z) = \lambda_{B_z}(z)$ , where  $\lambda_{B_z}$  is the density of the hyperbolic metric in  $B_z$ . In particular,

$$\mu_D(z) = \sup_{a, b \in \partial B_z} \frac{|a - b|}{|a - z| |z - b|}$$

and if  $\gamma_z$  is any hyperbolic geodesic in  $B_z$  passing through  $z$ , then

$$\mu_D(z) = \frac{|a(z) - b(z)|}{|a(z) - z| |z - b(z)|},$$

where  $a(z) \in \partial B_z$  and  $b(z) \in \partial B_z$  are the endpoints of  $\gamma_z$ . Hence using monotonicity of the Ferrand metric we obtain that

$$\mu_D(z) = \sup_{a, b \in \partial B_z} \frac{|a - b|}{|a - z| |z - b|} = \sigma_{B_z}(z) \geq \sigma_D(z)$$

and  $\mu_D(z) = \sigma_D(z)$  if and only if  $z$  lies on a circular geodesic. Using Jung's Theorem we also obtain that  $\mu_D(z) \leq (2/\sqrt{3})\sigma_D(z)$  (see [14] for details).

We also need the following lower bound for the Ferrand (and hence for the K–P) distance. Given a domain  $D \subset \bar{\mathbb{C}}$  with  $\text{card}(\partial D) \geq 2$  and points  $z, w \in D$ , we consider the following distance function

$$s_D(z, w) = \log \left( 1 + \sup_{a, b \in \partial D} \frac{|a - b| |z - w|}{|a - z| |b - w|} \right).$$

The function  $s_D$ , introduced by Seittenranta [19], defines a metric in  $D$  and since

$$\lim_{w \rightarrow z} \frac{s_D(z, w)}{|z - w|} = \sup_{a, b \in \partial D} \frac{|a - b|}{|a - z| |z - b|} = \sigma_D(z) \quad \text{for each } z \in D \cap \mathbb{C},$$

the Ferrand metric is the length (a.k.a. inner or internal) metric of Seittenranta's metric. Hence we have the aforementioned lower bound for the Ferrand and the K–P distances  $\sigma_D(z, w)$  and  $\mu_D(z, w)$ :

$$s_D(z, w) \leq \sigma_D(z, w) \leq \mu_D(z, w) \quad \text{for all } z, w \in D.$$

In particular, the length of a curve in Seittenranta's metric is smaller than its length in the K–P metric.

Next we show that each circular geodesic is a geodesic line for both the Ferrand and the K–P metric, justifying its name. Let  $\gamma$  be a circular geodesic in  $D$  with endpoints  $a, b$  and  $z, w \in \gamma$  be arbitrary points. Let  $\gamma(z, w)$  be the subarc of  $\gamma$  joining  $z$  and  $w$ . We

need to show that

$$\sigma_D(z, w) = \int_{\gamma(z, w)} \sigma_D(\xi) |d\xi| \quad \text{and} \quad \mu_D(z, w) = \int_{\gamma(z, w)} \mu_D(\xi) |d\xi|.$$

An easy observation shows that

$$s_D(z, w) = \log \left( 1 + \frac{|a - b| |z - w|}{|a - z| |b - w|} \right) = \log \frac{|z - b| |w - a|}{|z - a| |w - b|}$$

for all  $z, w \in \gamma$  such that the points  $a, z, w, b$  are in this order on  $\gamma$ . In particular,  $s_D(z_1, z_3) = s_D(z_1, z_2) + s_D(z_2, z_3)$  for all points  $z_1, z_2, z_3$  in this order on  $\gamma$  and, as a consequence, the  $s_D$ -length of  $\gamma(z, w)$  is equal to  $s_D(z, w)$ . Then

$$\sigma_D(z, w) \leq \int_{\gamma(z, w)} \sigma_D(\xi) |d\xi| = \int_{\gamma(z, w)} \frac{|a - b|}{|a - \xi| |\xi - b|} |d\xi| = s_D(z, w) \leq \sigma_D(z, w)$$

and, similarly

$$\mu_D(z, w) \leq \int_{\gamma(z, w)} \mu_D(\xi) |d\xi| = \int_{\gamma(z, w)} \frac{|a - b|}{|a - \xi| |\xi - b|} |d\xi| = s_D(z, w) \leq \mu_D(z, w).$$

Thus,  $\gamma$  is a geodesic for both the Ferrand and the K–P metric.

Next we discuss another type of geodesic for the K–P metric. As we have mentioned above, given  $z \in D$ , the extremal disc  $B_z$  is also the extremal disc for all points in  $\hat{K}_z$ , where  $\hat{K}_z$  is the hyperbolic convex hull of the set  $\partial B_z \cap \partial D$  in  $B_z$ . In particular,  $\mu_D(\xi) = \lambda_{B_z}(\xi)$  for all  $\xi \in \hat{K}_z$  and as a result, all the hyperbolic geodesics in  $B_z \cap \hat{K}_z$ , which are circular arcs, are also geodesics in the K–P metric  $\mu_D$ . Hence through every point of  $D$  there passes a K–P geodesic which is a circular arc. Notice also that the interior of  $\hat{K}_z$  is non-empty if and only if  $\text{card}(\partial B_z \cap \partial D) \geq 3$ .

Now we are ready to present the result on the isometries of the K–P metric.

**Theorem 3.1.** *Let  $f: D \rightarrow \bar{\mathbb{C}}$  be a K–P isometry. Assume that  $D' = f(D)$  contains a point  $z'$  so that  $\text{card}(\partial B_{z'} \cap \partial D') \geq 3$ . Then  $f$  is the restriction of a Möbius transformation.*

*Proof.* Recall that  $f$  is conformal and  $f^{-1}$  is also a K–P isometry. Put  $z = f^{-1}(z')$ . Let  $\gamma$  be a circular arc containing  $z$  which is also a K–P geodesic segment with the property that  $f(\gamma)$  is contained in the interior of  $\hat{K}_{z'}$ . Since all the geodesics in  $\hat{K}_{z'}$  are circular arcs, so is  $f(\gamma)$ . Using auxiliary Möbius transformations, if necessary, we can assume  $B_z = B_{z'} = B^2(0, 1)$ ,  $z = z' = 0$  and that both  $\gamma$  and  $f(\gamma)$  are subarcs of the real interval  $(-1, 1)$ . Then the fact that  $f$  is an isometry implies that  $f$  is identity on  $\gamma$  and hence it is also identity on  $D$ , up to a Möbius map. This completes the proof.  $\square$

There is an alternative way to prove the above theorem based on the following result for holomorphic functions, which can be thought of as an extension of Schwarz' Lemma. This approach also extends to prove a similar theorem for the Ferrand metric (see Theorem 4.8).

**Theorem 3.2** (Fact 2.1, [14]). *Let  $D$  and  $D'$  be hyperbolic regions. Assume  $f$  is holomorphic in some neighborhood of  $a \in D$  and takes values in  $D'$ . Let  $\lambda$  and  $\lambda'$  be the densities of the hyperbolic metric in  $D$  and  $D'$ , respectively, and  $f^*[\lambda'](z) = \lambda'(f(z))|f'(z)|$  be the pullback of  $\lambda'$  in  $D$  to a neighborhood of  $a$ . Suppose that for all  $z$  near  $a$ ,  $f^*[\lambda'](z) \leq \lambda(z)$  with equality holding at  $z = a$ . Then  $f: D \rightarrow D'$  is a holomorphic covering projection; in particular,  $f^*[\lambda'] = \lambda$ .*

*Analytic proof of Theorem 3.1.* Let  $f: D \rightarrow D'$  be a K-P isometry, hence conformal. Observe first that

$$f^*[\mu_{D'}] = \mu_{D'}(f(z))|f'(z)| = \lim_{w \rightarrow z} \frac{\mu_{D'}(f(w), f(z)) |f(w) - f(z)|}{|f(w) - f(z)|} = \lim_{w \rightarrow z} \frac{\mu_D(w, z)}{|w - z|} = \mu_D(z).$$

The assumption in the theorem implies that there is a point  $b = f(a)$  contained in  $G'$ , where  $G'$  is the interior of  $\hat{K}_b$  and  $K_b = B_b \cap \partial D'$ . Then in  $f^{-1}(G') \cap B_a$  we have

$$f^*[\lambda_{B_b}] = f^*[\mu_{D'}] = \mu_D \leq \lambda_{B_a}$$

with equality holding at the point  $z = a$  (see the proof of [14, Theorem 4.10]). Theorem 3.2 now implies that  $f$  maps  $B_a$  conformally onto  $B_b$  and hence is a Möbius transformation.  $\square$

#### 4. ISOMETRIES OF $\mu_D$ , $\sigma_D$ AND $k_D$ ; NEW RESULTS

If a disc touches the boundary of a domain in exactly  $k$  points, then we call it  $k$ -*extremal*. In this section we are only interested in domains in which every extremal disc is 2-extremal – we call such a domain also 2-extremal, as there is no danger of confusion. Examples of 2-extremal domains include parallel strips, angular sectors with angular openings strictly less than  $\pi$ , annuli and many other domains and their images under Möbius mappings.

**Theorem 4.1.** *If  $D$  is 2-extremal domain in  $\mathbb{R}^2$ , then circular geodesics foliate  $D$ . In particular,  $\mu_D = \sigma_D$ .*

*Proof.* We will show that each point of  $D$  lies on a circular geodesic and that circular geodesics of  $D$  are disjoint. Indeed, given an arbitrary point  $x \in D$ , since  $\text{card}(B_x \cap \partial D) = 2$ , the interior of the set  $\hat{K}_x$  is empty and hence  $\hat{K}_x$  is a circular geodesic containing  $x$ . Next if  $\gamma_1$  and  $\gamma_2$  are two circular geodesics in  $D$  and if  $x \in \gamma_1 \cap \gamma_2$ , then the endpoints of  $\gamma_1$  and  $\gamma_2$  belong to the set  $\partial B_x \cap \partial D$ . Since  $\text{card}(B_x \cap \partial D) = 2$ , we conclude that  $\gamma_1 = \gamma_2$ . The second part of the theorem now follows from the fact that  $\mu_D(x) = \sigma_D(x)$  whenever  $x$  lies on a circular geodesic (see Section 3).  $\square$

Herron, Ibragimov and Minda proved that every planar 2-extremal domain is either simply or doubly connected, see [14]. Given a 2-extremal disc  $B$  in a domain  $D$ , we denote the unique circular geodesic in  $B$  by  $\gamma(B)$ . The (Euclidean) midpoint of the circular geodesic is called the hyperbolic center of  $B$  and denoted by  $\text{HC}(B)$ . Let  $E(D)$  be the set of all 2-extremal discs in  $D$ .

**Definition 4.2.** The set of hyperbolic centers of discs in  $E(D)$  is called the *hyperbolic medial axis* of the domain  $D$  and is denoted by  $\text{HMA}(D)$ .

The hyperbolic medial axis is a modification of the usual medial axis, whose definition was presented in Section 2. In certain respects the hyperbolic medial axis is better behaved than the medial axis e.g. this is the case for smoothness and localization properties. A more thorough investigation of these issues is underway [12].

**Theorem 4.3.** *If  $D \subset \mathbb{R}^2$  is a 2-extremal domain, then  $\text{HMA}(D)$  is locally the graph of a  $C^1$  curve. If  $B$  is a 2-extremal disc in  $D$ , then the circular geodesic  $\gamma(B)$  and  $\text{HMA}(D)$  are orthogonal at the hyperbolic center  $\text{HC}(B)$ .*

*Proof.* Let  $B$  be a 2-extremal disc corresponding to the boundary point  $a$  and  $b$ . Let  $B_a$  be the disc in  $B$  with  $a$  and  $\text{HC}(B)$  as boundary points;  $B_b$  is defined similarly. Note that  $B_a$  and  $B_b$  are horodiscs in  $B$ . The circle  $\partial B_a$  is tangent to  $\partial B$  at  $a$ , so it is orthogonal to  $\gamma(B)$  there, and hence also at  $\text{HC}(B)$ . Thus both  $\partial B_a$  and  $\partial B_b$  are orthogonal to  $\gamma(B)$  at  $\text{HC}(B)$  and, in particular,  $\text{HMA}(D) \cap U \subset U \setminus (B_a \cup B_b)$  for some sufficiently small neighborhood  $U$  of  $\text{HC}(B)$ . It is clear that  $\text{HMA}(D)$  is orthogonal to  $\gamma(B)$  and has smoothness  $C^1$  at  $\text{HC}(B)$ .  $\square$

For metric densities which are at least  $C^2$  smooth it is well-known that geodesics are locally unique, i.e. through a given point in a given direction there is only one geodesic. For metrics defined by densities with less smoothness this is not the case. For instance for the quasihyperbolic metric in the strip  $\{x \in \mathbb{R}^2: |x_2| < 1\}$  we know that a geodesic consists of a circular arc, a segment lying in the real axis and a second circular arc (any two of these three pieces may of course be degenerate). In particular, geodesics are not locally unique on the real axis in the real direction.

It was shown in Theorem 4.1 that there is a unique circular geodesic through every point in a 2-extremal domain. We next prove a stronger statement: there is no geodesic which is tangent to a circular geodesic.

**Lemma 4.4.** *Smooth geodesics of the K–P metric are locally unique in 2-extremal domains in the direction of the circular geodesic.*

*Proof.* Using an auxiliary Möbius mapping we may restrict ourselves to the circular geodesic  $(-1, 1) \subset \mathbb{R}$ , and more specifically, we show that there is no other geodesic through the origin which is parallel to the real axis there.

As before we denote by  $\mu_D$  the density of the K–P metric in our domain. By  $\tilde{\mu}$  we denote the density of the K–P metric in the domain  $\{x \in \mathbb{R}^2: |x_1| < 1\}$ . Obviously,  $\tilde{\mu}(x) = 2(1 - x_1^2)^{-1}$ . As in the proof of Theorem 4.3 we find that the level sets of  $\mu_D$  are constrained by a pair of balls. We restrict our attention to a small neighborhood of the origin. Then the radii of these balls are greater than some constant  $r > 0$ , so we see that the level-sets of  $\mu_D$  are approximated by the level-sets of  $\tilde{\mu}$  near the real axis. More precisely,

$$\begin{aligned} \frac{1}{\mu_D(x)} &\geq \frac{1}{\tilde{\mu}(|x_1| + \Delta x)} = 1 - \left(|x_1| + r(1 - \sqrt{1 - x_2^2})\right)^2 \\ &\geq 1 - \left(|x_1| + rx_2^2\right)^2 = 1 - x_1^2 + O(|x_1|x_2^2 + x_2^4). \end{aligned}$$

Here  $\Delta x$  is the maximal distance between the level-set of  $\mu_D$  and  $\tilde{\mu}$  at distance  $x_2$  from the real axis. A similar lower bound applies, so we have

$$|\mu_D(x) - (1 - x_1^2)^{-1}| \leq C|x_1|x_2^2$$

provided  $x_2 = O(x_1)$ .

Now suppose that there would be a second smooth geodesic through the origin, which is parallel to the real axis. Locally such a geodesic can be represented by

$$y = f(x) = c_2x^2 + O(x^3).$$

We also define  $F: \mathbb{R} \rightarrow \mathbb{R}^2$  by  $F(x) = (x, f(x))$ . We assume that  $c_2 > 0$ ; the cases of negative coefficient or lower order leading term are similar. We will show that for small enough  $\epsilon > 0$ , the segment  $L_1 = [0, F(\epsilon)]$  is shorter than the curve  $L_2 = \{F(x): 0 < x < \epsilon\}$ . Thus the latter curve is certainly not a geodesic, which proves local uniqueness. We also introduce the function  $G: \mathbb{R} \rightarrow \mathbb{R}^2$  which parameterizes  $L_1$ :  $G(x) = (x, \frac{x}{\epsilon}f(\epsilon))$ .

We start by calculating the length of  $L_2$ :

$$\mu_D(L_2) = \int_{L_2} \mu_D(z) dz = \int_0^\epsilon \mu(F(x)) \sqrt{1 + f'(x)^2} dx.$$

We know that

$$\mu(F(x)) = 1 - x^2 + O(x(c_2x^2)^2) = 1 - x^2 + O(x^5).$$

Thus we find that

$$\begin{aligned} \mu(L_2) &= \int_0^\epsilon \mu(F(x))(1 + \frac{1}{2}f'(x)^2 + O(f'(x)^4)) dx \\ &= \int_0^\epsilon (1 - x^2 + O(x^5))(1 + 2c_2^2x^2 + O(x^4)) dx \\ &= \int_0^\epsilon (1 + (2c_2^2 - 1)x^2 + O(x^4)) dx \\ &= \epsilon + (\frac{2}{3}c_2^2 - \frac{1}{3})\epsilon^3 + O(\epsilon^5). \end{aligned}$$

For  $L_1$  we calculate

$$\begin{aligned} \mu(L_1) &= \int_0^\epsilon \mu(G(x)) \sqrt{1 + (f(\epsilon)/\epsilon)^2} dx \\ &= (1 + \frac{1}{2}c_2^2\epsilon^2 + O(\epsilon^4)) \int_0^\epsilon (1 - x^2 + O(x(\frac{x}{\epsilon}f(\epsilon))^2)) dx \\ &= (1 + \frac{1}{2}c_2^2\epsilon^2)(\epsilon - \frac{1}{3}\epsilon^3) + O(\epsilon^4) \\ &= \epsilon + (\frac{1}{2}c_2^2 - \frac{1}{3})\epsilon^3 + O(\epsilon^4). \end{aligned}$$

A comparison with the expression for  $\mu(L_2)$  above shows that  $\mu(L_1) < \mu(L_2)$  whenever  $\epsilon$  is small enough, so  $L_2$  is not a geodesic.  $\square$

**Lemma 4.5.** *Let  $D$  be 2-extremal and doubly connected. Then a simple  $C^1$  curve  $\gamma$  which is not contractible is orthogonal to a circular geodesic at some point.*

*Proof.* The claim clearly holds in the special case  $D^c \subset \mathbb{R}$ . Thus we assume that  $D$  has a non-degenerate boundary component. Since our domain is doubly connected, it is a ring domain of the type  $G \setminus K$ , where  $G$  is open and  $K$  is a closed subset of  $G$ . We assume without loss of generality that  $\infty \notin \overline{G}$ .

By Theorem 4.1,  $D$  is foliated by circular geodesics. We think of the circular geodesics as directed curves which start at  $K$  and denote the circular geodesic through  $x$  by  $C_x$ , and the tangent of this curve at  $x$  by  $T_x$ . If  $\gamma$  is not orthogonal to any of the circular geodesics, then either  $T_x \cdot \nabla\gamma(x) > 0$  for all  $x$ , or  $T_x \cdot \nabla\gamma(x) < 0$  for all  $x$ .

A point  $x \in \gamma$  divides  $C_x$  in two parts whose lengths are denoted by  $l_K(x)$  and  $l_G(x)$ . Let  $L_r$  be the set of points  $x \in D$  such that  $l_G(x) = r l_K(x)$ . As in Theorem 4.3, we find that the simple closed curve  $L_r$  is  $C^1$  and orthogonal to all circular geodesics. Let  $x_0 \in \gamma$  and  $r = l_G(x_0)/l_K(x_0)$ . Then  $x_0 \in L_r$ . Now if  $T_x \cdot \nabla\gamma(x) > 0$  for all  $x$ , then we see that  $\gamma$  will not cross  $L_r$  again. Therefore  $\gamma$  cannot be a closed curve, which is a contradiction. The same conclusion holds if  $T_x \cdot \nabla\gamma(x) < 0$  for all  $x$ . Thus there must be a point of orthogonality between the curves.  $\square$

We are now ready to prove the main result of this paper. Note that this result combined with the results from Section 3 take care of the isometry problem for the K–P metric except in some cases of simply connected planar domains.

**Theorem 4.6.** *Let  $D$  be 2-extremal and doubly connected. Then every isometry of  $\mu_D$  is a Möbius mapping.*

*Proof.* As in the previous proof, we may assume that  $D = G \setminus K$ , where  $G$  is open and bounded, and  $K$  is a closed subset of  $G$ . So every circular geodesic connects  $\partial G$  to  $\partial K$ . By Theorem 4.3 for each 2-extremal disc  $B$  the hyperbolic medial axis  $\text{HMA}(D)$  is orthogonal to the circular geodesic  $\gamma(B)$  at the hyperbolic center  $\text{HC}(B)$ . We recall that every isometry is a conformal mapping in the usual, Euclidean sense. Hence, if we show that the isometry coincides with a Möbius map on an arc of a circle, then it follows that our isometry is Möbius.

Let  $f$  denote the isometry under consideration. By [14, Theorem B] we know that every isometry of a domain which is not 2-extremal is Möbius. Note that  $f^{-1}$  is also an isometry. Thus, it is enough for us to consider the case when  $f(D)$  is also 2-extremal. Now, Theorem 4.3 shows that  $\text{HMA}(D)$  is a  $C^1$  curve and thus image of  $\text{HMA}(D)$  under  $f$  is a simple closed  $C^1$  curve in  $f(D)$ . By Lemma 4.5,  $f(\text{HMA}(D))$  is orthogonal to a circular geodesic  $C'$  at some point, say  $f(x)$ . Since  $f^{-1}$  is an isometry, we find that  $f^{-1}(C')$  is a geodesic line in  $D$ . Since  $f^{-1}$  is conformal, we see that  $f^{-1}(C')$  is orthogonal to  $\text{HMA}(D)$  at  $x$ . But by Theorem 4.3,  $\text{HMA}(D)$  is orthogonal to a circular geodesic  $C$  at  $x$ , and since geodesics are unique by Lemma 4.4, it follows that  $C = f^{-1}(C')$ . Since  $f$  maps an arc of a circle to an arc of a circle, and is a K–P isometry, we easily see as in the first proof of Theorem 3.1 that  $f$  coincides with a Möbius map on  $C$ , which completes the proof.  $\square$

We end this section by presenting two results on the isometries of the Ferrand and the quasihyperbolic metrics.

**Corollary 4.7.** *Let  $K \subset \mathbb{R}^2$  be convex, closed and non-degenerate, and set  $D = \mathbb{R}^2 \setminus K$ . Then every isometry  $f: D \rightarrow \mathbb{R}^2$  of the quasihyperbolic metric is a similarity mapping.*

*Proof.* It is clear that  $\text{MA}(D) = \emptyset$ . From Proposition 2.3 we see that this implies  $\text{MA}(f(D)) = \emptyset$ . From this it easily follows that  $f(D) = \mathbb{R}^2 \setminus K'$ , where  $K'$  is convex. Moreover, we easily see that  $k_G = \mu_G$  if  $G$  is the complement of a convex closed set. Thus  $k_D = \mu_D$  and  $k_{f(D)} = \mu_{f(D)}$ . Therefore  $f$  is an isometry of the K–P metric, so the claim follows from the previous theorem.  $\square$

The next result deals with Ferrand isometries in the special case of a domain with a circular arc as part of its boundary. Although this is quite a restrictive assumption, we would like to point out that hereto no results whatsoever have been derived for the isometries of this metric.

**Theorem 4.8.** *Let  $D \subset \mathbb{R}^2$  be a domain and  $f: D \rightarrow \mathbb{R}^2$  be a Ferrand isometry. Assume that there exists a disc  $B \subset D$  with a property that  $\partial B \cap \partial D$  contains an arc  $\gamma$ . Then  $f$  is the restriction of a Möbius transformation.*

We will prove this claim using Theorem 3.2. In order to conform with the notation of that theorem, we will actually prove the following claim, which is easily seen to be equivalent with the previous theorem.

**Lemma 4.9.** *Let  $D \subset \mathbb{R}^2$  be a domain and  $f: D \rightarrow \mathbb{R}^2$  be a Ferrand isometry. Assume that there exists a disc  $B' \subset D'$ ,  $D' = f(D)$ , with a property that  $\partial B' \cap \partial D'$  contains an arc  $\gamma'$ . Then  $f$  is the restriction of a Möbius transformation.*

*Proof.* Since  $f$  is conformal we see as in the second proof of Theorem 3.1 that

$$f^*[\sigma_{D'}] = \sigma_{D'}(f(z))|f'(z)| = \sigma_D(z).$$

Let  $G'$  be the interior of the hyperbolic convex hull of  $\gamma'$  in  $B'$ . Then one can easily see that  $\sigma_{D'}(x) = \lambda_{B'}(x)$  for all  $x \in G'$ . Let  $G = f^{-1}(G')$ . First we claim that there exists a point  $a \in G$  with  $\sigma_D(a) = \mu_D(a)$ . Using this claim we obtain

$$f^*[\lambda_{B'}] = f^*[\sigma_{D'}] = \sigma_D \leq \mu_D \leq \lambda_{B_a}$$

with equality holding at the point  $x = a$ . The proof is now completed by using Theorem 3.2. Thus, it remains to prove the claim.

Observe first that if there exists points  $x, y \in G$  with  $\hat{K}_x \cap \hat{K}_y = \emptyset$ , then due to connectedness of  $G$  there exists a point  $a \in G \cap \partial \hat{K}_x$ , i.e.,  $a$  lies on a circular geodesic and hence  $\sigma_D(a) = \mu_D(a)$  (see Section 3). We can now assume that  $\hat{K}_x = \hat{K}_y$  for all  $x, y \in G$ . In particular, all the points of  $G$  share a common extremal disc, which we can assume to be the unit disc  $B$  about the origin. Put  $K = \partial B \cap \partial D$  and let  $\hat{K}$  be the hyperbolic convex hull of  $K$  in  $B$ . Note that  $\hat{K}_x = \hat{K}$  for each  $x \in G$ . Since  $G \subset \hat{K} \subset B$ ,  $B$  is not 2-extremal. Hence we have a conformal map  $f^{-1}$  of  $B'$  into  $B$  with a property that  $|f^{-1}(z)| \rightarrow 1$  as  $z \rightarrow \gamma'$ . Then Schwarz' Reflection Principle implies that  $f^{-1}$  has an analytic continuation onto  $\gamma'$  and by the identity theorem it can not map  $\gamma'$  onto a single point. Thus, the set  $f^{-1}(\gamma') \subset \partial B \cap \partial D$  contains an open arc, say  $\gamma$ . Then  $\sigma_D(a) = \mu_D(a)$  for each point of the hyperbolic convex hull of  $\gamma$ , as required.  $\square$

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