

THE MAXIMAL OPERATOR ON GENERALIZED ORLICZ SPACES

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ABSTRACT. In this note I present a sufficient condition for the boundedness of the maximal operator on generalized Orlicz spaces. The result includes as special cases the optimal condition for Orlicz spaces as well as the essentially optimal conditions for variable exponent Lebesgue spaces and the double-phase functional.

1. INTRODUCTION

Generalized Orlicz spaces $L^{\varphi(\cdot)}$ have been studied since the 1940's. A major synthesis of this research is given in the monograph of Musielak [13] from 1983, hence the alternative name Musielak–Orlicz spaces. These spaces are similar to the better known Orlicz spaces, but defined by a more general function $\varphi(x, t)$ which may vary with the location in space: the norm is defined by means of the integral

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)|) dx,$$

whereas in an Orlicz spaces φ would be independent of x , $\varphi(|f(x)|)$.

The special case of variable exponent Lebesgue space $L^{p(\cdot)}$, i.e. $\varphi(x, t) := t^{p(x)}$, was introduced by Orlicz [17] already in 1931, but lay dormant for many years. However, in the beginning of the new millennium, there was an explosion in the number of $L^{p(\cdot)}$ -papers. It was Diening [5] who opened the floodgate by proving the boundedness of the maximal operator under natural and essentially optimal conditions on the exponent (see also [4, 15, 18]). This result allowed for the development of harmonic analysis and related differential equations in the $L^{p(\cdot)}$ setting.

In this note I present the analogue of this result for $L^{\varphi(\cdot)}$ with a streamlined proof which is a simplification even in the Orlicz case (cf. Section 3). Furthermore, this general result has optimal or near optimal conditions in three important special cases:

- (1) Orlicz spaces, where the optimal condition of Gallardo [9] is recovered;
- (2) Variable exponent spaces, where the log-Hölder condition is recovered (cf. [18] regarding the optimality);
- (3) The *double phase* functional $\varphi(x, t) = t^p + a(x)t^q$ of Mingione and collaborators [1, 2, 3], where the sharp condition for the regularity of minimizers is recovered, namely $\frac{q}{p} < 1 + \frac{\alpha}{n}$ with $a \in C^\alpha$ (Theorem 4.7).

I hope that the results and techniques in this note will allow most of the results that have been derived in $L^{p(\cdot)}$ over the past 15 years to be established in $L^{\varphi(\cdot)}$ as well. With these techniques, the Riesz potential has been considered in [7] and the Dirichlet energy integral in [8].

Maeda, Mizuta, Ohno and Shimomura [10, 11, 16] have also recently studied the boundedness of the maximal operator in $L^{\varphi(\cdot)}$. Their results are special cases of ours,

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as they deal only with doubling φ and have other restricting assumptions as well, see Sections 2.1 and 5. Related differential equations have been studied recently by Baroni, Colombo and Mingione [1, 2, 3] and Giannetti and Passarelli di Napoli [14].

2. BACKGROUND

Definition 2.1. A convex, left-continuous function $\varphi: [0, \infty) \rightarrow [0, \infty]$ with $\varphi(0) = \lim_{t \rightarrow 0^+} \varphi(t) = 0$, and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ is called a Φ -function. The set of Φ -functions is denoted by Φ .

Definition 2.2. The set $\Phi(\mathbb{R}^n)$ consists of those $\varphi: \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty]$ with

- (1) $\varphi(y, \cdot) \in \Phi$ for every $y \in \mathbb{R}^n$;
- (2) $\varphi(\cdot, t) \in L^0(\mathbb{R}^n)$, the set of measurable functions, for every $t \geq 0$.

Also the functions in $\Phi(\mathbb{R}^n)$ will be called Φ -functions. In sub- and superscripts the dependence on x will be emphasized by $\varphi(\cdot): L^\varphi$ (Orlicz) vs $L^{\varphi(\cdot)}$ (Musielak–Orlicz).

Definition 2.3. Let $\varphi \in \Phi(\mathbb{R}^n)$ and define $\varrho_{\varphi(\cdot)}$ for $f \in L^0(\mathbb{R}^n)$ by

$$\varrho_{\varphi(\cdot)}(f) := \int_{\mathbb{R}^n} \varphi(x, |f(x)|) dx$$

The *generalized Orlicz space*, also called Musielak–Orlicz space, is defined as the set

$$L^{\varphi(\cdot)}(\mathbb{R}^n) = \{f \in L^0(\mathbb{R}^n) : \lim_{\lambda \rightarrow 0} \varrho_{\varphi(\cdot)}(\lambda f) = 0\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{\varphi(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_{\varphi(\cdot)}\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

Two functions φ and ψ are *equivalent* if there exists $L \geq 1$ such that $\psi(x, \frac{t}{L}) \leq \varphi(x, t) \leq \psi(x, Lt)$ for all x and t . Equivalent Φ -functions give rise to the same space with comparable norms. For further properties of such spaces see [6, Chapter 2] and [13].

The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq Cg$. The (*Hardy–Littlewood*) *maximal operator* is defined for $f \in L^0(\mathbb{R}^n)$ by

$$Mf(x) := \sup_{r>0} \int_{B(x,r)} |f(y)| dy,$$

where $B(x, r)$ is the ball with center x and radius r , and \int denotes the average integral. For a convex function φ Jensen's inequality states that

$$\varphi\left(\int_A |f(x)| dx\right) \leq \int_A \varphi(|f(x)|) dx.$$

2.1. Examples. Our theorem applies e.g. to the following Φ -functions, with suitable p :

$$\begin{aligned} \varphi_1(x, t) &= t^{p(x)} \log(1+t), & \varphi_2(x, t) &= t^p + a(x)t^q, & \varphi_3(x, t) &= t^p + a(x)t^p \log(e+t), \\ \varphi_4(x, t) &= e^{p(x)t} - 1, & \varphi_5(x, t) &= e^t t^{p(x)}, & \varphi_6(x, t) &= \infty \chi_{(1, \infty)}(t). \end{aligned}$$

PDE related to φ_1 [14] and φ_2 and φ_3 [1, 2, 3] have been studied recently. The latter three functions are not doubling and therefore not covered by [10, 11, 16], whereas the results of this paper work.

Our definition of Φ -functions presupposes convexity, in contrast to that of [10, 11, 16]. However, theirs is an empty generalization, as we show in Section 5 that any Φ -function satisfying their conditions ($\Phi 1$)–($\Phi 5$) is equivalent to a convex Φ -function.

3. ORLICZ SPACES

Lemma 3.1. *Let $\varphi: [0, \infty) \rightarrow [0, \infty]$ left-continuous function with $\varphi(0) = \lim_{t \rightarrow 0^+} \varphi(t) = 0$, and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. If $s \mapsto s^{-1}\varphi(s)$ is increasing, then there exists $\psi \in \Phi$ equivalent to φ .*

Proof. Let ψ be the greatest convex minorant of φ . Since $0 \leq \psi \leq \varphi$, it follows that $\psi(0) = \lim_{t \rightarrow 0^+} \psi(t) = 0$.

Suppose that $\varphi(s) > 0$. Then $\varphi(t) \geq \frac{t}{s}\varphi(s)$ for $t > s$. Thus the function $(\frac{t}{s} - 1)\varphi(s)$ is a convex minorant of φ on $[0, \infty)$ and since ψ is the greatest convex minorant we conclude that

$$\psi(t) \geq (\frac{t}{s} - 1)\varphi(s).$$

It follows that $\lim_{t \rightarrow \infty} \psi(t) = \infty$. Furthermore, this inequality implies that $\psi(2s) \geq \varphi(s)$. Since also $\psi \leq \varphi$, we see that $\varphi \simeq \psi$.

Finally, since ψ is convex, it is continuous except at the (possible) left-most point t with $\psi(s) = \infty$ for $s > t$. We force ψ to be left-continuous by (re)defining $\psi(s) = \lim_{t \rightarrow s^-} \psi(t)$. The properties above still hold; for $\psi(2s) \geq \varphi(s)$ we need the left-continuity of φ . \square

Lemma 3.2. *Let $\varphi \in \Phi$ and $\beta > 1$ be such that $s \mapsto s^{-\beta}\varphi(s)$ is increasing. Then there exists $\psi \in \Phi$ equivalent to φ such that $\psi^{1/\beta}$ is convex.*

Proof. The function $\varphi^{1/\beta}$ satisfies all the assumptions of Lemma 3.1. Hence there exists $\xi \in \Phi$ such that $\xi \simeq \varphi^{1/\beta}$. Set $\psi := \xi^\beta$. Since $\beta > 1$, $\psi \in \Phi$ and further $\psi \simeq \varphi$, as required. \square

Corollary 3.3. *Let $\varphi \in \Phi$ and $\beta > 1$ be such that $s \mapsto s^{-\beta}\varphi(s)$ is increasing. Then*

$$M : L^\varphi(\mathbb{R}^n) \rightarrow L^\varphi(\mathbb{R}^n)$$

is bounded.

Proof. Let $\psi \in \Phi$ be as in Lemma 3.2. It suffices to show that $M : L^\psi(\mathbb{R}^n) \rightarrow L^\psi(\mathbb{R}^n)$. Since $\psi^{1/\gamma}$ is convex, it follows from Jensen's inequality that

$$\psi(\epsilon Mf) = (\psi^{1/\gamma}(\epsilon Mf))^\gamma \leq (M(\psi^{1/\gamma}(\epsilon f)))^\gamma.$$

Let $f \in L^\psi(\mathbb{R}^n)$ and $\epsilon := \|f\|_\psi^{-1}$ so that $\varrho_\psi(\epsilon f) \leq 1$. Since M is bounded in $L^\gamma(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (M(\psi^{1/\gamma}(\epsilon f)))^\gamma dx \lesssim \int_{\mathbb{R}^n} (\psi^{1/\gamma}(\epsilon f))^\gamma dx = \int_{\mathbb{R}^n} \psi(\epsilon f) dx \leq 1.$$

Hence $\varrho_\psi(\epsilon Mf) \lesssim 1$, which implies that $\|\epsilon Mf\|_\psi \lesssim 1$. Dividing by ϵ , we find that $\|Mf\|_\psi \lesssim \frac{1}{\epsilon} = \|f\|_\psi$, which completes the proof. \square

4. GENERALIZED ORLICZ SPACES

For $B \subset \mathbb{R}^n$ define $\varphi_B^-(t) := \inf_{x \in B} \varphi(x, t)$ and $\varphi_B^+(t) := \sup_{x \in B} \varphi(x, t)$. We will use the following assumptions for some common constant $\sigma > 0$. The second corresponds in the $L^{p(\cdot)}$ case to local log-Hölder continuity.

- (A0) There exists $\beta > 0$ such that $\varphi(x, \beta) \leq 1$ and $\varphi(x, \sigma) \geq 1$ for every $x \in \mathbb{R}^n$.
- (A1) There exists $\beta \in (0, 1)$ such that

$$\varphi_B^+(\beta t) \leq \varphi_B^-(t)$$

for every $t \in [\sigma, (\varphi_B^-)^{-1}(\frac{1}{|B|})]$ and every ball B with $\frac{1}{|B|} \geq \varphi_B^-(\sigma)$.

(A2) There exists $\beta > 0$ and $h \in L^1_{\text{weak}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that, for every $t \in [0, \sigma]$,

$$\varphi(x, \beta t) \leq \varphi(y, t) + h(x) + h(y).$$

Remark 4.1. These conditions are invariant under equivalence of Φ -functions, which can be seen as follows. Suppose that $\varphi \simeq \psi$ with constant $L \geq 1$ and φ satisfies (A0)–(A2).

From (A0) we obtain $\psi(x, \beta/L) \leq \varphi(x, \beta) \leq 1$ and $\psi(x, L\sigma) \geq \varphi(\sigma) \geq 1$, so ψ satisfies (A0) with constant β/L and $\sigma' := L\sigma$ in place of σ . Suppose that $t \in [\sigma', (\psi_B^-)^{-1}(\frac{1}{|\beta|})]$. Then

$$\psi_B^+(\frac{\beta}{L^2}t) \leq \varphi_B^+(\frac{\beta}{L}t) \leq \varphi_B^-(\frac{t}{L}) \leq \psi_B^-(t)$$

since $\frac{t}{L} \in [\sigma, (\varphi_B^-)^{-1}(\frac{1}{|\beta|})]$ so that (A1) of φ could be used. Thus (A1) holds for ψ , as well. For (A2) we estimate, when $t \in [0, \sigma']$,

$$\psi(x, \beta t/L^2) \leq \varphi(x, \beta t/L) \leq \varphi(y, t/L) + h(x) + h(y) \leq \psi(y, t) + h(x) + h(y).$$

Remark 4.2. Suppose that $\varphi_\infty(t) := \lim_{x \rightarrow \infty} \varphi(x, t)$ exists for every $t \in [0, 1]$ and that

$$h(x) := \sup_{t \in [0, 1]} |\varphi(x, t) - \varphi_\infty(t)| \in L^1_{\text{weak}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Then by the triangle inequality we obtain that

$$\varphi(x, t) \leq |\varphi(x, t) - \varphi_\infty(t)| + |\varphi_\infty(t) - \varphi(y, t)| + \varphi(y, t) \leq h(x) + h(y) + \varphi(y, t)$$

so (A2) holds. This assumption corresponds to Nekvinda's [15] decay condition in $L^{p(\cdot)}$.

Lemma 4.3. *Let $\varphi \in \Phi(\mathbb{R}^n)$. Then φ_B^- satisfies the Jensen-type inequality*

$$\varphi_B^-\left(\frac{1}{2} \int_B f \, dx\right) \leq \int_B \varphi_B^-(f) \, dx.$$

Proof. Let ψ be the greatest convex minorant of φ_B^- . Since $t \mapsto \frac{\varphi_B^-(t)}{t}$ is increasing, we conclude as in Lemma 3.1 that $\varphi_B^-(s) \leq \psi(2s)$. By Jensen's inequality for ψ ,

$$\varphi_B^-\left(\frac{1}{2} \int_B f \, dx\right) \leq \psi\left(\int_B f \, dx\right) \leq \int_B \psi(f) \, dx \leq \int_B \varphi_B^-(f) \, dx. \quad \square$$

Lemma 4.4. *Let $\varphi \in \Phi(\mathbb{R}^n)$ satisfy assumptions (A0)–(A2). If B is a ball and $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\varrho_{\varphi(\cdot)}(f \chi_{\{|f| > \sigma\}}) \leq 1$, then*

$$\varphi\left(x, \frac{\beta}{4} \int_B |f| \, dy\right) \leq \left(1 + \frac{1}{\sigma}\right) \int_B \varphi(y, f) \, dy + h(x) + \int_B h(y) \, dy.$$

Proof. Fix a ball B . Assume without loss of generality that $f \geq 0$, and denote $f_1 := f \chi_{\{f > \sigma\}}$ and $f_2 := f - f_1$. Since φ is convex and increasing,

$$\varphi\left(x, \frac{\beta}{4} \int_B f \, dy\right) \leq \varphi\left(x, \frac{\beta}{2} \int_B f_1 \, dy\right) + \varphi\left(x, \beta \int_B f_2 \, dy\right).$$

Consider first the part f_1 when $\frac{1}{|\beta|} \geq \varphi_B^-(\sigma)$ and define

$$\bar{\varphi}(x, t) := \begin{cases} \varphi(x, \sigma)t & \text{when } t \leq \sigma, \\ \varphi(x, t) & \text{when } t > \sigma. \end{cases}$$

Since φ is convex, $\varphi \leq \bar{\varphi}$, and since $f_1 \notin (0, \sigma)$, $\varphi(y, f_1(y)) = \bar{\varphi}(y, f_1(y))$. Therefore it suffices to prove the second inequality in

$$\varphi\left(x, \frac{\beta}{2} \int_B f_1 \, dy\right) \leq \bar{\varphi}\left(x, \frac{\beta}{2} \int_B f_1 \, dy\right) \leq \int_B \bar{\varphi}(y, f_1) \, dy = \int_B \varphi(y, f_1) \, dy.$$

Note that $\bar{\varphi}$ satisfies (A1) on all of $[0, (\varphi_B^-)^{-1}(\frac{1}{|B|})]$. By Lemma 4.3,

$$\varphi_B^- \left(\frac{1}{2} \int_B f_1 dy \right) \leq \int_B \varphi_B^-(f_1) dy \leq \int_B \varphi(y, f_1) dy \leq \frac{1}{|B|}.$$

Therefore we can use (A1) and Lemma 4.3 to conclude that

$$\bar{\varphi} \left(x, \frac{\beta}{2} \int_B f_1 dy \right) \leq \bar{\varphi}_B^+ \left(\frac{\beta}{2} \int_B f_1 dy \right) \leq \bar{\varphi}_B^- \left(\frac{1}{2} \int_B f_1 dy \right) \leq \int_B \varphi(y, f_1) dy.$$

Suppose then that $\frac{1}{|B|} \leq \varphi_B^-(\sigma)$. Now

$$\int_B f_1 dy \leq \int_B \varphi_B^-(\sigma) f_1 dy \leq \int_B \varphi(y, f_1) dy \leq 1.$$

By convexity, (A0) and convexity again, we conclude that

$$\varphi \left(x, \beta \int_B f_1 dy \right) \leq \varphi(x, \beta) \int_B f_1 dy \leq \int_B \varphi(y, \sigma) f_1 dy \leq \frac{1}{\sigma} \int_B \varphi(y, f_1) dy.$$

For f_2 we use the convexity of $\varphi(x, \cdot)$ and (A2):

$$\varphi \left(x, \beta \int_B f_2 dy \right) \leq \int_B \varphi(x, \beta f_2) dy \leq \int_B \varphi(y, f_2) dy + \int_B h(x) + h(y) dy.$$

Adding the estimates for f_1 and f_2 , we conclude the proof. \square

Taking the supremum over balls B in the previous lemma, and noticing that $h(x) \leq Mh(x)$, we obtain the following corollary:

Corollary 4.5. *Let $\varphi \in \Phi(\mathbb{R}^n)$ satisfy assumptions (A0)–(A2) and let $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\varrho_{\varphi(\cdot)}(f \chi_{\{|f|>\sigma\}}) \leq 1$. Then*

$$\varphi(x, \frac{\beta}{4} Mf) \lesssim M(\varphi(\cdot, f)) + Mh(x).$$

Theorem 4.6. *Let $\varphi \in \Phi(\mathbb{R}^n)$ satisfy assumptions (A0)–(A2). Suppose that $\beta > 1$ is such that $s \mapsto s^{-\beta} \varphi(x, s)$ is increasing for every $x \in \mathbb{R}^n$. Then the maximal operator is bounded,*

$$M : L^{\varphi(\cdot)}(\mathbb{R}^n) \rightarrow L^{\varphi(\cdot)}(\mathbb{R}^n).$$

Proof. Let $\psi(x, \cdot) \in \Phi$ be related to $\varphi(x, \cdot)$ as in Lemma 3.2, for every $x \in \mathbb{R}^n$. Since the Φ -functions φ and ψ are equivalent, it suffices to show that $M : L^{\psi(\cdot)}(\mathbb{R}^n) \rightarrow L^{\psi(\cdot)}(\mathbb{R}^n)$. Note that ψ also satisfies assumptions (A0)–(A2), by Remark 4.1.

Let $f \in L^{\psi(\cdot)}(\mathbb{R}^n)$ and choose $\epsilon > 0$ such that $\varrho_{\psi(\cdot)}(\epsilon f) \leq 1$. When $t > \sigma$, $\psi(x, t) \geq 1$ by (A0), so that $\psi(x, t)^{1/\gamma} \leq \psi(x, t)$. Thus $\varrho_{\psi^{1/\gamma}(\cdot)}(\epsilon f \chi_{\{|\epsilon f|>\sigma\}}) \leq 1$ and we can apply Corollary 4.5 to ϵf with the Φ -function $\psi^{1/\gamma}$:

$$\psi(x, \frac{\beta}{4} \epsilon Mf(x))^{1/\gamma} \lesssim M\left(\psi^{1/\gamma}(\cdot, \epsilon f)\right)(x) + Mh(x).$$

Raising both side to the power γ and integrating, we find that

$$\int_{\mathbb{R}^n} \psi(x, \frac{\beta}{4} \epsilon Mf(x)) dx \lesssim \int_{\mathbb{R}^n} M\left(\psi^{1/\gamma}(\cdot, \epsilon f)\right)(x)^\gamma dx + \int_{\mathbb{R}^n} Mh(x)^\gamma dx.$$

Note that $h \in L^1_{\text{weak}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset L^\gamma(\mathbb{R}^n)$. Since M is bounded on $L^\gamma(\mathbb{R}^n)$, we obtain that

$$\int_{\mathbb{R}^n} \psi(x, \frac{\beta}{4} \epsilon Mf(x)) dx \lesssim \int_{\mathbb{R}^n} \left(\psi^{1/\gamma}(x, \epsilon f)\right)^\gamma dx + \int_{\mathbb{R}^n} h(x)^\gamma dx = \varrho_{\psi(\cdot)}(\epsilon f) + \|h\|_\gamma^\gamma.$$

Hence $\varrho_{\psi(\cdot)}(\frac{\beta}{2}\epsilon Mf) \lesssim 1$, and the proof is completed by a scaling argument like Corollary 3.3. \square

As an example and application we consider the *double-phase* Φ -function studied by Baroni, Colombo and Mingione [1, 2, 3]. Note that the bound $\frac{q}{p} \leq 1 + \frac{\alpha}{n}$ is the same as that obtained by these researchers (for some of their results, the strict inequality is required).

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $\varphi(x, t) := t^p + a(x)t^q$, $q > p > 1$. If $a \in C^\alpha(\overline{\Omega})$ is non-negative, then the maximal operator is bounded on $L^{\varphi(\cdot)}(\Omega)$ when $\frac{q}{p} \leq 1 + \frac{\alpha}{n}$.*

Proof. We extend a by McShane extension to a function in $C^\alpha(\mathbb{R}^n)$. This extension can be multiplied by a smooth cut-off function $H \in C_0^\infty(\mathbb{R}^n)$ which equals 1 in Ω . Since $q > p > 1$ and $a \geq 0$ in \mathbb{R}^n , it follows that $t \mapsto t^{-p}\varphi(x, t)$ is increasing.

We show that (A0)–(A2) hold with $\sigma = 1$. For (A0), we note that $1 \leq \varphi(x, 1) \leq 1 + \|a\|_\infty$. If K is the support of H , then $\varphi \equiv t^p$ in $\mathbb{R}^n \setminus K$, so (A2) holds with $h := \|a\|_\infty \chi_K$.

Let us show that also condition (A1) holds. Note first that $\varphi(x, t) \approx \max\{t^p, a(x)t^q\}$. Denote $a_B^+ := \sup_{z \in B} a(z)$ and $a_B^- := \inf_{z \in B} a(z)$. It suffices to show that

$$\max\{t^p, a_B^+ t^q\} \lesssim \max\{t^p, a_B^- t^q\}$$

when $\varphi_B^-(t) < \frac{1}{|B|}$. We prove the inequality in the even greater range $t^p < \frac{1}{|B|}$.

The inequality $t^p \lesssim \max\{t^p, a_B^- t^q\}$ is trivial, so we only have to show that $a_B^+ \lesssim \max\{t^{p-q}, a_B^-\}$. Using the upper bound on t , we see that it is sufficient to prove that

$$a_B^+ \lesssim \max\{|B|^{\frac{q-p}{p}}, a_B^-\} \approx \text{diam}(B)^{\frac{q-p}{p}n} + a_B^-.$$

In view of the definition of α , this follows from the assumption $a \in C^\alpha$.

Note that the reverse implication does not hold, i.e. (A1) does not imply that $a \in C^\alpha(\overline{\Omega})$. Indeed, if we choose $a = \chi_\Omega + \chi_E$ for some measurable $E \subset \Omega$, then a is discontinuous but $\varphi_B^+ \leq 2\varphi_B^-$. On the other hand, the assumption is sharp in the sense that if $\frac{q}{p} > 1 + \frac{\alpha}{n}$, then $a \in C^\alpha$ does not imply (A1), as shown by the example $a(x) = |x|^\alpha$. \square

Remark 4.8. Baroni, Colombo and Mingione [2] considered the border-line ($p = q$) double-phase functional $\varphi(x, t) := t^p + a(x)t^p \log(e + t)$. Also in this case their regularity assumption $a \in C^{\log}(\Omega)$ implies the condition (A1), so our theorem yields the boundedness of the maximal operator.

Remark 4.9. It was pointed out in the introduction that in many situations the conditions on φ are optimal or near optimal. However, in terms of interaction of the variability in x and t the present assumptions are far from optimal. This is seen by considering $\varphi(x, t) = t^p w(x)$, the weighted Lebesgue space. In this case the optimal assumption for the boundedness is that $w \in A_p$ [12]. The difference is that in this well-structured situation it is possible to trade integrability of the function and dependence of φ on x . This is not possible in the general case, indeed not even in the $L^{\varphi(\cdot)}$ -case. It is of course possible to write $\varphi(x, t) = \psi(x, t)w(x)$ and have a more general condition for the weight (cf. [6, Section 5.8]), but this is not a very elegant solution. Whether a better solution exists is a question for future research.

5. REMARKS ON ALTERNATIVE CONDITIONS

In the papers [10, 11, 16], Maeda, Mizuta, Ohno and Shimomura considered Musielak–Orlicz spaces with six conditions on the Φ -function. The first four conditions are, for some constant $D > 1$:

($\Phi 1$) $\varphi: [0, \infty) \rightarrow [0, \infty)$ is continuous, $\varphi(0) = \lim_{t \rightarrow 0^+} \varphi(t) = 0$, and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$.

($\Phi 2$) $\frac{1}{D} \leq \varphi(x, 1) \leq D$.

($\Phi 3$) $\frac{\varphi(s)}{s}$ is almost increasing, i.e. $\frac{\varphi(s)}{s} \geq \frac{1}{D} \frac{\varphi(t)}{t}$ for every $s > t$.

($\Phi 4$) φ is doubling, i.e. $\varphi(2t) \leq D\varphi(t)$ for every $t > 0$.

We notice that assumption ($\Phi 1$) is ostensibly weaker than the assumptions in this note, since convexity is not assumed *a priori*. However, we show below that any function satisfying these conditions is equivalent to a convex function.

Assumption ($\Phi 4$) does not correspond to any assumption in this note.

5.1. Almost increasing and non-convex. Assumption ($\Phi 3$) seems to be less stringent than the one used in this paper, since it follows from convexity that $\frac{\varphi(s)}{s}$ is increasing, not merely almost increasing.

Let φ satisfy assumption ($\Phi 3$) and define $\psi(s) := s \sup_{t \leq s} \frac{\varphi(t)}{t}$. Clearly $\frac{\psi(s)}{s}$ is increasing and $\varphi \leq \psi$. By condition ($\Phi 3$), $\sup_{t \leq s} \frac{\varphi(t)}{t} \leq D \frac{\varphi(s)}{s}$. Therefore

$$\varphi(s) \leq \psi(s) \leq D\varphi(s) \leq \varphi(D^2s),$$

so the functions are equivalent. Thus, there is no added generality in considering almost increasing functions instead of increasing functions. Furthermore, since $s \mapsto \frac{\psi(s)}{s}$ is increasing there exists a convex $\xi \in \Phi$ is equivalent to ψ by Lemma 3.1.

5.2. Decay condition. Condition ($\Phi 5$) in [10, 11, 16] is essentially the same as (A1) in this paper. However, their decay condition ($\Phi 6$) seems more general, until we combine it with ($\Phi 2$), which is a stronger version of (A0). The former condition is as follows:

($\Phi 6$) there exist a function $g \in L^1(\mathbb{R}^n)$ and a constant $B_\infty \geq 1$ such that $0 \leq g(x) < 1$ for all $x \in \mathbb{R}^n$ and

$$B_\infty^{-1} \varphi(x, t) \leq \varphi(x', t) \leq B_\infty \varphi(x, t)$$

whenever $|x'| \geq |x|$ and $g(x) \leq t \leq 1$.

Let us show how this relates to condition (A2). First we define

$$\varphi_\infty(t) := \limsup_{x \rightarrow \infty} \varphi(x, t).$$

If $t \in [g(x), 1]$, then $\varphi(x, t) \leq B_\infty \varphi_\infty(t)$. If $t \in [0, g(x)]$, then $\varphi(x, t) \leq D\varphi(x, 1)t \leq DAg(x)$ by conditions ($\Phi 2$) and ($\Phi 3$). Hence

$$\varphi(x, t) \leq B_\infty \varphi_\infty(t) + DAg(x)$$

for all $t \in [0, 1]$. Similarly we may establish that

$$\varphi_\infty(t) \leq B_\infty \varphi(y, t) + DAg(y),$$

and thus we conclude that

$$\varphi(x, t) \lesssim \varphi(y, t) + g(x) + g(y),$$

as required by assumption (A2).

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