THE MAXIMAL OPERATOR ON GENERALIZED ORLICZ SPACES — CORRIGENDUM

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ABSTRACT. In this note some flaws in the proofs of the paper [P. Hästö: The maximal operator on generalized Orlicz spaces, *J. Funct. Anal.* **269** (2015), 4038–4048] are corrected.

A flaw in the proof of the main result in [2] was pointed out by *Tetsu Shimomura* (private communication). It is amended in this note. The results are indicated with the same numbers as the ones they replace.

Let $\varphi \in \Phi(\mathbb{R}^n)$ be a Φ-function. For $B \subset \mathbb{R}^n$ define $\varphi_B^-(t) := \inf_{x\in B} \varphi(x,t)$ and $\varphi_B^+(t) := \sup_{x\in B} \varphi(x,t)$. We will use the following assumptions for some common constant $\sigma > 0$. Note that (A2) is strengthened from [2] by replacing weak-L^1 with L^1. In (A0), the previous β has been denoted by $\beta\sigma$ for consistency with (A1) and (A2).

(A0) There exists $\beta \in (0,1)$ such that $\varphi(x,\beta\sigma) \leqslant 1 \leqslant \varphi(x,\sigma)$ for every $x \in \mathbb{R}^n$.

(A1) There exists $\beta \in (0,1)$ such that

$$\varphi_B^+(\beta t) \leqslant \varphi_B^-(t)$$

for every $t \in \big[\sigma, (\varphi_B^-)^{-1}(\frac{1}{|B|})\big]$.

(A2) There exists $\beta > 0$ and $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that, for every $t \in [0,\sigma]$,

$$\varphi(x,\beta t) \leqslant \varphi(y,t) + h(x) + h(y).$$

These conditions are invariant under equivalence of Φ-functions [2, Remark 4.1].

Remark A. Suppose that $\varphi_B^-(t) < b$. Let us show that $t \leqslant (\varphi_B^-)^{-1}(b)$. Since φ_B^- is defined as an infimum, there exists $x \in B$ such that $\varphi(x,t) < b$. Applying the increasing φ^{-1} to both sides, we obtain that $\varphi^{-1}(x,b) \geqslant \varphi^{-1}(x,\varphi(x,t))$. Denote $t_0 := \sup\{t \,|\, \varphi(x,t) = 0\}$. Now $\varphi^{-1}(x,\varphi(x,t)) = t$ if $t > t_0$ and zero otherwise [1, (2.2)]. In the latter case, $t \leqslant t_0 < \varphi^{-1}(x,b)$. Thus $t \leqslant \varphi^{-1}(x,b)$ in both cases, and since $\varphi^{-1}(x,b) \leqslant (\varphi_B^-)^{-1}(b)$, it follows that $t \leqslant (\varphi_B^-)^{-1}(b)$.

Lemma 4.3. *Let $\varphi \in \Phi(\mathbb{R}^n)$ with $t \mapsto t^{-\gamma}\varphi(x,t)$ increasing, $\gamma \geqslant 1$. Then the following Jensen-type inequality holds:*

$$\varphi_B^-\bigg(\frac{1}{2}\fint_B f\,dx\bigg)^{\frac{1}{\gamma}} \leqslant \fint_B \varphi_B^-(f)^{\frac{1}{\gamma}}\,dx.$$

Proof. Denote $\bar\varphi := (\varphi_B^-)^{\frac{1}{\gamma}}$ and let ψ be the greatest convex minorant of $\bar\varphi$. Since $t \mapsto \frac{\bar\varphi(t)}{t}$ is increasing, we find that $\bar\varphi(t) \geqslant \frac{t}{s}\bar\varphi(s)$ for $t > s$. Thus the function $t \mapsto (\frac{t}{s} - 1)\bar\varphi(s)$ is a convex minorant of $\bar\varphi$ on $[0,\infty)$ and since ψ is the greatest convex minorant we conclude

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that $\psi(t) \geqslant (\frac{t}{s}-1)\bar\varphi(s)$. With $t=2s$, we obtain that $\bar\varphi(s)\leqslant\psi(2s)$. By Jensen's inequality for ψ,

$$\varphi_B^-\Big(\frac{1}{2}\fint_B f\,dx\Big)^{\frac{1}{\gamma}} \leqslant \psi\Big(\fint_B f\,dx\Big) \leqslant \fint_B \psi(f)\,dx \leqslant \fint_B \varphi_B^-(f)^{\frac{1}{\gamma}}\,dx. \qquad \square$$

Lemma 4.4. *Let $\varphi\in\Phi(\mathbb{R}^n)$ satisfy assumptions (A0)–(A2) with $t\mapsto t^{-\gamma}\varphi(x,t)$ increasing and $\gamma\geqslant 1$. There exists $\beta'>0$ such that, for a ball B and $f\in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\varrho_{\varphi(\cdot)}(f\chi_{\{|f|>\sigma\}})<1$,*

$$\varphi\Big(x,\beta'\fint_B |f|\,dy\Big)^{\frac{1}{\gamma}} \lesssim \fint_B \varphi(y,f)^{\frac{1}{\gamma}}\,dy + h(x)^{\frac{1}{\gamma}} + \fint_B h(y)^{\frac{1}{\gamma}}\,dy.$$

Proof. Fix a ball B. Assume without loss of generality that $f\geqslant 0$ and that $\varphi^{\frac{1}{\gamma}}$ is convex [2, Lemma 3.2]. Denote $f_1 := f\chi_{\{f>\sigma\}}$, $f_2 := f-f_1$, and $A_i := \fint_B f_i\,dy$. Since $\varphi^{\frac{1}{\gamma}}$ is convex and increasing,

$$\varphi\Big(x,\frac{\beta}{4}\fint_B f\,dy\Big)^{\frac{1}{\gamma}} \leqslant \varphi\big(x,\tfrac{\beta}{2}A_1\big)^{\frac{1}{\gamma}} + \varphi\big(x,\beta A_2\big)^{\frac{1}{\gamma}}.$$

Consider first the part f_1 when $\frac{1}{|B|}>\varphi_B^-(\sigma)$. By Lemma 4.3 (with $\gamma=1$),

$$\varphi_B^-\big(\tfrac12 A_1\big) \leqslant \fint_B \varphi_B^-(f_1)\,dy \leqslant \fint_B \varphi(y,f_1)\,dy < \tfrac{1}{|B|}.$$

If $t=\frac12 A_1\geqslant\sigma$, then by Remark A we can use (A1) to infer that $\varphi\big(x,\frac{\beta}{2}A_1\big)\leqslant\varphi_B^-\big(\frac12 A_1\big)$. From this we continue with Lemma 4.3 to conclude that

$$\varphi\big(x,\tfrac{\beta}{2}A_1\big)^{\frac{1}{\gamma}} \leqslant \varphi_B^-\big(\tfrac12 A_1\big)^{\frac{1}{\gamma}} \leqslant \fint_B \varphi_B^-(f_1)^{\frac{1}{\gamma}}\,dy \leqslant \fint_B \varphi(y,f_1)^{\frac{1}{\gamma}}\,dy.$$

Next consider $\frac12 A_1<\sigma$. By convexity of $\varphi^{\frac{1}{\gamma}}$, (A0) and convexity again, we conclude that

$$\varphi\big(x,\tfrac{\beta}{2}A_1\big)^{\frac{1}{\gamma}} \leqslant \varphi\big(x,\beta\sigma\big)^{\frac{1}{\gamma}}\cdot\tfrac{A_1}{2\sigma} \leqslant \tfrac{1}{2\sigma}\fint_B f_1\,dy \leqslant \tfrac12\fint_B \varphi(y,f_1)^{\frac{1}{\gamma}}\,dy, \tag{4.5}$$

where, in the last step, we used also that $f_1>\sigma$ when it is non-zero.

Suppose then that $\frac{1}{|B|}\leqslant\varphi_B^-(\sigma)$. We show that $A_1\leqslant\sigma$ and then finish with (4.5). Now

$$A_1 = \frac{1}{|B|}\int_B f_1(y)\,dy \leqslant \int_B \varphi_B^-(\sigma)f_1\,dy \leqslant \int_B \varphi(y,\sigma)f_1\,dy \leqslant \sigma\int_B \varphi(y,f_1)\,dy < \sigma.$$

For f_2 we use the convexity of $\varphi(x,\cdot)^{\frac{1}{\gamma}}$ and (A2):

$$\varphi\big(x,\beta A_2\big)^{\frac{1}{\gamma}} \leqslant \fint_B \varphi(x,\beta f_2)^{\frac{1}{\gamma}}\,dy \lesssim \fint_B \varphi(y,f_2)^{\frac{1}{\gamma}}\,dy + \fint_B h(x)^{\frac{1}{\gamma}} + h(y)^{\frac{1}{\gamma}}\,dy.$$

Adding the estimates for f_1 and f_2, we conclude the proof. $\square$

Taking the supremum over balls B in the previous lemma, and noticing that $h(x)^{\frac{1}{\gamma}}\leqslant M(h^{\frac{1}{\gamma}})(x)$, we obtain the following corollary:

Corollary 4.6. *Let $\varphi\in\Phi(\mathbb{R}^n)$ satisfy assumptions (A0)–(A2) and let $f\in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\varrho_{\varphi(\cdot)}(f\chi_{\{|f|>\sigma\}})<1$. Then there exists $\beta'>0$ such that*

$$\varphi(x,\beta' Mf)^{\frac{1}{\gamma}} \lesssim M\big(\varphi(\cdot,f)^{\frac{1}{\gamma}}\big) + M(h^{\frac{1}{\gamma}}).$$

Theorem 4.7. *Let $\varphi \in \Phi(\mathbb{R}^n)$ satisfy assumptions (A0)–(A2). Suppose that $\beta > 1$ is such that $s \mapsto s^{-\beta}\varphi(x,s)$ is increasing for every $x \in \mathbb{R}^n$. Then the maximal operator is bounded,*

$$M : L^{\varphi(\cdot)}(\mathbb{R}^n) \to L^{\varphi(\cdot)}(\mathbb{R}^n).$$

Proof. Let $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ and choose $\epsilon > 0$ such that $\varrho_{\varphi(\cdot)}(\epsilon f) < 1$. By Corollary 4.6,

$$\varphi(x, \beta'\epsilon Mf)^{\frac{1}{\gamma}} \lesssim M\big(\varphi(\cdot, \epsilon f)^{\frac{1}{\gamma}}\big) + M(h^{\frac{1}{\gamma}}).$$

Raising both side to the power γ and integrating, we find that

$$\int_{\mathbb{R}^n} \varphi(x, \beta'\epsilon Mf(x))\, dx \lesssim \int_{\mathbb{R}^n} M\big(\varphi^{\frac{1}{\gamma}}(\cdot, \epsilon f)\big)(x)^\gamma\, dx + \int_{\mathbb{R}^n} M\big(h^{\frac{1}{\gamma}}\big)(x)^\gamma\, dx.$$

Since M is bounded on $L^\gamma(\mathbb{R}^n)$, we obtain that

$$\int_{\mathbb{R}^n} \varphi(x, \beta'\epsilon Mf(x))\, dx \lesssim \int_{\mathbb{R}^n} \big(\varphi^{\frac{1}{\gamma}}(x, \epsilon f)\big)^\gamma\, dx + \int_{\mathbb{R}^n} h(x)^{\frac{1}{\gamma}\gamma}\, dx = \varrho_{\varphi(\cdot)}(\epsilon f) + \|h\|_1.$$

Hence $\varrho_{\varphi(\cdot)}(\beta'\epsilon Mf) \lesssim 1$, and the proof is completed by a scaling argument like [2, Corollary 3.3]. ☐

REFERENCES

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