

Obstacle problems and superharmonic functions with nonstandard growth

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Abstract

In this article we study solutions and supersolutions of a variable exponent $p(x)$ -Laplace equation, and the corresponding obstacle problem, as well as related superharmonic functions. The relationship between these function classes closely parallels the classical case. However, integrability properties of superharmonic functions require stronger assumptions.

Key words: nonstandard growth, obstacle problem, $p(x)$ -Laplace equation, superharmonic function, subsolution, supersolution, variable exponent

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1 Introduction

The main purpose of this work is to study obstacle problems and superharmonic functions related to partial differential equations with nonstandard growth conditions. The equation we have in mind is

$$(1) \quad \operatorname{div}(p(x)|\nabla u|^{p(x)-2}\nabla u) = 0,$$

where $p(\cdot)$ is a measurable function such that

$$1 < \inf_{x \in \mathbb{R}^n} p(x) \leq p(x) \leq \sup_{x \in \mathbb{R}^n} p(x) < \infty.$$

Roughly speaking, a solution to the obstacle problem is a minimal supersolution of equation (1); Equation (1) is the Euler–Lagrange equation of the variational integral

$$\int |\nabla u|^{p(x)} dx.$$

Furthermore, if $p(\cdot)$ is constant we have the standard p -Laplace equation and p -Dirichlet integral.

In recent years there has been a growing interest in nonlinear equations with nonstandard growth, which are related in a natural way to spaces $L^{p(\cdot)}$ with variable exponent. The first investigations of such problems were by the Italian school and had as their starting point the calculus of variations, see, e.g., [1,22]. Recently, several authors have approached the same problem more in the spirit of nonlinear differential equations, e.g. in [3,5,8,9,12,13]. The latter approach is also taken in this article. An overview of the development of the variable exponent theory can be found in [7], along with an extensive bibliography on the subject.

This paper is divided into seven sections. After recalling some basic facts about variable exponent spaces in Section 2, we move on to elementary properties of (weak) solutions and supersolutions of equation (1). These results in Section 3 follow from the same proofs as in the fixed exponent case, hence, the proofs are not included here. We also introduce the obstacle problem, apparently for the first time in the nonstandard growth case. The real work begins in Section 4. We use a recent version of the Moser iteration technique, see [13], to derive a Harnack estimate for solutions of the obstacle problem. This allows us to prove that the solution of the obstacle problem with continuous obstacle is itself continuous. In Section 5 we study the limiting behaviour of sequences of supersolutions. We are able to recover classical results for increasing and locally bounded sequences, uniformly convergent sequences and solutions of obstacle problems. In Section 6 we introduce superharmonic functions based on the comparison principle, and show that the relationship between supersolutions and superharmonic functions is the same as in the classical case. Finally, in Section 7 we study integrability properties of

superharmonic functions. This part of the theory differs from the constant exponent case since a stronger finiteness assumption is necessary to establish integrability.

2 Preliminaries

A measurable function $p: \mathbb{R}^n \rightarrow (1, \infty)$ is called a *variable exponent*, and we denote for $A \subset \mathbb{R}^n$

$$p_A^+ = \sup_{x \in A} p(x), \quad p_A^- = \inf_{x \in A} p(x), \quad p^+ = \sup_{x \in \mathbb{R}^n} p(x), \quad p^- = \inf_{x \in \mathbb{R}^n} p(x).$$

Throughout the paper it is assumed that $1 < p^- \leq p^+ < \infty$.

Let Ω be an open subset of $\mathbb{R}^n, n \geq 2$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u defined on Ω for which

$$\int_{\Omega} |u(x)|^{p(x)} dx < \infty.$$

The Luxemburg norm on this space is defined as

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space studied in [23]. For a constant function p the variable exponent Lebesgue space coincides with the standard Lebesgue space.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

For basic results on variable exponent spaces we refer to [21].

An interesting feature of variable exponent Sobolev spaces is that smooth functions need not to be dense. This was observed by Zhikov in connection with Lavrentiev phenomenon, see [25]. However, when the exponent satisfies a logarithmic Hölder continuity property, or briefly “ p is *log-Hölder continuous*”, then the maximal operator is bounded and consequently smooth functions are dense, see [17,24]. Recall that the log-Hölder condition means that there is a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x - y|)}$$

for all $x, y \in \Omega$ with $|x - y| \leq 1/2$. The exponent p is log-Hölder continuous in an open set Ω if and only if there exists a constant $C > 0$ such that

$$|B|^{p_B^- - p_B^+} \leq C$$

for every ball $B \subset \Omega$, [6]. Under this condition smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$, [10].

In this work we do not study variable exponent spaces themselves, but rather related differential equations. For our purposes, the most important facts about the variable exponent Lebesgue spaces are the following: If E is a measurable set with a finite measure, p and q are variable exponents satisfying $q(x) \leq p(x)$ for almost every $x \in E$, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$ and the norm of the embedding cannot exceed $1 + |E|$. In particular this implies that every function $u \in W^{1,p(\cdot)}(\Omega)$ also belongs to $W_{loc}^{1,p^-}(\Omega)$ and to $W^{1,p_B^-}(B)$, where $B \subset \Omega$ is an open ball. We use also a variable exponent version of Hölder's inequality:

$$\int_{\Omega} fg \, dx \leq C \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)},$$

where the constant C depends only on p^- and p^+ . Here $1/p(x) + 1/p'(x) = 1$ for every x . For all these facts we refer to [21].

In the rest of this article we use the convention that $\Omega \subset \mathbb{R}^n$ is an open and bounded set, $B = B(x, r)$ is an open ball and p is log-Hölder continuous exponent in Ω with $1 < p^- \leq p^+ < \infty$.

We end this section by recalling that $f_+ = \max\{f, 0\}$ and $f_- = \min\{f, 0\}$. By a continuous function we always mean a real-valued continuous function, whereas a semicontinuous function is allowed to be extended real-valued, i.e. to take values in the extended real line $[-\infty, \infty]$.

The letter C denotes various positive constants whose exact values are unimportant and may vary with each usage.

3 Basic properties of solutions

We say that a function $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is a (weak) *supersolution* (of equation (1)) in Ω , if

$$(2) \quad \int_{\Omega} p(x) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx \geq 0$$

for every non-negative test function $\varphi \in C_0^\infty(\Omega)$. A function u is a *subsolution* in Ω if $-u$ is a supersolution in Ω , and a *solution* in Ω if it is both a super- and a subsolution in Ω .

The dual of $L^{p(\cdot)}(\Omega)$ is the space $L^{p'(\cdot)}(\Omega)$ obtained by conjugating the exponent pointwise. This together with our definition of $W_0^{1,p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ implies that we can also test with functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

Existence of solutions has been discussed in [9,12,16]. Under our conditions on p , every solution is locally Hölder continuous, see [2,3,8].

Let us present a simple example. Consider the one-dimensional case on the interval $[0, 1]$ with exponent $p = 3\chi_{[0,1/2]} + 2\chi_{[1/2,1]}$. This is a modified version of an example presented in [11]. The question is: When is the linear function $u(x) = ax$, $a > 0$, a supersolution? Let ϕ be a non-negative smooth test function with zero boundary values. Then

$$\begin{aligned} \int_0^1 p(x)|u'|^{p(x)-2}u'\phi' dx &= \int_0^{1/2} 3a^2\phi' dx + \int_{1/2}^1 2a\phi' dx \\ &= 3a^2(\phi(1/2) - \phi(0)) + 2a(\phi(1) - \phi(1/2)) \\ &= (3a - 2)a\phi(1/2). \end{aligned}$$

Since $\phi(1/2) \geq 0$, we see that u is a supersolution if $a \geq 2/3$, and a subsolution if $a \leq 2/3$. This is at variance with the constant exponent case, where it is trivial to see that the linear function is always a solution.

Let $\psi : \Omega \rightarrow [-\infty, \infty)$ be a function, called an obstacle; $w \in W^{1,p(\cdot)}(\Omega)$ is a function which gives the boundary values. Define

$$\mathcal{K}_{\psi,w}(\Omega) = \{u \in W^{1,p(\cdot)}(\Omega) : u - w \in W_0^{1,p(\cdot)}(\Omega), u \geq \psi \text{ a.e. in } \Omega\}.$$

We say that a function $u \in \mathcal{K}_{\psi,w}(\Omega)$ is a *solution of the obstacle problem* $\mathcal{K}_{\psi,w}(\Omega)$ if

$$\int_{\Omega} p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(v - u) dx \geq 0$$

for every $v \in \mathcal{K}_{\psi,w}(\Omega)$.

All the basic properties of solutions of the obstacle problem follow as in [15, Section 3]. This is based on the fact that the function $\mathcal{A}(x, \xi) = |\xi|^{p(x)-2}\xi$ satisfies the relevant properties (3.4)–(3.6) of [15, p. 56]. Note, in particular, that the following condition holds

$$(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) \geq 0$$

whenever $\xi_1, \xi_2 \in \mathbb{R}^n$.

A function $u \in \mathcal{K}_{\psi,w}(\Omega)$ is a solution of the obstacle problem if and only if it is a minimizer of the Dirichlet energy integral among the functions in $\mathcal{K}_{\psi,w}(\Omega)$. This is

shown in the usual way, but using Young's inequality directly rather than Hölder's. Explicitly, if $u \in \mathcal{K}_{\psi,w}(\Omega)$ is a solution of the obstacle problem, then

$$\begin{aligned} \int_{\Omega} p(x)|\nabla u|^{p(x)} dx &\leq \int_{\Omega} p(x)|\nabla u|^{p(x)-1}|\nabla v| dx \\ &\leq \int_{\Omega} p(x)\left(\frac{1}{p'(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|\nabla v|^{p(x)}\right) dx \\ &= \int_{\Omega} (p(x) - 1)|\nabla u|^{p(x)} + |\nabla v|^{p(x)} dx \end{aligned}$$

for every $v \in \mathcal{K}_{\psi,w}(\Omega)$. Moving all the terms with u to the left-hand-side, we see that u is a minimizer. Suppose next that $u \in \mathcal{K}_{\psi,w}$ is a minimizer and let $v \in \mathcal{K}_{\psi,w}(\Omega)$. Since $\mathcal{K}_{\psi,w}(\Omega)$ is convex, we have $u + \varepsilon(v - u) \in \mathcal{K}_{\psi,w}(\Omega)$ for each $\varepsilon \in [0, 1]$. This implies

$$\int_{\Omega} \frac{1}{\varepsilon} \left(|\nabla u + \varepsilon \nabla(v - u)|^{p(x)} - |\nabla u|^{p(x)} \right) dx \geq 0$$

for each $\varepsilon \in [0, 1]$. Letting $\varepsilon \rightarrow 0$ gives

$$\int_{\Omega} p(x)|\nabla u|^{p(x)-2} \nabla u \cdot \nabla(v - u) dx \geq 0.$$

Thus u is a solution of the obstacle problem $\mathcal{K}_{\psi,w}(\Omega)$.

Since solutions of the obstacle problem are minimizers of the Dirichlet energy integral, existence and uniqueness follow provided that the boundary value function w is bounded or p is, for instance, continuous, see [12,16].

If $u \in \mathcal{K}_{\psi,w}(\Omega)$ and $\phi \in C_0^\infty(\Omega)$ is non-negative, then $u + \phi \in \mathcal{K}_{\psi,w}(\Omega)$. Thus it directly follows that every solution of the obstacle problem is a supersolution.

The proofs of the following theorems are similar to the corresponding proofs on pp. 61–62 in [15], and are not repeated here.

Theorem 1 *Let $u \in \mathcal{K}_{\psi,w}(\Omega)$ be a solution of the obstacle problem. If $v \in W^{1,p(\cdot)}(\Omega)$ is a supersolution in Ω such that $\min\{u, v\} \in \mathcal{K}_{\psi,w}(\Omega)$, then $v \geq u$ a.e. in Ω .*

Theorem 2 *If u and v are two supersolutions, then $\min\{u, v\}$ is also a supersolution.*

Theorem 3 *Let $u \in \mathcal{K}_{\psi,w}(\Omega)$ be a solution of the obstacle problem. Then*

$$u \leq \operatorname{ess\,sup}_{x \in \Omega} \max\{\psi(x), w(x)\}.$$

If u is a solution with boundary values w , then

$$\operatorname{ess\,inf}_{\Omega} w \leq u \leq \operatorname{ess\,sup}_{\Omega} w.$$

We say that $u \geq 0$ on $\partial\Omega$ in the Sobolev sense, if $\min\{u, 0\} \in W_0^{1,p(\cdot)}(\Omega)$. Similarly $u \geq v$ on $\partial\Omega$ in the Sobolev sense if $\min\{u - v, 0\} \in W_0^{1,p(\cdot)}(\Omega)$.

Lemma 4 *Let u be a supersolution and v a subsolution such that $u \geq v$ on $\partial\Omega$ in the Sobolev sense. Then $u \geq v$ a.e. in Ω .*

Theorem 5 *Let $u \in \mathcal{K}_{\psi,u}(\Omega)$ be a solution of the obstacle problem and let $D \subset \Omega$ be open. If there is a subsolution v in D with $\psi \leq v \leq u$ a.e. in D , then u is a solution in D .*

4 Harnack estimates and regularity of solutions of the obstacle problem

In this section we prove Harnack type estimates for solutions of the obstacle problem and show that a solution of the obstacle problem is continuous provided the obstacle itself is continuous.

Lemma 6 (Caccioppoli type estimate) *Assume u is a solution of the obstacle problem $\mathcal{K}_{\psi,w}(\Omega)$ with a non-positive obstacle ψ . Let $B \Subset \Omega$, $R \geq 0$ and $\eta \in C_0^\infty(B)$ with $0 \leq \eta \leq 1$. Then*

$$\int_B (u_+ + R)^\gamma |\nabla u_+|^{p_B^-} \eta^{p_B^+} dx \leq C \int_B ((u_+ + R)^\gamma \eta^{p_B^+} + (u_+ + R)^{\gamma+p(x)} |\nabla \eta|^{p(x)}) dx$$

holds for every $\gamma \geq 0$. Here the constant C depends only on p^- and p^+ .

Proof: We want to test with the function $u - \phi$ where $\phi = u_+ \eta^{p_B^+}$. This is clearly above the non-positive obstacle ψ and has the same boundary values as u , thus, it is enough to show that $\phi \in W^{1,p(\cdot)}(\Omega)$. Since

$$|\nabla \phi| \leq |\eta^{p_B^+} \nabla u_+ + u_+ p_B^+ \eta^{p_B^+-1} \nabla \eta| \leq |\nabla u| + C|u_+|,$$

we observe that $|\nabla \phi| \in L^{p(\cdot)}(\Omega)$, and $\phi \in L^{p(\cdot)}(\Omega)$, so $u - \phi$ is a valid test function.

Since $u - \phi \in \mathcal{K}_{\psi,u}(\Omega)$, we find

$$\begin{aligned} 0 &\leq \int_\Omega p(x) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla(-\phi) dx \\ &= - \int_\Omega p(x) |\nabla u|^{p(x)-2} \nabla u \cdot (\eta^{p_B^+} \nabla u_+ + u_+ p_B^+ \eta^{p_B^+-1} \nabla \eta) dx \\ &= - \int_\Omega p(x) |\nabla u_+|^{p(x)} \eta^{p_B^+} dx - \int_\Omega p(x) |\nabla u|^{p(x)-2} u_+ p_B^+ \eta^{p_B^+-1} \nabla u \cdot \nabla \eta dx. \end{aligned}$$

This implies that

$$(3) \quad p_B^- \int_\Omega |\nabla u_+|^{p(x)} \eta^{p_B^+} dx \leq p_B^+ \int_\Omega p(x) |\nabla u_+|^{p(x)-1} u_+ \eta^{p_B^+-1} |\nabla \eta| dx.$$

We denote the right hand side of (3) by I .

By Young's inequality we obtain

$$\begin{aligned} I &\leq p_B^+ \int_{\Omega} \varepsilon^{1-p(x)} (u_+ |\nabla \eta| \eta^{p_B^+ - p_B^+ / p'(x) - 1})^{p(x)} + \varepsilon \frac{p(x)}{p'(x)} (|\nabla u_+|^{p(x)-1} \eta^{p_B^+ / p'(x)})^{p'(x)} dx \\ &\leq p_B^+ \varepsilon^{1-p_B^+} \int_{\Omega} (u_+)^{p(x)} |\nabla \eta|^{p(x)} \eta^{p_B^+ - p(x)} dx + p_B^+ (p_B^+ - 1) \varepsilon \int_{\Omega} |\nabla u_+|^{p(x)} \eta^{p_B^+} dx. \end{aligned}$$

Combining this with the inequality (3) we have

$$(4) \quad \begin{aligned} p_B^- \int_{\Omega} |\nabla u_+|^{p(x)} \eta^{p_B^+} dx &\leq p_B^+ \varepsilon^{1-p_B^+} \int_{\Omega} (u_+)^{p(x)} |\nabla \eta|^{p(x)} \eta^{p_B^+ - p(x)} dx \\ &\quad + p_B^+ (p_B^+ - 1) \varepsilon \int_{\Omega} |\nabla u_+|^{p(x)} \eta^{p_B^+} dx. \end{aligned}$$

Choosing $\varepsilon = \min \{1, p_B^- / 2p_B^+ (p_B^+ - 1)\}$ we can absorb the last term in (4) in the left hand side. We obtain

$$(5) \quad \int_{\Omega} |\nabla u_+|^{p(x)} \eta^{p_B^+} dx \leq 2 \frac{p_B^+}{p_B^-} \left(2 \frac{p_B^+}{p_B^-} (p_B^+ - 1) + 1 \right)^{p_B^+ - 1} \int_B (u_+)^{p(x)} |\nabla \eta|^{p(x)} dx.$$

For every $\gamma > 0$ we have

$$\begin{aligned} \int_{\Omega} |u_+ + R|^{\gamma} |\nabla u_+|^{p(x)} \eta^{p_B^+} dx &= \gamma \int_0^{\infty} (t + R)^{\gamma-1} \int_{\{x \in \Omega: u(x) > t\}} |\nabla u_+|^{p(x)} \eta^{p_B^+} dx dt \\ &= \gamma \int_0^{\infty} (t + R)^{\gamma-1} \int_{\{x \in \Omega: u(x) > t\}} |\nabla(u - t)_+|^{p(x)} \eta^{p_B^+} dx dt. \end{aligned}$$

Since u is a solution of the obstacle problem $\mathcal{K}_{\psi, u}$, $u - t$ is a solution of the obstacle problem $\mathcal{K}_{\psi - t, u - t}$. Hence, by inequality (5), we have

$$\begin{aligned} \int_{\Omega} |u_+ + R|^{\gamma} |\nabla u_+|^{p(x)} \eta^{p_B^+} dx &\leq C \gamma \int_0^{\infty} (t + R)^{\gamma-1} \int_{\{x \in \Omega: u(x) > t\}} |\nabla(u - t)_+|^{p(x)} |\nabla \eta|^{p(x)} dx \\ &\leq C \gamma \int_0^{\infty} (t + R)^{\gamma-1} \int_{\{x \in \Omega: u(x) > t\}} |u_+ + R|^{p(x)} |\nabla \eta|^{p(x)} dx \\ &= C \int_{\Omega} |u_+ + R|^{p(x) + \gamma} |\nabla \eta|^{p(x)} dx. \end{aligned}$$

The inequality $\|f g^{p_B^-}\|_{L^1} \leq \|f\|_{L^1} + \|f g^{p_B^-}\|_{L^1}$, [2, Proposition 1.1], with $g = |\nabla u_+|$ and $f = (u_+ + R)^{\gamma} \eta^{p_B^+}$ yields the result. \square

We also need a Caccioppoli type estimate for the negative parts of supersolutions.

Lemma 7 Assume u is a supersolution in Ω . Let $B \Subset \Omega$ and $\eta \in C_0^\infty(B)$ with $0 \leq \eta \leq 1$. Then we have

$$\int_B |\nabla u_-|^{p(x)} \eta^{p_B^+} dx \leq C \int_B |u_-|^{p(x)} |\nabla \eta|^{p(x)} dx.$$

Here the constant C depends only on p_B^- and p_B^+ .

Proof: We will test the supersolution u by the non-negative function $-u_- \eta^{p_B^+}$. We find that

$$\begin{aligned} 0 &\leq \int_\Omega p(x) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla (-u_- \eta^{p_B^+}) dx \\ &= - \int_\Omega p(x) |\nabla u|^{p(x)-2} \nabla u \cdot (\eta^{p_B^+} \nabla u_- + u_- p_B^+ \eta^{p_B^+-1} \nabla \eta) dx \\ &= - \int_\Omega p(x) |\nabla u_-|^{p(x)} \eta^{p_B^+} dx - \int_\Omega p(x) |\nabla u|^{p(x)-2} u_- p_B^+ \eta^{p_B^+-1} \nabla u \cdot \nabla \eta dx. \end{aligned}$$

This implies

$$p_B^- \int_\Omega |\nabla u_-|^{p(x)} \eta^{p_B^+} dx \leq p_B^+ \int_\Omega p(x) |\nabla u_-|^{p(x)-1} |u_-| \eta^{p_B^+-1} |\nabla \eta| dx.$$

This inequality is similar to inequality (3), hence, the rest of the proof mimics the proof of Lemma 6. Details are left to the reader. \square

We define $v_r(x) = u_+(x) + r$ for $0 < r \leq R$ and

$$\Phi(f, q, E) = \left(\int_E f^q dx \right)^{1/q}.$$

Lemma 8 Assume that u is a solution of the obstacle problem $\mathcal{K}_{\psi, u}(\Omega)$ with a non-positive obstacle ψ . Let $0 < \rho < r \leq R$. Then the inequality

$$\begin{aligned} \Phi\left(v_r, \frac{n}{n-1} \beta, B_\rho\right) &\leq C^{1/\beta} \left(\frac{r}{\rho}\right)^{(n-1)/\beta} (1 + \beta)^{p_{B_R}^+/\beta} \left(2 + \|v_R\|_{L^{q's}(B_R)}^{q'(p_{B_R}^+ - p_{B_R}^-)}\right)^{1/(q'\beta)} \\ &\quad \cdot \left(\frac{r}{r-\rho}\right)^{p_{B_R}^+/\beta} \Phi(v_r, \beta q, B_r) \end{aligned}$$

holds for any $\beta \geq p_{B_r}^-$, $1 < q < np_{B_R}^-/(n-1)$ and $s > p_{B_R}^+ - p_{B_R}^-$. The constant C depends only on n , s , $p_{B_R}^-$, $p_{B_R}^+$ and log-Hölder constant of p .

Proof: Lemma 6 with $\gamma = \beta - p_{B_r}^-$ implies that

$$(6) \quad \int_{B_r} v_r^{\beta - p_{B_r}^-} |\nabla u_+|^{p_{B_r}^-} \eta^{p_{B_r}^+} dx \leq C \int_{B_r} (v_r^{\beta - p_{B_r}^-} \eta^{p_{B_r}^+} + v_r^{\beta - p_{B_r}^- + p(x)} |\nabla \eta|^{p(x)}) dx.$$

We choose $\eta \in C_0^\infty(B_r)$ so that $0 \leq \eta \leq 1$, $\eta = 1$ in B_ρ and $|\nabla\eta| \leq C/(r-\rho)$. By the log-Hölder continuity of p we obtain

$$(7) \quad |\nabla\eta|^{p(x)} \leq Cr^{-p(x)} \left(\frac{r}{r-\rho} \right)^{p_{B_r}^+} \leq Cr^{-p_{B_r}^-} \left(\frac{r}{r-\rho} \right)^{p_{B_r}^+}.$$

Using inequality (6) with this choice of η for the second inequality, we obtain

$$\begin{aligned} & \int_{B_r} |\nabla(v_r^{\beta/p_{B_r}^-} \eta^{p_{B_r}^+/p_{B_r}^-})|^{p_{B_r}^-} dx \\ & \leq C \int_{B_r} |\beta|^{p_{B_r}^-} v_r^{\beta-p_{B_r}^-} |\nabla u_+|^{p_{B_r}^-} \eta^{p_{B_r}^+} dx + C \int_{B_r} v_r^\beta \eta^{p_{B_r}^+-p_{B_r}^-} |\nabla\eta|^{p_{B_r}^-} dx \\ & \leq C |\beta|^{p_{B_r}^-} \int_{B_r} (v_r^{\beta-p_{B_r}^-} \eta^{p_{B_r}^+} + v_r^{\beta-p_{B_r}^-+p(x)} |\nabla\eta|^{p(x)}) dx + C \int_{B_r} v_r^\beta \eta^{p_{B_r}^+-p_{B_r}^-} |\nabla\eta|^{p_{B_r}^-} dx \\ & \leq C(1+|\beta|)^{p_{B_r}^-} \left(\int_{B_r} v_r^{\beta-p_{B_r}^-} dx + \int_{B_r} v_r^{\beta-p_{B_r}^-+p(x)} |\nabla\eta|^{p(x)} dx + \int_{B_r} v_r^\beta |\nabla\eta|^{p_{B_r}^-} dx \right). \end{aligned}$$

Now our goal is to estimate each integral in the brackets by $\left(\int_{B_r} v_r^{q\beta} dx \right)^{1/q}$. Since $v_r^{-p_{B_r}^-} \leq r^{-p_{B_r}^-}$, we have for the first integral by Hölder's inequality

$$\int_{B_r} v_r^{\beta-p_{B_r}^-} dx \leq r^{-p_{B_r}^-} \left(\int_{B_r} v_r^{q\beta} dx \right)^{1/q}.$$

For the second integral we estimate by (7), Hölder's inequality and [13, Lemma 3.7] that

$$\begin{aligned} & \int_{B_r} v_r^{\beta-p_{B_r}^-+p(x)} |\nabla\eta|^{p(x)} dx \\ & \leq Cr^{-p_{B_r}^-} \left(\frac{r}{r-\rho} \right)^{p_{B_r}^+} \int_{B_r} v_r^{\beta-p_{B_r}^-+p(x)} dx \\ & \leq Cr^{-p_{B_r}^-} \left(\frac{r}{r-\rho} \right)^{p_{B_r}^+} \left(\int_{B_r} v_r^{q'(p(x)-p_{B_r}^-)} dx \right)^{1/q'} \left(\int_{B_r} v_r^{q\beta} dx \right)^{1/q} \\ & \leq Cr^{-p_{B_r}^-} \left(\frac{r}{r-\rho} \right)^{p_{B_r}^+} \left(\int_{B_r} (1+v_r)^{q'(p_{B_r}^+-p_{B_r}^-)} dx \right)^{1/q'} \left(\int_{B_r} v_r^{q\beta} dx \right)^{1/q} \\ & \leq Cr^{-p_{B_r}^-} \left(\frac{r}{r-\rho} \right)^{p_{B_R}^+} \left(1 + \|v_r\|_{L^{q'(B_r)}^{q'(p_{B_r}^+-p_{B_r}^-)}} \right)^{1/q'} \left(\int_{B_r} v_r^{q\beta} dx \right)^{1/q} \\ & \leq Cr^{-p_{B_r}^-} \left(\frac{r}{r-\rho} \right)^{p_{B_R}^+} \left(2 + \|v_R\|_{L^{q'(B_R)}^{q'(p_{B_R}^+-p_{B_R}^-)}} \right)^{1/q'} \left(\int_{B_r} v_r^{q\beta} dx \right)^{1/q}. \end{aligned}$$

Finally, for the third integral we have by Hölder's inequality

$$\int_{B_r} v_r^\beta |\nabla \eta|^{p_{B_r}^-} dx \leq Cr^{-p_{B_r}^-} \left(\frac{r}{r-\rho} \right)^{p_{B_r}^-} \left(\int_{B_r} v_r^{q\beta} dx \right)^{1/q}.$$

Combining these estimates gives us that

$$\begin{aligned} & \int_{B_r} |\nabla(v_r^{\beta/p_{B_r}^-} \eta^{p_{B_r}^+/p_{B_r}^-})|^{p_{B_r}^-} dx \\ & \leq C(1+\beta)^{p_{B_r}^+} \left(2 + \|v_r\|_{L^{q's}(B_R)}^{q'(p_{B_r}^+ - p_{B_r}^-)} \right)^{1/q'} r^{-p_{B_r}^-} \left(\frac{r}{r-\rho} \right)^{p_{B_r}^+} \left(\int_{B_r} v_r^{q\beta} dx \right)^{1/q}. \end{aligned}$$

Using the Sobolev inequality ($1 < a < \infty$)

$$\left(\int_{B_r} |f|^{an/(n-1)} dx \right)^{(n-1)/an} \leq Cr \left(\int_{B_r} |\nabla f|^a dx \right)^{1/a},$$

with $f = v_r \eta$ and $a = p_{B_r}^-$ gives

$$\begin{aligned} & \left(\int_{B_\rho} v_r^{\beta n/(n-1)} dx \right)^{(n-1)/n} \leq C \left(\frac{r}{\rho} \right)^{n-1} \left(\int_{B_r} (v_r^{\beta/p_{B_r}^-} \eta^{p_{B_r}^+/p_{B_r}^-})^{np_{B_r}^-/(n-1)} dx \right)^{(n-1)/n} \\ & \leq C \left(\frac{r}{\rho} \right)^{n-1} r^{p_{B_r}^-} \int_{B_r} |\nabla(v_r^{\beta/p_{B_r}^-} \eta^{p_{B_r}^+/p_{B_r}^-})|^{p_{B_r}^-} dx \\ & \leq C \left(\frac{r}{\rho} \right)^{n-1} (1+\beta)^{p_{B_r}^+} \left(2 + \|v_r\|_{L^{q's}(B_R)}^{q'(p_{B_r}^+ - p_{B_r}^-)} \right)^{1/q'} \left(\frac{r}{r-\rho} \right)^{p_{B_r}^+} \left(\int_{B_r} v_r^{q\beta} dx \right)^{1/q}. \end{aligned}$$

This is the desired result. \square

Theorem 9 Assume u is a solution of the obstacle problem $\mathcal{K}_{\psi,u}(\Omega)$ with a non-positive obstacle ψ , $0 < \rho < r < \text{dist}(x, \partial\Omega)$, $1 < q < n/(n-1)$ and $s > p_{B_r}^+ - p_{B_r}^-$. Then for any $t > 0$ we have

$$\text{ess sup}_{x \in B(x,\rho)} u_+(x) \leq C \left(\frac{r}{r-\rho} \right)^C \left(\int_{B(x,r)} |u_+(x)|^t dx \right)^{1/t} + Cr,$$

where the constant C depends only on n, t, s, p^-, p^+ and log-Hölder constant of p and the $L^{q's}(B_r)$ norm of u .

Since the exponent $p(\cdot)$ is uniformly continuous, we can choose for example $q's = p_{\Omega}^-$ by using balls that are small enough. Thus the constant in the above estimate is finite in a scale that depends only on $p(\cdot)$, not on u . We will also use this observation in the proof of Theorem 16.

Proof: For $j = 0, 1, 2, \dots$ we write $r_j = \rho + 2^{-j}(r - \rho)$ and $\xi_j = \left(\frac{n}{q(n-1)}\right)^j q p_{B_r}^-$. Lemma 8 yields

$$\Phi(v_r, \xi_{j+1}, B_{r_{j+1}}) \leq C^{1/\xi_j} (1 + \xi_j)^{p_{B_r}^+/\xi_j} \left(\frac{r_j}{r_j - r_{j+1}}\right)^{p_{B_r}^+/\xi_j} \Phi(v_r, \xi_j, B_{r_j}).$$

By iterating this inequality, we have

$$\begin{aligned} \operatorname{ess\,sup}_{x \in B_\rho} v_r(x) &\leq \prod_{j=0}^{\infty} \left[C^{1/\xi_j} (1 + \xi_j)^{p_{B_r}^+/\xi_j} \left(2^j \frac{r}{r - \rho}\right)^{p_{B_r}^+/\xi_j} \right] \Phi(v_r, q p_{B_r}^-, B_r) \\ &\leq C^{\sum_{j=0}^{\infty} \frac{1}{\xi_j}} 2^{\sum_{j=0}^{\infty} j \frac{p_{B_r}^+}{\xi_j}} \left(\frac{r}{r - \rho}\right)^{\sum_{j=0}^{\infty} \frac{p_{B_r}^+}{\xi_j}} \Phi(v_r, q p_{B_r}^-, B_r) \prod_{j=0}^{\infty} (1 + \xi_j)^{p_{B_r}^+/\xi_j}. \end{aligned}$$

We note that the exponents of C , 2 and $r/(r - \rho)$ are all convergent series.

We estimate the remaining product by using the fact that $\xi_j > 1$ when $j > j_0$ and $\xi_j \leq 1$ when $j \leq j_0$ for some j_0 , so that

$$\begin{aligned} \prod_{j=0}^{\infty} (1 + \xi_j)^{p_{B_r}^+/\xi_j} &\leq 2^{\sum_{j=0}^{j_0} \frac{p_{B_r}^+}{\xi_j}} (2\xi_j)^{\sum_{j=j_0+1}^{\infty} \frac{p_{B_r}^+}{\xi_j}} \\ &\leq 2^{\sum_{j=0}^{j_0} \frac{p_{B_r}^+}{\xi_j}} (2q p_{B_r}^-)^{\sum_{j=j_0+1}^{\infty} \frac{p_{B_r}^+}{\xi_j}} \left(\frac{n}{q(n-1)}\right)^{\frac{p_{B_r}^+}{q p_{B_r}^-} \sum_{j=j_0+1}^{\infty} j \left(\frac{q(n-1)}{n}\right)^j} \\ &\leq (2q p_{B_r}^-)^{\sum_{j=0}^{\infty} \frac{p_{B_r}^+}{\xi_j}} \left(\frac{n}{q(n-1)}\right)^{\frac{p_{B_r}^+}{q p_{B_r}^-} \sum_{j=0}^{\infty} j \left(\frac{(n-1)q}{n}\right)^j}. \end{aligned}$$

Since the series in the above estimates are convergent by the root test, we obtain for some $\xi > 0$

$$(8) \quad \operatorname{ess\,sup}_{x \in B_\rho} v_r(x) \leq C \left(\frac{r}{r - \rho}\right)^\xi \Phi(v_r, q p_{B_r}^-, B_r).$$

By the Sobolev imbedding, the right hand side of (8) is finite since $1 < q < n/(n - 1)$.

Using (8) and [15, Lemma 3.38] with exponents ∞ and $q p_{B_r}^-$, we find

$$\operatorname{ess\,sup}_{x \in B_\rho} v_r(x) \leq C \left(\frac{r}{r - \rho}\right)^C \Phi(v_r, t, B_r)$$

for $0 < t < q p_{B_r}^-$. Here C depends on n , t , q and $p_{B_r}^-$ as well as C and ξ in (8). For $t > q p_{B_r}^-$ the inequality follows by Hölder's inequality. This completes the proof. \square

Every solution of the obstacle problem is a supersolution. Hence we obtain by [13, Theorem 3.15] that there exists a constant $q_0 > 0$ so that

$$\left(\int_{B_{2R}} u^{q_0} dx \right)^{1/q_0} \leq C(\operatorname{ess\,inf}_{B_R} u(x) + R).$$

Here the constant C depends on n, p, q and the $L^{q's}(B_{4R})$ -norm of u ($s > p_B^+ - p_B^-$).

Next we study the continuity of solutions of the obstacle problem, adapting the proof from the fixed exponent case, [15, Theorem 3.67].

Theorem 10 *Let $\psi: \Omega \rightarrow [-\infty, \infty)$ be continuous. Then the solution u of the obstacle problem $\mathcal{K}_{\psi,w}(\Omega)$ is continuous. Moreover, u is a solution in the open set $\{x \in \Omega : u(x) > \psi(x)\}$.*

Proof: First, recall that the solution of the obstacle problem is a supersolution in Ω . Hence, there is a lower semicontinuous function u^* defined in Ω by

$$u^*(x) = \operatorname{ess\,lim\,inf}_{y \rightarrow x} u(y)$$

such that $u^*(x) = u(x)$ for almost every $x \in \Omega$, see [13, Theorem 4.1]. For simplicity we denote the lower semicontinuous representative again by u . We need to prove that

$$(9) \quad \operatorname{ess\,lim\,sup}_{y \rightarrow x} u(y) \leq u(x)$$

for every $x \in \Omega$.

Since $u \in \mathcal{K}_{\psi,w}(\Omega)$ we have that

$$u(x) = \operatorname{ess\,lim\,inf}_{y \rightarrow x} u(y) \geq \operatorname{ess\,lim\,inf}_{y \rightarrow x} \psi(y) = \psi(x)$$

for every $x \in \Omega$, as the obstacle ψ is continuous.

Theorem 9 yields that $\operatorname{ess\,sup}_{B_R} u < \infty$ for every B_R with $2R < \operatorname{dist}(x, \partial\Omega)$, and we observe that $\sup_{B_R} \psi < \infty$. Fix $\varepsilon > 0$ and choose $B_R = B(x, R)$ so that $2R < \operatorname{dist}(x, \partial\Omega)$ and

$$\sup_{B_R} \psi \leq u(x) + \varepsilon \quad \text{and} \quad \inf_{B_R} u > u(x) - \varepsilon.$$

Theorem 9 implies that

$$\begin{aligned}
\operatorname{ess\,sup}_{B_r}(u - (u(x) + \varepsilon)) &\leq \operatorname{ess\,sup}_{B_r}(u - (u(x) + \varepsilon))_+ \\
&\leq C \int_{B_{2r}} (u - (u(x) + \varepsilon))_+ dy + Cr \\
&\leq C \int_{B_{2r}} (u - \min\{u, u(x) + \varepsilon\}) dy + Cr
\end{aligned}$$

for every $r < R < \frac{1}{2} \operatorname{dist}(x, \partial\Omega)$. Moreover, we have that

$$\min\{u(y), u(x) + \varepsilon\} \geq \min\{\inf_{B_r} u, u(x) + \varepsilon\} \geq u(x) - \varepsilon$$

for every $y \in B_{2r}$. Hence

$$\int_{B_{2r}} (u - \min\{u, u(x) + \varepsilon\}) dy \leq \int_{B_{2r}} u(y) dy - u(x) + \varepsilon.$$

Since u is (locally) bounded above, we obtain as in the proof of [13, Theorem 4.1] that

$$\lim_{r \rightarrow 0} \int_{B(x, 2r)} u(y) dy = u(x)$$

for every $x \in \Omega$. Altogether, the estimates above imply that

$$\begin{aligned}
\operatorname{ess\,lim\,sup}_{y \rightarrow x} u(y) &= \lim_{r \rightarrow 0} \operatorname{ess\,sup}_{B_r} u(y) \\
&\leq u(x) + \varepsilon + C \lim_{r \rightarrow 0} \int_{B(x, 2r)} u(y) dy + C \lim_{r \rightarrow 0} r - Cu(x) + C\varepsilon
\end{aligned}$$

from which (9) follows by letting $\varepsilon \rightarrow 0^+$.

The second assertion follows from Theorem 5, see [15, Theorem 3.67] for the proof.

□

5 Convergence theorems

This section is devoted to convergence properties of both supersolutions and solutions of the obstacle problem. Note that supersolutions are locally bounded from below. This is true because subsolutions are locally bounded from above, which can be seen from the proof of [2, Theorem 1].

Theorem 11 *Assume (u_i) is an increasing sequence of supersolutions in Ω and that $u = \lim_{i \rightarrow \infty} u_i$ is locally bounded. Then u is a supersolution in Ω .*

Proof: Fix open sets $D \Subset G \Subset \Omega$. Since u is locally bounded, we may assume that $u < 0$ in G . Since the sequence is increasing and u_1 is locally bounded from below in Ω , we find that $-\infty < \inf_G u_1 \leq u_1 < 0$ in G . It follows from the Caccioppoli estimate, Lemma 7, that the sequence ∇u_i is uniformly bounded in $L^{p(\cdot)}(G)$. By reflexivity there is a subsequence (u_i) such that $u_i \rightharpoonup u \in W^{1,p(\cdot)}(G)$ weakly in $W^{1,p(\cdot)}(G)$.

Choose $\eta \in C_0^\infty(G)$ so that $0 \leq \eta \leq 1$ and $\eta = 1$ in D . Using $\phi_i = \eta(u - u_i)$ as a nonnegative test function for the supersolution u_i in G , we obtain

$$\begin{aligned} 0 &\leq \int_G p(x) |\nabla u_i|^{p(x)-2} \nabla u_i \cdot \nabla \phi_i \, dx \\ &= \int_G p(x) |\nabla u_i|^{p(x)-2} (u - u_i) \nabla u_i \cdot \nabla \eta \, dx \\ &\quad + \int_G p(x) |\nabla u_i|^{p(x)-2} \eta \nabla u_i \cdot (\nabla u - \nabla u_i) \, dx. \end{aligned}$$

This and the variable exponent Hölder inequality imply that

$$\begin{aligned} & - \int_G p(x) |\nabla u_i|^{p(x)-2} \eta \nabla u_i \cdot (\nabla u - \nabla u_i) \, dx \\ & \leq p^+ \int_G |u - u_i| |\nabla u_i|^{p(x)-1} |\nabla \eta| \, dx \\ & \leq Cp^+ \sup_G |\nabla \eta| \|u - u_i\|_{L^{p(\cdot)}(G)} \|\nabla u_i\|_{L^{p'(\cdot)}(G)}. \end{aligned}$$

Since $(|\nabla u_i|)$ is uniformly bounded in $L^{p(\cdot)}(G)$, we find that the last term on the right hand side is bounded. Using $(2|u| + 1)^{p^+}$ as a majorant, we find, by the dominated convergence theorem, that $\|u - u_i\|_{L^{p(\cdot)}(G)} \rightarrow 0$ as $i \rightarrow \infty$. Moreover, since $p(x) |\nabla u|^{p(x)-2} \eta \nabla u \in L^{p'(\cdot)}(G)$, weak convergence implies that

$$\int_G p(x) |\nabla u|^{p(x)-2} \eta \nabla u \cdot (\nabla u - \nabla u_i) \, dx \rightarrow 0.$$

Adding these estimates gives that

$$\int_G p(x) \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla u_i|^{p(x)-2} \nabla u_i \right) \cdot (\nabla u - \nabla u_i) \eta \, dx.$$

But the integrand is non-negative, so we conclude, for a suitable subsequence, still denoted (u_i) , that

$$\left(|\nabla u|^{p(x)-2} \nabla u - |\nabla u_i|^{p(x)-2} \nabla u_i \right) \cdot (\nabla u - \nabla u_i) \eta \rightarrow 0$$

almost everywhere in G . Since $\eta \equiv 1$ in D , we conclude that $\nabla u_i \rightarrow \nabla u$ almost everywhere in this set.

Since the sequence (∇u_i) is uniformly bounded in $L^{p(\cdot)}(G)$, the sequence $|\nabla u_i|^{p(x)-2} \nabla u_i$ is uniformly bounded in $L^{p'(\cdot)}(G)$. Therefore we find a subsequence, denoted still

by u_i , which converges weakly in $L^{p(\cdot)}(G)$. But the weak limit must coincide with the point-wise limit, hence

$$p(x)|\nabla u_i|^{p(x)-2}\nabla u_i \rightarrow p(x)|\nabla u|^{p(x)-2}\nabla u$$

weakly in $L^{p(\cdot)}(D)$. Now let ϕ be a smooth test function with support in D . Then

$$\begin{aligned} 0 &\leq \int_{\Omega} p(x)|\nabla u_i(x)|^{p(x)-2}\nabla u_i(x) \cdot \nabla \phi(x) dx \\ &= \int_D p(x)|\nabla u_i(x)|^{p(x)-2}\nabla u_i(x) \cdot \nabla \phi(x) dx \\ &\rightarrow \int_D p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot \nabla \phi(x) dx \\ &= \int_{\Omega} p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot \nabla \phi(x) dx. \end{aligned}$$

Hence u is a supersolution, and the proof is completed. \square

Theorem 12 *Assume (u_i) is a decreasing and locally bounded sequence of supersolutions in Ω . Then $u = \lim_{i \rightarrow \infty} u_i$ is a supersolution in Ω .*

Proof: As in the proof of Theorem 11 it follows that $u \in W_{\text{loc}}^{1,p(\cdot)}(\Omega)$. Pick an open set $D \Subset \Omega$, and let v be the solution of the obstacle problem $\mathcal{K}_{u,u}(D)$. Then $v \geq u$ and Theorem 1 implies $v \leq u_i$ for each i . Thus $v = u$ almost everywhere in D and therefore u is a supersolution in Ω . \square

Corollary 13 *Assume (u_i) is a sequence of continuous solutions in Ω such that $u_i \rightarrow u$ locally uniformly in Ω . Then u is a solution in Ω .*

Proof: Let $D \Subset G \Subset \Omega$. Now $u_i \rightarrow u$ uniformly in G and thus u is continuous in G . For every $j = 3, 4, \dots$ we choose i_j so that

$$\sup_D |u_{i_j} - u| \leq j^{-3}.$$

We write $v_j = u_{i_j} - 1/j$. Now (v_j) is an increasing sequence converging pointwise in D . Since u is continuous in G , it is bounded in D and thus by Theorem 11 it is a supersolution. By Theorem 12 we find that $-u$ is a supersolution and hence u is a solution in Ω . \square

The following theorem can be proved in a similar fashion as when p is constant, [15, Theorem 3.79].

Theorem 14 *Assume (ψ_i) is a decreasing sequence in $W^{1,p(\cdot)}(\Omega)$ such that $\psi_i \rightarrow \psi$ in $W^{1,p(\cdot)}(\Omega)$. Let $u_i \in W^{1,p(\cdot)}(\Omega)$ be a solution of the obstacle problem $\mathcal{K}_{\psi_i,w}(\Omega)$.*

Then the sequence (u_i) is decreasing and the limit function $u = \lim_{i \rightarrow \infty} u_i$ is a solution of the obstacle problem $\mathcal{K}_{\psi,w}(\Omega)$.

Theorem 15 Assume (ψ_i) and (u_i) are increasing sequences of functions in Ω such that $\psi_i \rightarrow \psi$ in $W^{1,p(\cdot)}(\Omega)$ and $u_i \in W^{1,p(\cdot)}(\Omega)$ is a solution of the obstacle problem $\mathcal{K}_{\psi_i,u_i}(\Omega)$. Then the limit function $u = \lim_{i \rightarrow \infty} u_i$ is the solution of the obstacle problem $\mathcal{K}_{\psi,u}(\Omega)$ provided that $u \in W^{1,p(\cdot)}(\Omega)$.

Proof: Minor modifications to the proof of Theorem 11 imply that u is a supersolution. Let v be the solution of the obstacle problem $\mathcal{K}_{\psi,u}(\Omega)$. Since u is a supersolution it follows by Theorem 1 that $u \geq v$ in Ω . Theorem 1 also implies that $v \geq u_i$ in Ω , and hence $v \geq u$ in Ω . Therefore $u = v$, and the proof is complete. \square

Theorem 16 (Harnack's Principle) Let (h_i) be an increasing sequence of solutions and set $h = \lim_{i \rightarrow \infty} h_i$. If $h \in L^t_{loc}(\Omega)$ for some $t > 0$, then h is a solution.

Proof: We see by a partition of unity argument that being a solution is a local property in the sense that h is a solution if for each point $x \in \Omega$ there is a ball $B = B(x, r)$ such that h is a solution in B . Choosing a small enough radius we can assume that the Harnack inequality holds for the functions h_j in B with a constant independent of j since $h \in L^t_{loc}(\Omega)$ [13, Theorem 3.17]. This implies that the functions h_j are equicontinuous, and using the Arzela–Ascoli theorem we can assume that they converge locally uniformly. Now Corollary 13 implies that h is a solution. \square

6 Supersolutions and the comparison principle

In this section, we prove that functions for which the comparison principle holds are monotone limits of supersolutions.

We say that a function $u : \Omega \rightarrow (-\infty, \infty]$ is *superharmonic* in Ω if

- (1) u is lower semicontinuous,
- (2) u is finite almost everywhere and
- (3) The comparison principle holds: Let $D \Subset \Omega$ be an open set. If h is a solution in D , continuous in \overline{D} and $u \geq h$ on ∂D , then $u \geq h$ in D .

Further, we say that u has the *truncation property* if

- (1) u is finite almost everywhere, and
- (2) $\min\{u, \lambda\}$ is a supersolution for all $\lambda \in \mathbb{R}$.

Theorem 17 *Let u be a supersolution in Ω with the property*

$$(10) \quad u(x) = \operatorname{ess\,lim\,inf}_{y \rightarrow x} u(y)$$

for all $x \in \Omega$. Then u is superharmonic.

Proof: Note first that u is locally bounded below as a supersolution (see the beginning of Section 5), lower semicontinuous by (10) and finite almost everywhere since $u \in W^{1,p(\cdot)}(\Omega)$.

To establish the comparison principle, let $D \Subset \Omega$ be open, assume that h is a solution in D and continuous in \overline{D} such that $u \geq h$ on ∂D and take $\varepsilon > 0$. For any open set $G \Subset D$, such that $u + \varepsilon > h$ in $D \setminus G$, the function $\min\{u + \varepsilon - h, 0\}$ is compactly supported in G since the set $\{u - h \leq -\varepsilon\}$ is closed by the lower semicontinuity of u . By Lemma 4, $u + \varepsilon \geq h$ almost everywhere in G , and then also in D . By assumption (10), $u + \varepsilon \geq h$ everywhere in D , and letting $\varepsilon \rightarrow 0$ completes the proof. \square

Corollary 18 *For a supersolution u there is a superharmonic function v such that $u = v$ almost everywhere.*

Proof: Combine the above result with [13, Theorem 4.1]. \square

Corollary 19 *For a function u which has the truncation property there is a superharmonic function v such that $u = v$ almost everywhere.*

Proof: By [13, Theorem 4.1] there is a function v with property (10) such that $u = v$ almost everywhere. By Theorem 17 the comparison principle holds for $\min\{v, \lambda\}$ since the truncations of v are supersolutions with property (10). This implies that the comparison principle holds for v . \square

We can easily modify a well-known example to show that the class of superharmonic functions is not the same as the class of supersolutions. For this example, let p be a radial function in the open unit ball $B = B(0, 1)$ with values strictly between 1 and n . Then

$$v(x) = \int_{|x|}^1 (p(r)r^{n-1})^{-1/(p(r)-1)} dr$$

defines a superharmonic function in B , which is solution in $B \setminus \{0\}$, but not a supersolution in B .

Let us verify these claims. We first note that the gradient of v is locally bounded in $B \setminus \{0\}$, so that v belongs to $W_{loc}^{1,p(\cdot)}(B \setminus \{0\})$. We easily calculate that

$$\nabla v(x) = (p(x)|x|^{n-1})^{-1/(p(x)-1)} \frac{x}{|x|}$$

for $x \neq 0$. Thus we find that

$$\operatorname{div} (|\nabla v(x)|^{p(x)-2} \nabla v(x)) = \operatorname{div} \left(\frac{x}{|x|^n} \right) = 0,$$

so that v is even a strong solution in $B \setminus \{0\}$.

Since $p < n$ in a neighborhood of the origin, we see that

$$|\nabla v(x)|^{p(x)} = (p(x)|x|^{n-1})^{-p(x)/(p(x)-1)} \geq C|x|^{-n}.$$

Thus it follows that $v \notin W_{loc}^{1,p(\cdot)}(B)$, and so v is not a supersolution in B . From what has been said before, it is clear that $\max\{v(x), \lambda\}$ is a supersolution for all $\lambda > 0$. Thus it follows from Corollary 19 that v is superharmonic in B .

Next we show that superharmonic functions can be approximated (locally) from below by continuous supersolutions. Moreover, these supersolutions are superharmonic due to the previous results.

Theorem 20 *Let u be a superharmonic function in Ω and $D \Subset \Omega$ an open set. Then there is an increasing sequence of continuous supersolutions (u_i) in D such that $u = \lim_{i \rightarrow \infty} u_i$ everywhere in D . Moreover, the functions u_i are superharmonic in D .*

Proof: Let $D \Subset \Omega$ be open. Since u is lower semicontinuous in Ω , there is an increasing sequence of smooth functions (ψ_i) in Ω such that $u = \lim_{i \rightarrow \infty} \psi_i$ everywhere in \overline{D} .

Let $u_i \in \mathcal{K}_{\psi_i, \psi_i}(D)$. Each u_i is a supersolution and by Theorem 10 the functions u_i are continuous in D and the sets $A_i = \{x \in \Omega : u_i(x) > \psi_i(x)\}$ are open for every $i = 1, 2, \dots$

Since $\min\{u_i, u_{i+1}\} \in \mathcal{K}_{\psi_i, \psi_i}(D)$ and $\min\{u_i, u_{i+1}\}$ is a supersolution in D (Theorem 2), we obtain by Theorem 1 and the continuity of the functions u_i that $u_i \leq u_{i+1}$ everywhere in D . Hence the sequence (u_i) is increasing.

By Theorem 10 each u_i is a solution in A_i . Since ψ_i and u_i are continuous in $\overline{A_i}$ we have $\psi_i = u_i$ on ∂A_i . Since $\psi_i \leq u$, the comparison principle yields $u_i \leq u$ in A_i . Since $u_i = \psi_i \leq u$ in $D \setminus A_i$, it follows that $u_i \leq u$ in D . Thus

$$u = \lim_{i \rightarrow \infty} \psi_i \leq \lim_{i \rightarrow \infty} u_i \leq u$$

everywhere in Ω .

Every u_i is superharmonic by Theorem 17 and the proof is complete. \square

Corollary 21 *Locally bounded superharmonic functions are supersolutions.*

Proof: For a superharmonic function u there is an increasing sequence (u_i) of supersolutions such that $u = \lim u_i$ by Theorem 20. Thus u is a supersolution by Theorem 11. \square

Corollary 22 *Superharmonic functions have the truncation property.*

Proof: Let $\lambda \in \mathbb{R}$. The comparison principle holds for $\min\{u, \lambda\}$ if it holds for u . Thus $\min\{u, \lambda\}$ is a supersolution by the previous corollary. \square

7 Local integrability of superharmonic functions

In this section we study the local integrability of superharmonic functions by employing the truncation property. Using the weak Harnack inequality directly as in [20] is not an option to us since the constants in Harnack type estimates depend on the L^p norm of the solution. This problem is less severe in the method presented in [18], which we adapt to prove that a superharmonic function is integrable over a compact set K provided that it is integrable over the complement of K with respect to some open set (Theorem 26). Both methods can also be found in [15, Chapter 7].

Throughout this section we assume that u is a lower semicontinuous function with the truncation property and define pointwise

$$Du = \lim_{k \rightarrow \infty} \nabla \min\{u, k\}.$$

The proof of the following lemma is the same as that of [15, Lemma 7.9], and we omit it.

Lemma 23 *Assume that $D \subset \Omega$ is open, u is superharmonic in Ω and that v is superharmonic in D . If the function*

$$s = \begin{cases} \min\{u, v\} & \text{in } D, \\ u & \text{in } \Omega \setminus D \end{cases}$$

is lower semicontinuous, then it is superharmonic in Ω .

Let $f \in W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})$ and let h be the continuous solution with Sobolev boundary values f . We say that the set Ω is regular if

$$\lim_{x \rightarrow x_0} h(x) = f(x_0)$$

for all $x_0 \in \partial\Omega$ and all f . For our purposes it suffices to note that “simple” sets such as polyhedra, balls, etc. are regular and general open sets can be exhausted by regular ones. This follows from the results of [4]. In fact Alkhutov and Krasheninnikova studied slightly different equations than we; they omitted the factor ‘ $p(x)$ ’ in (2). It is easy to check that their results hold also in our case.

Lemma 24 (Poisson modification) *Suppose that $D \Subset \Omega$ is open and regular and that $u \in L^t_{loc}(D)$ for some $t > 0$. Then there is a superharmonic function $P(u, D)$ such that $P(u, D)$ is a solution in D and $P(u, D) \leq u$ in Ω .*

Proof: Choose an increasing sequence of smooth functions (φ_i) that converge pointwise to u in \bar{D} (possible by the semicontinuity of u). Let h_i be the unique continuous solution in D such that $h_i = \varphi_i$ on ∂D . The sequence (h_i) is increasing by Theorem 1 and $h_i \leq u$ in D by the comparison principle since $h_i = \varphi_i \leq u$ on ∂D . Thus the function $h = \lim_{i \rightarrow \infty} h_i$ is a solution in D by Theorem 16.

We define

$$P(u, D) = \begin{cases} h & \text{in } D, \\ u & \text{in } \Omega \setminus D. \end{cases}$$

By Lemma 23 it suffices to verify that $P(u, D)$ is lower semicontinuous to complete the proof. To see this, consider a point $y \in \partial D$ and observe that $P(u, D)(y) = u(y)$ and

$$\begin{aligned} \liminf_{x \rightarrow y} P(u, D)(x) &\geq \min\{\liminf_{x \rightarrow y} h(x), \liminf_{x \rightarrow y} u(x)\} \\ &\geq \min\{\liminf_{x \rightarrow y} h(x), u(y)\} \end{aligned}$$

by the lower semicontinuity of u . Using the regularity of D , we obtain $\liminf_{x \rightarrow y} h(x) \geq \liminf_{x \rightarrow y} h_i(x) = \varphi_i(y)$. This holds for all i , so that

$$\liminf_{x \rightarrow y} h(x) \geq \lim_{i \rightarrow \infty} \varphi_i(y) = u(y),$$

and thus $\liminf_{x \rightarrow y} P(u, D)(x) \geq P(u, D)(y)$. \square

Theorem 25 *Assume that $p^-_{\Omega} < n$. If u is nonnegative and $\min\{u, k\} \in W_0^{1,p(\cdot)}(\Omega)$ for $k = 1, 2, 3, \dots$, then*

- (1) $u \in L^s(\Omega)$ for $0 < s < \kappa(p^-_{\Omega} - 1)$ and
- (2) $Du \in L^{q(p^-_{\Omega}-1)}(\Omega)$ for $0 < q < \frac{\kappa p^-_{\Omega}}{\kappa(p^-_{\Omega} - 1) + 1}$,

where $\kappa = \frac{n}{n - p^-_{\Omega}}$.

Proof: By [15, Lemma 7.43] it suffices to prove the estimate

$$\int_{\Omega} |\nabla \min\{u, k\}|^{p_{\Omega}^-} dx \leq Mk$$

with a constant M independent of k .

Let $a_k = \int_{\{k-1 \leq u \leq k\}} p(x) |Du|^{p(x)} dx$. We have

$$\begin{aligned} \int_{\{k-1 \leq u \leq k\}} |\nabla \min\{u, k\}|^{p_{\Omega}^-} dx &\leq \int_{\{k-1 \leq u \leq k\}} |\nabla \min\{u, k\} + 1|^{p(x)} dx \\ &\leq C \int_{\{k-1 \leq u \leq k\}} (|\nabla \min\{u, k\}|^{p(x)} + 1) dx \leq C(a_k + |\Omega|). \end{aligned}$$

If we can prove that the sequence (a_k) is decreasing, then the integral on the left hand side in the above inequality would be bounded by a constant independent of k and we get the needed estimate with $M = C(a_1 + |\Omega|)$ by adding up the inequalities.

Let $v_k = (1 - |u - k|)_+$. The function v_k is an admissible test function and $v_k = 1 - k + u$ when $k - 1 \leq u \leq k$, $v_k = 1 + k - u$ when $k \leq u \leq k + 1$ and $v_k = 0$ otherwise. We have

$$\begin{aligned} 0 &\leq \int_{\Omega} p(x) |\nabla \min\{u, k + 1\}|^{p(x)-2} \nabla \min\{u, k + 1\} \cdot \nabla v_k dx \\ &= \int_{\{k-1 \leq u \leq k\}} p(x) |Du|^{p(x)} dx - \int_{\{k \leq u \leq k+1\}} p(x) |Du|^{p(x)} dx = a_k - a_{k+1} \end{aligned}$$

since $\min\{u, k + 1\}$ is a supersolution. Thus $a_{k+1} \leq a_k$ and the claim follows. \square

In the proof of the next theorem, we need to find a solution h of the generalized Dirichlet problem in a regular bounded open set D . This means that h needs to be a solution in D such that

$$(11) \quad \lim_{x \rightarrow x_0} h(x) = f(x_0)$$

for all $x_0 \in \partial D$ where f is a continuous function defined only on ∂D . This can be done by first extending f to the whole of \mathbb{R}^n by topology and then choosing a sequence (f_i) of smooth functions converging uniformly and monotonously to f in \overline{D} . Let h_i be the unique continuous solution in D with Sobolev boundary values f_i . The same reasoning as in the proof of Lemma 24 shows that $h = \lim_{i \rightarrow \infty} h_i$ is a solution in D . The limit property (11) then follows because the boundary regularity estimate [4, Theorem 1.2.] is stable under uniform convergence. For another way of establishing the existence of h , see [4, Theorem 4.1.].

Theorem 26 *Let $K \subset D \Subset \Omega$, where K is compact and D is open. Assume that u is a superharmonic function in Ω which is locally integrable on $D \setminus K$ to some power $t > 0$. Then u and Du are integrable over K to the powers given by Theorem 25.*

Proof: We can assume without loss of generality that D and K are regular since general sets can be exhausted by regular ones. We can also assume that $u \geq 1$ on \overline{D} since u is locally bounded below.

Choose a regular open set U such that $K \subset U \Subset D$. The Poisson modification $P(u, D \setminus K)$ is a solution in $D \setminus K$. Thus by solving the generalized Dirichlet problem in $D \setminus \overline{U}$ we find a function h which is a solution in $D \setminus \overline{U}$, $h = P(u, D \setminus K)$ on ∂U and $h = 0$ on ∂D . By the comparison principle $h \leq P(u, D \setminus K)$ on $D \setminus \overline{U}$ since $P(u, D \setminus K) = u \geq 1$ on ∂D . Now Lemma 23 implies that the function

$$v = \begin{cases} P(u, D \setminus K) & \text{in } U \\ h & \text{in } D \setminus \overline{U} \end{cases}$$

is superharmonic in D . By Corollary 22 the truncations of v are in $W_0^{1,p(\cdot)}(D)$. Since $v = u$ in K the claim follows from Theorem 25. \square

References

- [1] E. Acerbi and G. Mingione: Regularity results for stationary electro-rheological fluids, *Arch. Ration. Mech. Anal.* **164** (2002), no. 3, 213–259.
- [2] Yu. A. Alkhutov: The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with nonstandard growth condition, *Differ. Uravn.* **33** (1997), no. 12, 1651–1660, 1726. Translation in *Differential Equations* **33** (1997), no. 12, 1653–1663.
- [3] Yu. A. Alkhutov: On the Hölder continuity of $p(x)$ -harmonic functions, *Mat. Sb.* **196** (2005), no. 2, 3–28. Translation in *Sb. Math.* **196** (2005), no. 1–2, 147–171.
- [4] Yu. A. Alkhutov and O.V. Krasheninnikova: Continuity at boundary points of solutions of quasilinear elliptic equations with a non-standard growth condition, *Izv. Ross. Akad. Nauk. Ser. Mat.*, **68** (2004), no. 6, 3–60. Translation in *Izv. Math.* **68** (2004), no. 6, 1063–1117.
- [5] S. Antontsev and S. Shmarev: A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, *Nonlinear Anal.* **60** (2005), no. 3, 515–545.
- [6] L. Diening: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Math. Inequal. Appl.* **7** (2004), no. 2, 245–254.
- [7] L. Diening, P. Hästö and A. Nekvinda: Open problems in variable exponent Lebesgue and Sobolev spaces, *FSDONA04 Proceedings* (Drábek and Rákosník (eds.); Milovy, Czech Republic, 2004), 38–58.
- [8] X.-L. Fan and D. Zhao: A class of De Giorgi and Hölder continuity, *Nonlinear Anal.* **36** (1999), 295–318.

- [9] X.-L. Fan and Q.-H. Zhang: Existence of solutions for $p(x)$ -Laplacian Dirichlet problem, *Nonlinear Anal.* **52** (2003), 1843–1852.
- [10] P. Harjulehto: Variable exponent Sobolev spaces with zero boundary values, submitted. Manuscript available at <http://mathstat.helsinki.fi/analysis/varsobgroup/>.
- [11] P. Harjulehto, P. Hästö and M. Koskenoja: The Dirichlet energy integral on intervals in variable exponent Sobolev spaces, *Z. Anal. Anwend.* **22** (2003) no. 4, 911–923.
- [12] P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen: The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values, *Potential Anal.*, to appear.
- [13] P. Harjulehto, J. Kinnunen and T. Lukkari: Unbounded supersolutions for nonlinear equations with nonstandard growth, *Bound. Value Probl.*, to appear.
- [14] J. Heinonen and T. Kilpeläinen: A -superharmonic functions and supersolutions of degenerate elliptic equations, *Ark. Mat.* **26** (1988), no. 1, 87–105.
- [15] J. Heinonen, T. Kilpeläinen, and O. Martio: *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, Oxford, 1993.
- [16] P. Hästö: On the variable exponent Dirichlet energy integral, *Comm. Pure Appl. Anal.* **5** (2006), no. 3, 413–420.
- [17] P. Hästö: On the density of smooth functions in variable exponent Sobolev space, *Rev. Mat. Iberoamericana*, to appear.
- [18] T. Kilpeläinen and J. Malý: Degenerate elliptic equations with measure data and nonlinear potentials, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **19** (1992), no. 4, 591–613.
- [19] J. Kinnunen and O. Martio: Nonlinear potential theory on metric spaces, *Illinois J. Math.* **46** (2002), no. 3, 857–883.
- [20] P. Lindqvist: On the definition and properties of p -superharmonic functions, *J. Reine Angew. Math.* **365** (1986), 67–79.
- [21] O. Kováčik and J. Rákosník: On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41(116)** (1991), 592–618.
- [22] G. Mingione and D. Mucci: Integral functionals and the gap problem: sharp bounds for relaxation and energy concentration, *SIAM J. Math. Anal.* **36** (2005), no. 5, 1540–1579.
- [23] J. Musielak: *Orlicz spaces and modular spaces*, Springer-Verlag, Berlin, 1983.
- [24] S. Samko: Denseness of $C_0^\infty(\mathbb{R}^n)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbb{R}^n)$, pp. 333–342 in *Direct and Inverse Problems of Mathematical Physics* (Newark, DE, 1997), Int. Soc. Anal. Appl. Comput. **5**, Kluwer Acad. Publ., Dordrecht, 2000.
- [25] V. V. Zhikov: On some variational problems, *Russian J. Math. Physics* **5** (1997), 105–116.