# OPEN PROBLEMS IN VARIABLE EXPONENT LEBESGUE AND SOBOLEV SPACES

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Abstract. In this article we provide an overview of several open problems in variable exponent spaces. The problems are related to boundedness of the maximal operator, interpolation theory, density of smooth functions and Sobolev embeddings. We also extend a result on complex interpolation to the variable exponent setting and give an example of a continuous exponent $p$ for which $W^{1,p(\cdot)}$ does not embed into $L^{p^*(\cdot)}$.

## 1. Introduction

In this paper we give a detailed description of some problems in variable exponent Lebesgue and Sobolev spaces that have been the focus of intense research over the past few years. Our selection of topics is based on our personal interests, so we make no attempt at completeness in this respect. We have, however, strived to make the bibliography on variable exponent spaces as complete as possible, and the large number of publications listed certainly attests to the boost in interest that the field has experienced recently.

We start by sketching the development of the field from 1931 till about the turn of the millennium. In Section 2 we consider problems in variable exponent Lebesgue spaces, specifically the boundedness of the Hardy–Littlewood maximal operator, and interpolation theory. In Section 3 we consider two problems in variable exponent Sobolev spaces: the density of continuous functions and the Sobolev embedding. In Appendix A we prove some results on complex interpolation.

Variable exponent Lebesgue spaces appeared in the literature for the first time already in a 1931 article by W. Orlicz [105]. In this article the following question is considered: let $(p_i)$ (with $p_i > 1$) and $(x_i)$ be sequences of real numbers such that $\sum x_i^{p_i}$ converges. What are the necessary and sufficient conditions on $(y_i)$ for $\sum x_i y_i$ to converge? It turns out that the answer is that $\sum (\lambda y_i)^{p_i'}$ should converge for some $\lambda > 0$ and $p_i' = p_i/(p_i - 1)$. This is exactly Hölder's inequality in the space $\ell^{p_i}$. Orlicz also considered the variable exponent function space $L^{p(x)}$ on the real line, and proved the Hölder inequality in this setting.

However, after this one paper, Orlicz abandoned the study of variable exponent spaces, to concentrate on the theory of the function spaces that now bear his name (but see also [141]). In the theory of Orlicz spaces, one defines the space $L^{\varphi}$ to consist of those measurable

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functions $u\colon \Omega \to [0,\infty)$ for which

$$\int_\Omega \varphi(\lambda|u(x)|)\,dx < \infty$$

for some $\lambda > 0$ ($\varphi$ has to satisfy certain conditions, which we will not get into here). If we allow $\varphi$ to depend also on $x$, we end up with a more general class of spaces. Again, assuming that $\varphi$ satisfies certain conditions, such spaces are called modular. They were first systematically studied by H. Nakano [138, 139]. In the appendix [p. 284] of the first of these books, Nakano mentions explicitly variable exponent Lebesgue spaces as an example of the more general spaces he considers. The duality property mentioned above is again observed.

Following the work of Nakano, modular spaces were investigated by several people, most importantly by groups at Sapporo (Japan), Voronezh (U.S.S.R.), and Leiden (the Netherlands). Somewhat later, a more explicit version of these spaces, modular function spaces, were investigated by Polish mathematicians, like H. Hudzik [125]–[135] A. Kamińska [135]–[137] and J. Musielak [140]. For a comprehensive presentation of modular function spaces, see the monograph [140].

Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers, notably I. Sharapudinov. These investigations originated in a paper by I. Tsenov from 1961 [117]. The question raised by Tsenov and solved by Sharapudinov [114] is the minimization of

$$\int_a^b |u(x) - v(x)|^{p(x)} dx,$$

where $u$ is a fixed function and $v$ varies over a finite dimensional subspace of $L^{p(\cdot)}([a,b])$. In [114] Sharapudinov also introduced the Luxemburg norm for the Lebesgue space and showed that this space is reflexive if the exponent satisfies $1 < p^- \le p^+ < \infty$. In the mid-80's V. Zhikov [122] started a new line of investigation, that was to become intimately related to the study of variable exponent spaces, namely he considered variational integrals with non-standard growth conditions.

The next major step in the investigation of variable exponent spaces was the paper by O. Kováčik and J. Rákosník in the early 90's [93]. This paper established many of the basic properties of Lebesgue and Sobolev spaces in $\mathbb{R}^n$. During the following ten years there were many scattered efforts to understand these spaces. At the turn of the millennium several factors contributed to start a period of systematic intense study of variable exponent spaces.

- The "correct" condition (log-Hölder continuity) for regularly varying exponents was found, which allowed researchers to prove a multitude of results, starting with the boundedness of the maximal operator.
- A connection was made between variable exponent spaces and variational integrals with non-standard growth and coercivity conditions.
- It was found that these non-standard variational problems are related to modeling of so-called electrorheological fluids. Moreover, progress in physics over the past ten year have made the study of fluid mechanical properties of these fluids an important issue.
- Several new groups emerged and eventually found each other. These groups are connected to Faro, Freiburg, Helsinki, Hiroshima Lanzhaou, Parma, Prague, and Tbilisi.

Some of the new developments from the last five years are the content of the rest of this paper.

## 2. Lebesgue spaces

2.1. **Preliminaries.** For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B^n(x,r)$ the open ball with center $x$ and radius $r$. All cubes $Q \subset \mathbb{R}^n$ considered in this paper are assumed to have sides parallel to the axes. By $\Omega$ we always denote a non-empty open subset of $\mathbb{R}^n$. For an integrable function defined on a set $A$ of finite non-zero measure we denote

$$\langle u \rangle_A = \fint_A u(x)\,dx = \frac{1}{|A|}\int_A u(x)\,dx.$$

Let $p : \Omega \to [1,\infty)$ be a measurable bounded function, called a variable exponent on $\Omega$, and denote $p^+ = \operatorname{ess\,sup} p(x)$ and $p^- = \operatorname{ess\,inf} p(x)$. We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular $\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx$ is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0\colon \varrho_{p(\cdot)}(u/\lambda) \le 1\}.$$

One central property of these spaces (since $p$ is bounded) is that $\varrho_{p(\cdot)}(u_i) \to 0$ if and only if $\|u_i\|_{p(\cdot)} \to 0$, so that the norm and modular topologies coincide. This and many other basic results were proven in [93]. By $C^{0,1/|\log t|}$ we denote the space of log-*Hölder continuous* functions $p$, i.e. functions which satisfy

$$|p(x) - p(y)| \le \frac{c}{|\log|x-y||}$$

for all points with $|x-y| < \frac{1}{2}$. Some other names that have been used for these functions are 0-Hölder continuous, Dini-Lipschitz continuous, and weak Lipschitz continuous.

2.2. **The maximal operator.** Most of the problems in the development of the theory of $L^{p(\cdot)}$ spaces arise from the fact that these spaces are virtually never translation invariant. The use of convolution is also limited: it was shown in [23] that Young's inequality $\|f * g\|_{p(\cdot)} \le c\,\|f\|_{p(\cdot)}\|g\|_1$ holds if and only if $p$ is constant. These two problems significantly restrict the techniques available and slowed down the development of the theory.

Despite the failure of Young's inequality it was discovered by Samko [111] that it is possible to mollify in $L^{p(\cdot)}$. If one assumes that $p$ is log-Hölder continuous, then $f * \varphi_\varepsilon$ converges point-wise and in norm to $f$, where $\varphi \in C_0^\infty(\mathbb{R}^n)$ has unit mass and $\varphi_\varepsilon(t) = \varepsilon^{-n}\varphi(t/\varepsilon)$. This property can be reduced to the boundedness of the Hardy-Littlewood maximal operator $M$, which was an important open problem for a long time. Eventually, based on earlier work by Diening and Nekvinda [23, 102], the following was proven by Cruz-Uribe, Fiorenza and Neugebauer [20, Theorem 1.5]:

**Theorem 2.1.** *Suppose that* $1 < p^- \le p^+ < \infty$ *and that* $p$ *is* log-*Hölder continuous and decays as* $|p(x) - p_\infty| \le c/\log(e + |x|)$. *Then* $M$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to* $L^{p(\cdot)}(\mathbb{R}^n)$.

The conditions in the theorem are in fact optimal in the sense of modulus of continuity and decay [20, 106] – if one of the two conditions is weakened, then there exists an exponent $p$ which satisfies the weaker conditions for which $M$ is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

On the other hand, the boundedness of $M$ on $L^{p(\cdot)}$ does not imply that $p$ satisfies the log-Hölder continuity or the decay condition. As a first result in this direction, Nekvinda proved the existence of a function $p$ which is not log-Hölder continuous such that $M$ is still bounded on $L^{p(\cdot)}$. It can be shown that if $M$ is bounded, then $1/p \in BMO_{1/|\log t|}$, i.e. for all small cubes

$$M^\sharp_Q\big(\tfrac{1}{p}\big) = \fint_Q \Big|\tfrac{1}{p(x)} - \big\langle\tfrac{1}{p}\big\rangle_Q\Big|\,dx \le \frac{c}{|\log(\operatorname{diam} Q)|}.$$

But unfortunately $1/|\log t|$ does not satisfy the Dini condition $\int \psi(t)/t\,dt < \infty$ and therefore $BMO_{1/|\log t|}$ is not embedded into $C^{1/|\log t|}$, see [4]. Due to this gap it was conjectured by Diening [25] that there exists a discontinuous exponent without a limit at infinity such that $M$ is still bounded on $L^{p(\cdot)}$. Lerner showed in [94] that if the exponent $q$ satisfies

$$M^\sharp_Q\big(\tfrac{1}{q}\big) \le c\,\min\Big\{\tfrac{1}{|\log|\operatorname{diam} Q||}, |\log|\operatorname{diam} Q||, \log(e + |\operatorname{center} Q|)\Big\}$$

for all cubes $Q$ and $p(x) = p_0 + q(x)$ for some large $p_0 > 1$, then $M$ is bounded on $L^{p(\cdot)}$. His example $q(x) = \sin(\log(e + |\log(|x|)|))$ proves the conjecture from [25]. He then raised the question if the condition above on $q$ (although not necessary) is sufficient for the boundedness of $M$ on $L^{q(\cdot)}$.

A characterization of those exponents $p$ for which $M$ is bounded was recently given by Diening. In [25] he studied the boundedness of the maximal function in the broader context of Orlicz–Musielak spaces. He proposed a generalization of the concept of Muckenhoupt classes for weighted Orlicz spaces to more general Orlicz–Musielak spaces $L^\varphi$. It states that $\varphi$ is of class $\mathcal{A}$ (or $\varphi \in \mathcal{A}$, for short) if the averaging operator $T_{\mathcal{Q}}\colon f \mapsto \sum_{Q\in\mathcal{Q}} \chi_Q \fint_Q f\,dx$ is uniformly bounded on $L^\varphi(\mathbb{R}^n)$ with respect to all families $\mathcal{Q}$ of disjoint cubes. For weighted Orlicz spaces, i.e. $\varphi(x,t) = \psi(t)\omega(x)$ with some weight $\omega$ and some Young function $\psi$, the uniform boundedness assumption can be restricted to families of single cubes. In this case, "class $\mathcal{A}$" coincides with the concept of Muckenhoupt classes, more precisely $\varphi = \psi\omega \in \mathcal{A}$ if and only if $\omega \in A_\psi$. This concept provides the following characterizations of when the maximal operator is bounded:

**Theorem 2.2** (Theorem 8.1, [25]). *Let $1 < p^- \le p^+ < \infty$. The following are equivalent:*

(i) $t^{p(x)}$ *is of class $\mathcal{A}$;*
(ii) $M$ *is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$;*
(iii) $(M(|f|^q))^{\frac{1}{q}}$ *is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for some $q > 1$, ("left–openness");*
(iv) $M$ *is bounded on $L^{\frac{p(\cdot)}{q}}(\mathbb{R}^n)$ for some $q > 1$ ("left–openness");*
(v) $M$ *is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$.*

As another special case of the results in [25], we recover the well-known fact that (i), (ii), (iii), and (v) are equivalent for weighted Orlicz spaces. Nevertheless, it remains open if this is also the case for general Orlicz–Musielak spaces.

Let $\mathcal{P}(\mathbb{R}^n)$ denote the set of exponents $p$ with $1 < p^- \le p^+ < \infty$ such that $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Then it is natural to ask whether $\mathcal{P}(\mathbb{R}^n)$ is closed under some simple operations. Theorem 2.2 shows that $\mathcal{P}(\mathbb{R}^n)$ is closed under duality, in other words $p \in \mathcal{P}(\mathbb{R}^n)$ implies

$p' \in \mathcal{P}(\mathbb{R}^n)$. Moreover, if $p \in \mathcal{P}(\mathbb{R}^n)$ and $s \in [1, \infty)$, then

$$\|Mf\|_{sp(\cdot)}^s = \left\|(Mf)^s\right\|_{p(\cdot)} \le \left\|M(|f|^s)\right\|_{p(\cdot)} \le C \left\||f|^s\right\|_{p(\cdot)} = C \|f\|_{sp(\cdot)}^s,$$

which implies that $sp \in \mathcal{P}(\mathbb{R}^n)$. Theorem 2.2 (iv) shows that this is also true for all $s \in [1-\varepsilon, 1]$, where $\varepsilon = \varepsilon(p) > 0$. This raises the following question.

**Question 2.3.** Let $1 < p^- \le p^+ < \infty$ and let $M$ be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Does this imply that $M$ is bounded on $L^{sp(\cdot)}(\mathbb{R}^n)$ for every $s \in (1/p^-, \infty)$.

The following, related question was asked by Lerner [94, Question 1.5].

**Question 2.4.** Let $1 < p^- \le p^+ < \infty$ and let $M$ be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Does this imply that $M$ is bounded on $L^{\alpha+p(\cdot)}(\mathbb{R}^n)$ for every $\alpha \in (1-p^-, \infty)$.

Using Corollaries A.2 and A.5 from the appendix we immediately get the following closure property, which states that $\{1/p \colon p \in \mathcal{P}(\mathbb{R}^n)\}$ is convex.

**Corollary 2.5.** *Let $1 < p_j^- \le p_j^+ < \infty$ and $M$ be bounded on $L^{p_j(\cdot)}(\mathbb{R}^n)$ for $j = 0, 1$. For $\theta \in (0,1)$ define $p_\theta$ by*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

*Then $M$ is bounded on $L^{p_\theta(\cdot)}(\mathbb{R}^n)$.*

As a final remark on maximal operators, let us mention that it is also possible to study boundedness from $L^{p(\cdot)}$ to a space with smaller exponent, in particular to $L^1$, and this gives a much larger class of exponents for which the maximal operator is bounded, see [61, 79].

2.3. **Interpolation.** Another very interesting question is whether it is possible to transfer complex and real interpolation results to variable exponent Lebesgue spaces.

It was shown in [140] that the Riesz-Thorin theorem is valid on $L^{p(\cdot)}(\Omega)$ spaces, i.e. a linear operator $T$ which is bounded from $L^{p_j(\cdot)}(\Omega)$ to $L^{p_j(\cdot)}(\Omega)$, is also bounded from $L^{p_\theta(\cdot)}(\Omega)$ to $L^{p_\theta(\cdot)}(\Omega)$. Here $p_\theta$ is defined in the usual way, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. In the appendix we show the stronger result $\left[L^{p_0(\cdot)}(\Omega), L^{p_1(\cdot)}(\Omega)\right]_{[\theta]} \cong L^{p_\theta(\cdot)}(\Omega)$, where $[A_0, A_1]_{[\theta]}$ denotes the complex interpolation space of the Banach spaces $A_0$ and $A_1$.

In the view of Corollaries 2.5 and A.5 it is natural to ask if it is possible to generalize the interpolation theorem of Marcinkiewicz to variable exponent spaces. Let us first introduce a suitable notion of weak-type.

**Definition 2.6.** Let $p_0, p_1$ be variable exponents and $T$ be a sublinear operator that is bounded from $L^{p_0(\cdot)}(\Omega)$ to $L^{p_1(\cdot)}(\Omega)$. Then we say that $T$ is of weak-type $(p_0(\cdot), p_1(\cdot))$ if there exists $c > 0$ such that

$$\lambda \left\|\chi_{\{Tf > \lambda\}}\right\|_{p_1(\cdot)} \le c \|f\|_{p_0(\cdot)} \tag{2.7}$$

for all $\lambda > 0$ and all $f \in L^{p_0(\cdot)}(\Omega)$. Here $\chi_{\{Tf>\lambda\}}$ denotes the characteristic function of the set $\{x \colon (Tf)(x) > \lambda\}$.

**Question 2.8** (Marcinkiewicz Interpolation)**.** Let $T$ be a sublinear operator that is of weak type $(p_0(\cdot), p_0(\cdot))$ and $(p_1(\cdot), p_1(\cdot))$. Is $T$ then bounded from $L^{p_\theta(\cdot)}(\Omega)$ to $L^{p_\theta}(\Omega)$?

The notion of weak type certainly generalizes to Orlicz–Musielak spaces if we replace (2.7) by $\lambda \left\|\chi_{\{Tf>\lambda\}}\right\|_{\varphi_1} \leq c_0 \|f\|_{\varphi_0}$, when $T$ is of weak type $(\varphi_0, \varphi_1)$. It is easy to see that in the context of weighted Lebesgue spaces, i.e. $\varphi(x,t) = t^p\omega(x)$, we have $\varphi \in \mathcal{A}$ if and only if $M$ is of weak-type $(\varphi, \varphi)$. We therefore raise the following question:

**Question 2.9.** Let $p$ be a variable exponent. Under what conditions do "$t^{p(x)} \in \mathcal{A}$" and "$M$ is of weak-type $(p(\cdot), p(\cdot))$" agree.

## 3. Problems in Sobolev spaces

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the subspace of $L^{p(\cdot)}(\Omega)$ of functions $u$ whose distributional gradient exists almost everywhere and satisfies $|\nabla u| \in L^{p(\cdot)}(\Omega)$. The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

3.1. **Density of smooth functions.** Because of the density of smooth functions we have a nice way to understand classical function spaces as completions of a regular class of functions in a certain norm. It is also useful in many classical proofs, where it allows us to work with classical derivatives etc. Unfortunately, density of smooth functions in variable exponent Sobolev spaces is not a clear-cut issue, and, indeed, there exist spaces in which smooth functions are not dense.

For variable exponent Lebesgue spaces Kováčik and Rákosník [93, Theorem 2.11] showed that smooth functions are dense, provided only that the exponent is bounded. The first result for Sobolev spaces is due to Edmunds and Rákosník, who showed in [33, Theorem 1] that smooth functions are dense provided the exponent satisfies a certain monotony condition. The monotony condition is quite complicated, and was not so easy to relate to other results. In view of this, the later result by Samko [111, Theorem 3] (see also [13, 22]), which states that log-Hölder continuity of the exponent is sufficient for density, was an important advance. Recently these two conditions were merged into one result, which reads as follows:

**Theorem 3.1** (Theorem 3.2, [80])**.** *Let $\Omega \subset \mathbb{R}^n$ and associate to every $x \in \Omega$ four quantities:*

$$r_x \in \left(0, \tfrac{1}{2}\min\{1, d(x,\partial\Omega)\}\right),$$

$h_x \in (0,1)$, $\xi_x \in S^{n-1}$ *and* $K_x \in [0,\infty)$. *Suppose that for every $x \in \Omega$ and $y \in B^n(x, r_x)$ we have*

$$p(z) - p(y) \geqslant -\frac{K_x}{\log(1/|y-z|)}$$

*for every $z$ in the cone*

$$C(y) = \bigcup_{0<t\leq r_x} B^n(y + t\xi_x, h_x t).$$

*Then $C^\infty(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$.*

In [80, Theorem 4.10] some new types of sufficient conditions for the density of continuous functions are given. This result is based on the regularity of the level-sets of the exponent.

There is essentially only one counter-example to the density of continuous functions, due to Zhikov, [122, Section 1]. This simple space has the unit disk as its domain, and the exponent equals $p_1 < 2$ in the first and third quadrant, and $p_2 > 2$ in the remaining quadrants. In

[78, 124] this example was modified to a uniformly continuous exponent with modulus of continuity

$$\omega(t) = \frac{\log\log(1/t)}{\log(1/t)},$$

i.e. just worse than log-Hölder continuity. In these latter examples the modulus of continuity is very close to optimal, but the examples suffer from the strange short-coming that the saddle-point has to occur at a point $x$ with $p(x) = n$, the dimension. Therefore one may ask:

**Question 3.2.** Let $p_0 \in [1, n)$. Does there exist a continuous variable exponent $p$ with $p(0) = p_0$ whose only saddle-point is at the origin, such that continuous functions are not dense?

Of course one could also look at other types of counter-examples, where the critical feature is not a saddle-point. However, it seems to us that there are probably no such examples, so we prefer to state the question in the opposite direction:

**Question 3.3.** Suppose that $p\colon \Omega \to [1,\infty)$ is a variable exponent without saddle-points. Is $C(\Omega)$ or $C^\infty(\Omega)$ dense in $W^{1,p(\cdot)}(\Omega)$?

Another way to approach the problem is to consider closedness properties of the set $\mathcal{C}$ of exponents for which continuous functions are dense. In analogy with the situation for the maximal operator described previously, we can ask

**Question 3.4.** Suppose that $p \in \mathcal{C}$. Is it always true that $sp \in \mathcal{C}$ or $p + t \in \mathcal{C}$ for $s \geq 1$ and $t \geq 0$?

Notice that the example discussed above shows that the assumptions $s \geq 1$ and $t \geq 0$ are necessary in this question (for some exponents). The reason for this is that if we move the saddle point from level $n$ to $n + \varepsilon$, then the Sobolev functions themselves are continuous in a neighborhood of the saddle point, so the question of density becomes trivial.

*Remark* 3.5. Above we have sometimes considered the density of smooth functions, and sometimes the density of continuous functions. To-date there are no examples of variable exponent Sobolev spaces where continuous functions are dense, but smooth ones are not.

Instead of trying to find characterizations of variable exponent Sobolev spaces where smooth functions are dense, we can look at properties of such spaces:

**Question 3.6.** Suppose that $C(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$. What regularity properties of the Sobolev space does this imply? For instance, is the minimizer of the Dirichlet energy integral always continuous?

Although this question has not hereto received that much attention, some results have been derived under the assumption of density. For instance, Harjulehto, Hästö, Koskenoja and Varonen showed that the density of continuous functions is enough to guarantee that every Sobolev function has a quasicontinuous representative [71, Theorem 5.2].

One program to arrive at properties of Sobolev spaces with $p \in \mathcal{C}$ is based on the study of Sobolev spaces on metric measure spaces. The theory of these so-called Newtonian spaces has been developed only recently for the fixed exponent case (cf. [3]) and is based largely on techniques that are easier to adapt than classical methods. Moreover, in domains of

$\mathbb{R}^n$, variable exponent Newtonian spaces agree with variable exponent Sobolev spaces when Lipschitz functions are dense [77, Theorem 5.3]. This means that if we prove a result in Newtonian space, then it holds in Sobolev space provided Lipschitz functions are dense.

3.2. **Sobolev embeddings.** Initially, also the Sobolev embedding posed quite a problem in variable exponent spaces. Early results in [93, 108] did not recover the optimal conjugate exponent. At the turn of the century progress was made, and optimal results were achieved for Lipschitz and then Hölder continuous exponents [34, 35, 48]. The present state of the art is due to Diening and Samko who generalized the classical method based on Riesz potentials [24, 109, 110]. According to these results, log-Hölder continuity of the exponent is sufficient for the Sobolev embedding.

However, the question of necessity remains largely open. The only known counter-example is due to Kováčik and Rákosník [93, Example 3.2] and features a non-continuous exponent. It is quite easy to modify their example to give a continuous but not uniformly continuous exponent:

**Proposition 3.7.** *There exists a continuous exponent $p$ on a regular domain $\Omega$ such that*

$$W^{1,p(\cdot)}(\Omega) \not\hookrightarrow L^{p^*(\cdot)}(\Omega).$$

*Here $p^*$ denotes the point-wise Sobolev conjugate exponent, $p^*(x) = np(x)/(n - p(x))$.*

The exponent in the previous example is clearly not uniformly continuous (see the proof, and consider the origin). The question is whether it is possible to improve on this:

**Question 3.8.** Are there counter-examples to the Sobolev embedding in regular domains for uniformly continuous exponents?

*Proof of Proposition 3.7.* Fix $t$ and $s$ such that $1 < t < s < 2$. Let $\Omega$ be the intersection of the upper half-plane with the unit disk and define $f(\tau) = 2(\frac{\tau}{t} - 1)$. Denoting by $(r, \varphi)$ spherical co-ordinates in $\Omega$ (with $\varphi \in (0, \pi)$) we define two exponents as follows:

$$q(r,\varphi) = \begin{cases} t, & \text{if } \varphi \geq r^{f(s)} \\ s, & \text{if } \varphi < r^{f(s)} \end{cases}$$

(the exponent from [93, Example 3.2]) and

$$p(r,\varphi) = \begin{cases} t, & \text{if } \varphi \geq 1 \\ \tau, & \text{if } \varphi = r^{f(\tau)} \\ s, & \text{if } \varphi < r^{f(s)} \end{cases}$$

where $\tau \in (t, s)$. The result of Kováčik and Rákosník says that $W^{1,q(\cdot)}(\Omega) \not\hookrightarrow L^{q^*(\cdot)}(\Omega)$. We show that the same is true when $q$ is replaced by the continuous exponent $p$.

Like Kováčik and Rákosník we consider the function $u(x) = |x|^\mu$, where $\mu = \frac{s-2}{t}$. Since $p^* \geq q^*$ it follows that $L^{p^*(\cdot)}(\Omega) \subset L^{q^*(\cdot)}(\Omega)$, so from [93, Example 3.2] we conclude that $u \notin L^{p^*(\cdot)}(\Omega)$. We still need to show that it is in $W^{1,p(\cdot)}(\Omega)$.

We easily calculate that $|\nabla u(x)| = |\mu||x|^{\mu-1}$. Since $|\mu| > 1$, we find that

$$\int_\Omega |\nabla u(x)|^{p(x)} dx < \int_0^1 \int_0^\pi r^{(\mu-1)p(r,\varphi)} d\varphi\, r\, dr.$$

The parts of the domain where $p(x) = t$ or $p(x) = s$ are handled as in [93]; let us denote the integral over these parts by $C < \infty$. In the remaining parts $p(r, \varphi) = \tau$ when $\varphi = r^{f(\tau)}$; solving this relation for $\tau$ we find that $p(r, \varphi) = (\frac{1}{2}\frac{\log\varphi}{\log r} + 1)t$. Thus we have

$$\begin{aligned}\int_0^1 \int_0^\pi r^{(\mu-1)p(r,\varphi)} d\varphi\, r\, dr &\le C + \int_0^1 \int_0^1 e^{(\mu-1)(\frac{1}{2}\log\varphi + \log r)t} d\varphi\, r\, dr \\ &= C + \int_0^1 \int_0^1 \varphi^{(\mu-1)t/2} d\varphi\, r^{(\mu-1)t+1}\, dr \\ &= C + \int_0^1 \frac{2}{s-t} r^{(\mu-1)t+1}\, dr \\ &= C + \frac{2}{s-t}\frac{1}{s-t} < \infty.\end{aligned}$$

So we have shown that $|\nabla u| \in L^{p(\cdot)}(\Omega)$. For our function $u$ we find that $|u(x)| = \frac{1}{|\mu|}|\nabla u(x)||x| \le \frac{1}{|\mu|}|\nabla u(x)|$ and so it also follows that $u \in L^{p(\cdot)}(\Omega)$, and we are done. □

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## Appendix A. Complex Interpolation

In this section we prove some results on interpolation in Orlicz–Musielak spaces. Musielak [140, Chapter 13] has previously considered interpolation in this setting, but his proofs were longer and more complicated.

In the following we need some definitions from [140]. By $\mathbb{R}^+$ we denote the set of non-negative real numbers. Let $a\colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be a right-continuous, non-decreasing function for every $x \in \Omega$ with $a(x, 0) = 0$, $a(x, t) > 0$ for $t > 0$. Suppose that $\varphi\colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$\varphi(x, t) = \int_0^t a(x, u)\, du$$

for every $x \in \Omega$. If $\Omega = \mathbb{R}^n$ we additionally require that $\varphi(x, t)$ is Lebesgue measurable in $x$ for all $t \ge 0$. Then $\varphi$ is be said to be an *$N$–function* on $\Omega$. In this case we usually write $\varphi'(x, t)$ instead of $a(x, t)$.

We say that $\varphi$ satisfies the strong $\Delta_2$–condition if there exists $c_1 > 0$ such that $\varphi(x, 2t) \le c_1\, \varphi(x, t)$ for all $x \in \Omega$ and $t \ge 0$. In the following we always assume that $\varphi$ is an $N$-function which satisfies the strong $\Delta_2$ condition. The space

$$L^\varphi(\Omega) = \left\{ f\colon \Omega \to \mathbb{R}\colon \int_\Omega \varphi\big(x, |f(x)|\big)\, dx < \infty \right\}$$

equipped with the norm

$$\|f\|_\varphi = \inf\left\{ \lambda > 0\colon \int_\Omega \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}$$

is a Banach space. The spaces $L^\varphi(\Omega)$ are special Orlicz–Musielak spaces.

The following results are standard, see [140]. By $(\varphi')^{-1}\colon \Omega\times\mathbb{R}^+\to\mathbb{R}^+$ we denote the function

$$(\varphi')^{-1}(x,t)=\sup\{u\in\mathbb{R}^+\colon \varphi'(x,u)\le t\}.$$

Then $\varphi^*\colon \Omega\times\mathbb{R}^+\to\mathbb{R}^+$ with $\varphi^*(x,t)=\int_0^t(\varphi')^{-1}(x,u)\,du$ is again an $N$–function on $\Omega$. It is the complementary function of $\varphi$. Note that $(\varphi^*)^*=\varphi$. For all $x\in\Omega$ and $t,u\ge 0$, Young's inequality $tu\le\varphi(x,t)+\varphi^*(x,u)$ holds.

We say that $\varphi$ is a proper $N$-function if both $\varphi$ and $\varphi^*$ satisfy the strong $\Delta_2$ condition. In such a case it follows by Section 13 of [140] that $(L^{\varphi})^*(\mathbb{R}^n)\cong L^{\varphi^*}(\mathbb{R}^n)$ and $(L^{\varphi^*})^*(\mathbb{R}^n)\cong L^{\varphi}(\mathbb{R}^n)$. Let $\varphi^{-1}\colon\Omega\times\mathbb{R}^+\to\mathbb{R}^+$, denote the inverse of $\varphi(x,t)$ with respect to $t$, i.e. $t=\varphi^{-1}(\varphi(x,t))$. For two $N$-functions $\varphi_0$ and $\varphi_1$ and $\theta\in(0,1)$ we define the $\theta$–intermediate function $\varphi_\theta$ by

$$\varphi_\theta^{-1}(x,t)=\big(\varphi_0^{-1}(x,t)\big)^{1-\theta}\big(\varphi_1^{-1}(x,t)\big)^{\theta}.$$

Then $\varphi_\theta$ is also an $N$-function.

**Theorem A.1** (Complex interpolation)**.** *Let $\varphi_0$, $\varphi_1$ be proper $N$-functions. Then for all $0<\theta<1$ there holds*

$$\big[L^{\varphi_0}(\Omega),L^{\varphi_1}(\Omega)\big]_{[\theta]}\cong L^{\varphi_\theta}(\Omega).$$

*Moreover, $\|g\|_{[\theta]}\le\|g\|_{\varphi_\theta}\le 4\,\|g\|_{[\theta]}$.*

*Proof.* We proceed along the lines of [1], including some standard notation. We extend $\varphi_j:\Omega\times\mathbb{R}^+\to\mathbb{R}^+$ to $\varphi:\Omega\times\mathbb{C}\to\mathbb{R}^+$ via $\varphi(x,t)=\varphi(x,|t|)$. For $z\in\mathbb{C}$ with $0\le\operatorname{Re}z\le 1$ define $\varphi_z$ by

$$\varphi_z^{-1}(x,t)=\big(\varphi_0^{-1}(x,t)\big)^{1-z}\big(\varphi_1^{-1}(x,t)\big)^{z}.$$

Then $\varphi_z^{-1}$ is holomophic in $z$ on $S=\{z\in\mathbb{C}\colon 0<\operatorname{Re}z<1\}$ and continuous on $\overline{S}$. Let $\mathcal{F}$ be the space of functions on $\overline{S}$ with values in $L^{\varphi_0}(\Omega)+L^{\varphi_1}(\Omega)$ which are analytic on $S$ and bounded and continuous on $\overline{S}$ such that $f(it)$ and $f(1+it)$ tend to zero for $|t|\to\infty$.

For $g\in L^{\varphi_\theta}$ with $\|g\|_{\varphi_\theta}=1$, equivalently $\int\varphi_\theta(x,g(x))\,dx=1$, define

$$f_\varepsilon(z;x)=\exp(\varepsilon z^2-\varepsilon\theta^2)\,\varphi_z^{-1}\Big(x,\varphi_\theta\big(x,g(x)\big)\Big)\operatorname{sgn}g(x).$$

Then $f_\varepsilon(\theta)=g$. Since $\varphi_1$ satisfies the $\Delta_2$ condition, there exists $\alpha>1$ such that $\varphi(x,st)\le s^\alpha\,\varphi(x,t)$ for all $t\ge 0$, $s\ge 1$. Using this we derive

$$\begin{aligned}\int_\Omega\varphi_1(x,|f_\varepsilon(1+it,x)|)\,dx&=\int_\Omega\varphi_1\Big(x,\exp(\varepsilon-\varepsilon t^2-\varepsilon\theta^2)\,\varphi_{1+it}^{-1}\Big(x,\varphi_\theta\big(x,g(x)\big)\Big)\,dx\\&\le\int_\Omega\varphi_1\bigg(x,\exp(\varepsilon)\,\varphi_1^{-1}\Big(x,\varphi_\theta\big(x,g(x)\big)\Big)\bigg)\,dx\\&\le\big(\exp(\varepsilon)\big)^\alpha\int_\Omega\varphi_1\Big(x,\varphi_1^{-1}\Big(x,\varphi_\theta\big(x,g(x)\big)\Big)\,dx\\&=\exp(\alpha\varepsilon)\int_\Omega\varphi_\theta\big(x,g(x)\big)\,dx=\exp(\alpha\varepsilon).\end{aligned}$$

A similar calculation, but without the $\exp(\varepsilon)$ term, shows that

$$\int_\Omega \varphi_0(x, |f_\varepsilon(it,x)|)\,dx \le 1.$$

Thus $\|f_\varepsilon\|_{\mathcal{F}} = \sup_{t\in\mathbb{R}} \max\big\{\|f_\varepsilon(it,\cdot)\|_{\varphi_0}, \|f_\varepsilon(1+it,\cdot)\|_{\varphi_1}\big\} \le 1+\delta(\varepsilon)$. This and $f_\varepsilon(\theta) = g$ prove $\|g\|_{[\theta]} \le 1+\delta(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary we deduce $\|g\|_{[\theta]} \le 1$, whence $\|g\|_{[\theta]} \le \|g\|_{\varphi_\theta}$.

The converse inequality follows from the relation

$$\|g\|_{\varphi_\theta} \le 2 \sup\big\{\langle g,b\rangle \colon \|b\|_{(\varphi_\theta)^*} \le 1,\ b \text{ bounded with compact support}\big\},$$

In fact, for $\|g\|_{[\theta]} = 1$ and $b$ as above put

$$f_\varepsilon(z;x) = \exp(\varepsilon z^2 - \varepsilon\theta^2)\,(\varphi_z^*)^{-1}\Big(x, \varphi_\theta^*\big(x,b(x)\big)\Big)\operatorname{sgn} b(x).$$

Writing $F_\varepsilon(z) = \langle f_\varepsilon(z), g_\varepsilon(z)\rangle$ we have

$$\begin{aligned} F_\varepsilon(it) &\le \int_\Omega (\varphi_z^*)^{-1}\big(x,\varphi_\theta^*(x,b(x))\big)\varphi_z^{-1}\Big(x,\varphi_\theta\big(x,g(x)\big)\Big)\,dx \\ &\le \int_\Omega \varphi_\theta^*(x,b(x)) + \varphi_\theta(x,g(x))\,dx \le 2, \end{aligned}$$

where we have used Young's inequality. Analogously,

$$\begin{aligned} F_\varepsilon(1+it) &\le \exp(2\alpha\varepsilon)\int_\Omega (\varphi_z^*)^{-1}\big(x,\varphi_\theta^*(x,b(x))\big)\varphi_z^{-1}\Big(x,\varphi_\theta\big(x,g(x)\big)\Big)\,dx \\ &\le 2\,\exp(2\alpha\varepsilon). \end{aligned}$$

The Three-line Theorem implies that $F_\varepsilon(z) \le 2\,\exp(2\alpha\varepsilon)$ for all $z\in S$. This implies $\|g\|_{\varphi_\theta} \le 4\,\exp(2\alpha\varepsilon)$, whence $\|g\|_{\varphi_\theta} \le 4\,\|g\|_{[\theta]}$. □

**Corollary A.2.** *If $p_0$ and $p_1$ are variable exponents with $1 < p_j^- \le p_j^+ < \infty$, $j = 0,1$, then*

$$\big[L^{p_0(\cdot)}(\Omega), L^{p_1(\cdot)}(\Omega)\big]_{[\theta]} \cong L^{p_\theta(\cdot)}(\Omega).$$

*Proof.* Define $\varphi_j(x,t) = t^{p_j(x)}$. For variable exponents $p_0$ and $p_1$ with $1 < p_j^- \le p_j^+ < \infty$, $L^{\varphi_j}(\Omega) = L^{p_j(\cdot)}(\Omega)$. This proves the assertion. □

**Corollary A.3.** *Let $\varphi_0$, $\varphi_1$ be proper $N$-functions and let $S$ be a linear operator that is bounded from $L^{\varphi_j}(\Omega)$ to $L^{\varphi_j}(\Omega)$ for $j = 0,1$. Then the operator $S$ is also bounded from $L^{\varphi_\theta}(\Omega)$ to $L^{\varphi_\theta}(\Omega)$ for $0 < \theta < 1$.*

*Remark* A.4. Theorem A.1 can easily be generalized to other Orlicz–Musielak spaces. It is for example possible to replace $\Omega \subset \mathbb{R}^n$ by $\mathbb{N}$, i.e. to consider functions $\varphi\colon \mathbb{N}\times\mathbb{R}^+ \to \mathbb{R}^+$. The resulting Orlicz–Musielak space is usually called $l^\varphi(\mathbb{N})$. Then the result of Theorem A.1 remains valid. In particular, if $\varphi_j\colon \mathbb{N}\times\mathbb{R}^+ \to \mathbb{R}^+$ are proper $N$-functions, then $\big[l^{\varphi_0}(\mathbb{N}), l^{\varphi_1}(\mathbb{N})\big]_\theta \cong l^{\varphi_\theta}(\mathbb{N})$.

The proof of the following result is based on [2, Theorem 1.5.11].

**Corollary A.5.** *Let $\varphi_0$, $\varphi_1$ be proper $N$-functions and let $T$ be a sublinear operator that is bounded from $L^{\varphi_j}(\Omega)$ to $L^{\varphi_j}(\Omega)$ for $j = 0,1$. Then the operator $T$ is also bounded from $L^{\varphi_\theta}(\Omega)$ to $L^{\varphi_\theta}(\Omega)$ for $0 < \theta < 1$.*

*Proof.* Fix $f_0 \in L^{\varphi_\theta}(\Omega)$ with $\|f_0\|_{\varphi_\theta} \le 1$. Then by the Hahn-Banach extension theorem there exists a linear operator $V_0 \colon L^{\varphi_0}(\Omega) + L^{\varphi_1}(\Omega) \to L^{\varphi_0}(\Omega) + L^{\varphi_1}(\Omega)$ such that $|V_0 f_0| = |T f_0|$ and $|V_0 f| \le |T f|$ for all $f \in L^{\varphi_0}(\Omega) + L^{\varphi_1}(\Omega)$. Therefore,

$$\|V_0 f\|_{L^{\varphi_j}(\Omega)} \le \|T f\|_{L^{\varphi_j}(\Omega)} \le c_0 \|f\|_{L^{\varphi_j}(\Omega)},$$

for $j = 0, 1$ with $c_0$ independent of $f_0$. Thus by Theorem A.1 and Corollary A.3 it follows that $\|V_0 f\|_{L^{\varphi_\theta}(\Omega)} \le c_1 \|f\|_{L^{\varphi_\theta}(\Omega)}$. In particular, $\|T f_0\|_{L^{\varphi_\theta}(\Omega)} = \|V_0 f_0\|_{L^{\varphi_\theta}(\Omega)} \le c_1 \|f_0\|_{L^{\varphi_\theta}(\Omega)}$. Since $f_0$ was arbitrary, this proves the corollary. □

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