

## Counter-examples of regularity in variable exponent Sobolev spaces

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**ABSTRACT.** In this article an example is presented which demonstrates that continuous functions are not dense in some variable exponent Sobolev spaces. In contrast to previous examples, our example features an exponent which is uniformly continuous and near optimal. Using the same example we also show that the minimizer of the Dirichlet energy integral is not always continuous and that not quasievery point of a Sobolev function need be a Lebesgue point.

### 1. Introduction

Variable exponent Lebesgue and Sobolev spaces have been intensely investigated in the last couple of years. Advances have been seen recently for instance in the study of maximal and potential-type operators and singular integrals. Results are due to Cruz-Uribe, Fiorenza, Neugebauer, Diening, Růžička, Edmunds, Rákosník, Kokilashvili, Samko and Nekvinda [**CFN**, **D1**, **D2**, **DR1**, **DR2**, **ER2**, **KS1**, **KS2**, **KS3**, **N**]. In these studies the assumption that the variable exponent  $p$  is logarithmically Hölder continuous, i.e. that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|} \quad \text{for } |x - y| \leq \frac{1}{2},$$

has become somewhat of a canon. This is understandable, since the condition is more or less necessary for the local boundedness of the maximal operator, by an example of Pick and Růžička [**PR**]. The question of global boundedness of the maximal operator is not quite as well understood, see [**CFN**, **D3**, **N**]. The counter-examples presented in this article are all local in nature, but it is easy to see that they extend to spaces where the exponent is arbitrarily regular (e.g. constant) outside a compact set.

Differential equations with non-standard growth and coercivity conditions are one of the main motivations for the study of variable exponent spaces. Usually these studies have used the log-Hölder assumption (cf. [**AM**, **E**, **FF**, **FZ**] and references therein). However, recently Harjulehto, Hästö, Koskenoja and Varonen showed that it is also possible to study variational problems on variable exponent spaces under much weaker assumptions on the exponent [**HHKV1**, **HHKV2**]. It is

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2000 *Mathematics Subject Classification.* Primary 46E35; secondary 35J65.

*Key words and phrases.* Variable exponent, Sobolev space, density condition, smooth functions, Dirichlet problem, regularity, Lebesgue points.

therefore natural to ask whether also other properties derived under the log-Hölder assumption hold in a more general setting.

In contrast to the situation of the maximal operator, the necessary and sufficient conditions known for the density of smooth functions in Sobolev space were very far apart. Samko [S1, S2] and Diening [D1] have shown, independently, that log-Hölder continuity of the exponent is sufficient for the density of smooth functions. Harjulehto and Hästö [HH3] have shown that smooth functions are dense on open intervals of the real line irrespective of the variation of the exponent. The only previously known example of when there is no density is due to Zhikov [Z]. In this example, the exponent is discontinuous. When it comes to the study of regularity of minimizers of various variational problems, the situation is even worse. In this case there are no examples which show that one needs the log-Hölder assumption. The same is true for Lebesgue points.

In this article we take a large step towards correcting this problem. We construct a Sobolev space with a variable exponent which is uniformly continuous and has growth just slightly greater than allowed by the log-Hölder assumption (see (2.2)). We show that in this space continuous functions are not dense, that the minimizer of the Dirichlet energy integral is not continuous and that not quasievery point is a Lebesgue point. The proofs are given in Sections 2, 3 and 4, respectively.

This example goes a long way towards bridging the gap between the necessary and sufficient conditions in the density problem. However, this is only true when it comes to regularity at saddle points – Edmunds and Rákosník [ER1] have given a sufficient condition of a different kind, which allows no saddle points, but works even for some discontinuous exponents. This condition is discussed further in Section 2.2. It seems quite possible that the situation for the other two properties, continuity of the minimizer and Lebesgue points, is similar, but at present there are no results which do not rely on log-Hölder continuity assumption. In particular one could ask how much of the classical theory works in variable exponent spaces if one assumes that smooth functions are dense?

**Definitions.** For  $x \in \mathbb{R}^n$  and  $r > 0$  we denote by  $B(x, r)$  the open ball with center  $x$  and radius  $r$ . We abbreviate  $B(r) = B(0, r)$  and  $S(r) = \partial B(r)$ . By  $\Omega$  we always denote a non-empty open subset of  $\mathbb{R}^n$ .

Let  $p : \Omega \rightarrow [1, \infty)$  be a measurable bounded function, called the *variable exponent* on  $\Omega$ , and set  $p^+ = \text{ess sup}_{x \in \Omega} p(x)$  and  $p^- = \text{ess inf}_{x \in \Omega} p(x)$ . We define the *variable exponent Lebesgue space*  $L^{p(\cdot)}(\Omega)$  to consist of all measurable functions  $u : \Omega \rightarrow \mathbb{R}$  for which  $\varrho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx < \infty$ . We define the *Luxemburg norm* on this space by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leq 1\}.$$

The *variable exponent Sobolev space*  $W^{1,p(\cdot)}(\Omega)$  is the subspace of functions  $u \in L^{p(\cdot)}(\Omega)$  whose distributional gradient exists almost everywhere and satisfies  $|\nabla u| \in L^{p(\cdot)}(\Omega)$ . The norm  $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$  makes  $W^{1,p(\cdot)}(\Omega)$  a Banach space. One central property of these spaces (since  $p$  is bounded) is that  $\varrho_{p(\cdot)}(u_i) \rightarrow 0$  if and only if  $\|u_i\|_{p(\cdot)} \rightarrow 0$ . This and many other basic results are proven in [KR].

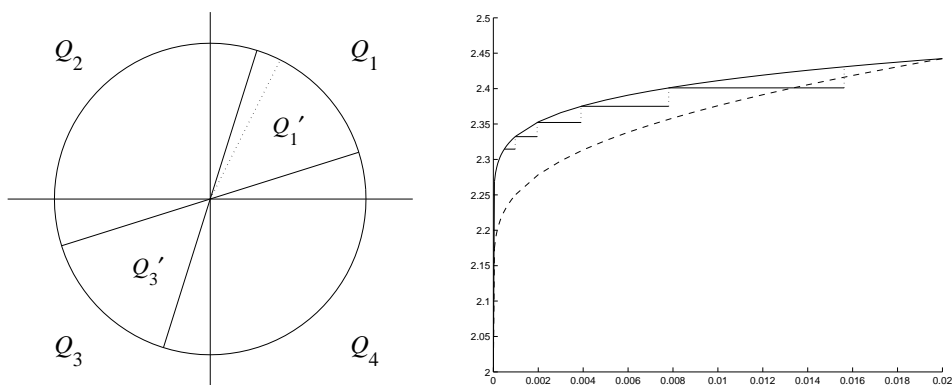
## 2. On the density of continuous functions

In this section we give an example of a variable exponent Sobolev space in which not every function can be approximated by continuous functions. For simplicity

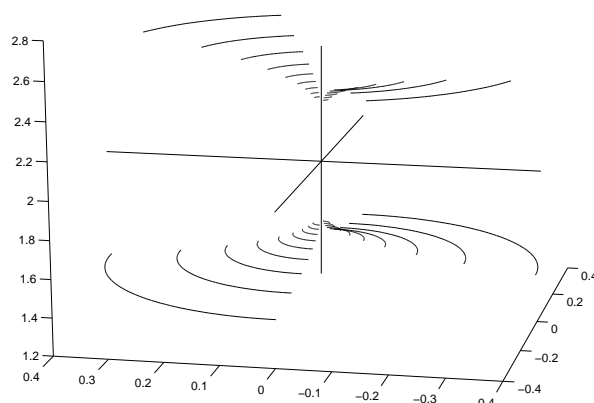
the example is given in the two-dimensional case; it is straightforward to generalize it to higher dimensions.

We start with the unit disk  $D$  and divide it into the four quadrants. In the example of Zhikov [Z, Section 1], the exponent is defined to be a constant  $\alpha_2 > 2$  on quadrants  $Q_1$  and  $Q_3$  and a constant  $\alpha_1 \in (1, 2)$  on the remaining two quadrants. In this case Zhikov proved that continuous functions are not dense in  $W^{1,\alpha(\cdot)}(D)$ . We will start from this and construct a uniformly continuous exponent for which the non-density still holds.

For technical reasons let's actually take  $D = B(1/4)$ . We further partition the first and third quadrants into three parts by lines through the origin, see Figure 1. For  $0 < \epsilon < 1$  we define an exponent  $q: D \rightarrow [1, \infty)$  as follows: On  $Q'_1$  and  $Q'_3$  we set  $q(x) = 2 + (1 + \epsilon) \log_2(i)/i$  for  $|x| \in [2^{-i}, 2^{1-i})$ . On  $Q_2$  and  $Q_4$  we set  $q(x) = 2 - (1 + \epsilon) \log_2(\log_2(1/|x|))/\log_2(1/|x|)$ . We define  $q(x) = 2$  on  $Q_1 \setminus Q'_1$  and  $Q_3 \setminus Q'_3$  (actually any value here will do). Clearly the exponent  $q$  is not continuous; we'll remedy this later. The exponent is sketched in Figure 2.



**Fig. 1.** **Left:** The partition of the disk  $B(1/4)$ . **Right:** A cross-sectional view of the exponent  $q$  along the dotted segment in the left figure. Also shown: a continuous version of  $q$  and, for comparison, a log-Hölder continuous exponent (dashed line).



**Fig. 2.** A contour sketch of the exponent  $q$  on  $Q'_1 \cup Q_2 \cup Q'_3 \cup Q_4$ .

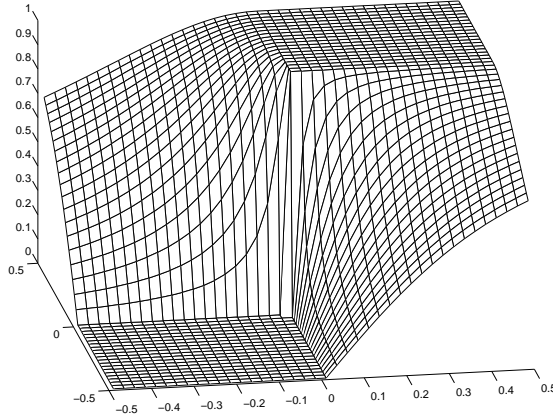
Define a function  $u: D \rightarrow [0, 1]$  by

$$u(x) = \begin{cases} 1 & \text{for } x \in Q_1 \\ x_2/|x| & \text{for } x \in Q_2 \\ 0 & \text{for } x \in Q_3 \\ x_1/|x| & \text{for } x \in Q_4 \end{cases}$$

where  $x = (x_1, x_2)$ . The function is shown in Figure 3. Since  $u$  is bounded, it is clear that  $u \in L^{q(\cdot)}(D)$ . We easily calculate  $|\nabla u(x)| = |x|^{-1}$  for  $x \in Q_2 \cup Q_4$ . Using the substitution  $s = \log_2(1/r)$  we calculate

$$\begin{aligned} \int_D |\nabla u(x)|^{q(x)} dx &= \pi \int_0^{1/4} r^{1-q(r)} dr = \pi \int_0^{1/4} 2^{-(1+\epsilon)\log_2(\log_2(1/r))+\log_2(1/r)} dr \\ &= \pi \log 2 \int_2^\infty s^{-1-\epsilon} ds = \pi \log(2) 2^{-\epsilon} \epsilon^{-1} < \infty. \end{aligned}$$

Therefore  $|\nabla u| \in L^{q(\cdot)}(D)$ , and so  $u \in W^{1,q(\cdot)}(D)$ .



**Fig. 3.** The function  $u$  which cannot be approximated by a continuous function.

Let us next show that  $u$  cannot be approximated by a continuous function in the space  $W^{1,q(\cdot)}(D)$ . Let  $v \in C(D) \cap W^{1,q(\cdot)}(D)$  with  $v(0) = a$ . We will use the estimate

$$\|u - v\|_{W^{1,q(\cdot)}(D)} \geq \|v\|_{W^{1,q(\cdot)}(Q'_3)} + \|1 - v\|_{W^{1,q(\cdot)}(Q'_1)}.$$

By symmetry, we may then assume that  $a \geq 1/2$  and consider only the first term on the right-hand-side of this inequality.

We can estimate  $|\nabla v|$  from below by the radial derivative, which we will denote by a prime. (Note that it makes sense to speak of the radial derivative, since  $v$  is classically differentiable almost everywhere in  $Q'_3$  by [St, Theorem VIII.1.1].) Then we have

$$\int_{Q'_3} |v(x)|^{q(x)} + |v'(x)|^{q(x)} dx \geq \int_{Q'_3} |v_1(x)|^{q(x)} + |v'_1(x)|^{q(x)} dx,$$

where  $v_1(x) = \min_{y \in [0,x]} v(y)$  and  $[0, x]$  denotes the segment between 0 and  $x$ . We may therefore assume that  $v$  is radially decreasing. Let  $L$  be the subset of

$S(1/5) \cap Q'_3$  where  $v(x) > 1/4$ . Then

$$\int_{Q'_3} |v(x)|^{q(x)} dx \geq 4^{-q^+} \frac{m_1(L)}{2\pi} m_2(B(1/5)),$$

where  $m_1$  denotes the 1-dimensional Lebesgue measure. On the other hand, on  $S = (S(1/5) \cap Q'_3) \setminus L$  the function  $v$  has value  $1/4$  or less. Therefore the function  $\phi(x) = (v(x) - 1/4)/a$  is continuous,  $\phi(0) = 1$  and  $\phi(x) \leq 0$  for  $x \in S$ . Therefore  $\phi$  can serve as a test function for the variational pseudocapacity. For a definition see the Section 2.1 below. Using the lemma of that subsection, we conclude that

$$\int_{Q'_3} |v'(x)|^{q(x)} dx \geq c m_1(S).$$

Combining these estimates we have shown that

$$\varrho_{q(\cdot)}(v - u) + \varrho_{q(\cdot)}(|\nabla(v - u)|) > c m_1(L) + c m_1(S) \geq c > 0$$

for all continuous functions  $v$ . Since  $\varrho_{p(\cdot)}(u_i) \rightarrow 0$  if and only if  $\|u_i\|_{p(\cdot)} \rightarrow 0$ , this means that  $\|u - v\|_{1,p(\cdot)} > c > 0$  for all continuous  $v$ . Therefore continuous functions are not dense in  $W^{1,q(\cdot)}(D)$ . But  $q$  is not a continuous exponent, so there is still some work to be done.

On the sets  $Q'_1$  and  $Q'_3$  we choose the exponent  $p$  radially symmetric and affine on the annuli  $B(2^{1-i}) \setminus B(2^{-i})$ , so that  $p(x) = q(x)$  on  $S(2^{1-i})$  for all  $i = 3, 4, \dots$ . We define  $p = q$  on  $Q_2 \cup Q_4$  and  $p(0) = 2$ . On  $Q_1 \setminus Q'_1$  and  $Q_3 \setminus Q'_3$  we define  $p$  affinely on the lines  $x_1 + x_2 = c$ . We want to show that changing the exponent in this way does not affect the properties we just proved for  $W^{1,q(\cdot)}(D)$ . Since  $p = q$  on  $Q_2 \cup Q_4$ , it is easy to see that  $u \in W^{1,p(\cdot)}(D)$  ( $u$  is as defined before). On  $Q'_1 \cup Q'_3$  we have  $p \geq q$ . We therefore have an embedding from  $L^{p(\cdot)}(Q'_1 \cup Q'_3)$  to  $L^{q(\cdot)}(Q'_1 \cup Q'_3)$  whose norm is at most  $1 + |D|$ , by [KR, Theorem 2.8]. If  $v$  is a continuous function this implies that

$$\|u - v\|_{W^{1,p(\cdot)}(B)} \geq \|u - v\|_{W^{1,p(\cdot)}(Q'_1 \cup Q'_3)} \geq \frac{\|u - v\|_{W^{1,q(\cdot)}(Q'_1 \cup Q'_3)}}{1 + |D|} \geq c > 0,$$

so we still can't approximate by continuous functions. Thus  $W^{1,p(\cdot)}(D)$  is the Sobolev space we were trying to construct. For future reference we summarize our result:

**PROPOSITION 2.1.** *There exists a variable exponent Sobolev space with uniformly continuous exponent such that continuous (or smooth) functions are not dense.*

Note that in our example we have

$$(2.2) \quad |p(x) - p(0)| \approx \frac{\log_2 \log_2(1/x)}{-\log x},$$

which is just barely worse than log-Hölder continuity, see Figure 1, Right.

**2.1. The variational pseudocapacity.** In the article [HH] a variational pseudocapacity was introduced in order to estimate the norm of the gradient of a function growing a certain amount. For completeness some definitions and details from that paper are included here.

Let  $F, E \subset \mathbb{R}^n$  be closed disjoint sets in  $D \subset \mathbb{R}^n$ . Let  $L(E, F; D)$  be the set of continuous functions  $f$  that satisfy  $f|_E = 1$ ,  $f|_F = 0$  and  $|\nabla f| \in L^{p(\cdot)}(D)$ . The *variational  $p(\cdot)$ -pseudocapacity* is defined by

$$\psi_{p(\cdot)}(F, E; D) = \inf_{f \in L(F, E; D)} \|\nabla f\|_{p(\cdot)}.$$

For  $L(F, E; D) = \emptyset$  we define  $\psi_{p(\cdot)}(F, E; D) = \infty$ . (Notice that this definition has built into it an assumption of density of continuous functions. However, since will be using the capacity in sets where  $p(x) > n$ , this still makes sense.)

The reason for the term pseudocapacity is that our set-function it is defined using the norm instead of the modular, as a normal capacity. This corresponds to introducing an exponent  $1/p$  to the capacity in the fixed exponent case. The pseudocapacity still has many of the usual properties of a capacity, see [HH, Theorem 4.2].

Let us now prove the lemma that was needed in our example. The proof is essentially part of the proof of Theorem 4.7, [HH].

LEMMA 2.3. *Let  $S$  be a subset of  $B(1/5)$  of positive  $(n-1)$ -measure. Let*

$$C = \bigcup_{x \in S} [0, x].$$

*Suppose that the exponent satisfies*

$$p(x) \geq n + (n-1 + \epsilon) \frac{\log_2 \log_2(1/|x|)}{\log_2(1/|x|)}$$

*in  $C$  for some fixed  $\epsilon > 0$ . Then*

$$\text{cap}_{p(\cdot)}(S, \{0\}; C) \geq c(n).$$

PROOF. We divide  $C$  into annuli,  $A_i = \{x \in C : 2^{-i} \leq |x| < 2^{1-i}\}$  for  $i = 3, 4, \dots$ . By [KR, Theorem 2.8],  $\|f\|_{q(\cdot)} \leq (1 + |C|)\|f\|_{p(\cdot)}$  for  $q$  satisfying  $q(x) \leq p(x)$  a.e. Therefore we see that it suffices to show that

$$\text{cap}_{q(\cdot)}(S', \{0\}; C) \geq c(n),$$

where

$$q(x) = n + (n-1 + \epsilon) \sum_{i=3}^{\infty} \frac{\log_2 i}{i} \chi_{A_i},$$

and  $\chi_A$  denotes the characteristic functions of  $A$ . For every function  $f \in W^{1, q(\cdot)}(C)$  we have

$$\|f\|_{1, q(\cdot)} \geq \min\{1, \varrho_{1, q(\cdot)}(f)\},$$

by [KR, (2.11)]. Therefore we see that it suffices to show that  $\varrho_{1, q(\cdot)}(f) > c(n)$  for every  $f \in L(S, \{0\}; C)$ .

The next step is crucial in making this lemma work even with possibly very irregular domains  $C$ . We estimate the gradient of  $f$  from below by the radial component of the derivative:  $|\nabla f| \geq |\partial f / \partial r|$ . (Again, we are using that  $f$  is classically differentiable almost everywhere in  $C$  by [St, Theorem VIII.1.1].) Therefore

$$\int_C |\nabla f(x)|^{q(x)} dx \geq \int_C \left| \frac{\partial f(x)}{\partial r} \right|^{q(x)} dx.$$

It is then easy to see that the function minimizing the integral should depend only on the distance from the origin, not on the direction. If  $f$  is such a function then

$$\int_C |\nabla f(x)|^{q(x)} dx = \int_S dm_{n-1} \int_0^1 |\nabla f(rs)|^{q(rs)} dr,$$

where  $s$  is any fixed element in  $S$ . Thus the problem at hand is essentially a one-dimensional one.

We choose  $r > 0$  such that

$$m_{n-1}(B(e_1/5, r) \cap S(0, 1/5)) = m_{n-1}(S).$$

Since  $S \subset S(0, 1/5)$  this is clearly possible. Define  $S' = B(e_1/5, r) \cap S(0, 1/5)$  and

$$C' = \bigcup_{x \in S'} [0, x].$$

Since  $m_{n-1}(S) = m_{n-1}(S')$ , the formula in the previous paragraph implies that

$$\int_C |\nabla f(x)|^{q(x)} dx = \int_{C'} |\nabla f(x)|^{q(x)} dx,$$

where  $f$  is radially symmetric.

After this reduction the problem at hand is exactly the same as the one solved in the proof of Theorem 4.7, [HH]. For completeness a sketch of that proof follows.

Since the exponent  $q$  is fixed on  $A_i$ , we can use fixed-exponent capacity estimates for each annulus. This turns out to equal  $c2^{i(q_i-n)}$ , where  $q_i$  is the value of  $q$  on  $A_i$ . This means that if the function  $f$  increases by 1 from the inner to the outer boundary of the annulus  $A_i$ , then  $\rho_{q_i}(|\nabla f|) \geq c2^{i(q_i-n)}$ . Therefore, if the function increases by  $f_i$  on  $A_i$ , then the modular of its gradient is at least  $cf_i^{q_i}2^{i(q_i-n)}$ . Thus we seek to choose the  $f_i$ 's so as to minimize

$$(2.4) \quad c \sum_{i=3}^{\infty} f_i^{q_i} 2^{i(q_i-n)}.$$

On the other hand the total increment of the function over all  $A_i$ 's is at least 1 (by definition), so we have  $\sum f_i = 1$ . It was shown in [HH, Lemma 4.5] that this constraint forces (2.4) to be greater than a positive constant independent of  $f$ .  $\square$

**2.2. The monotony condition of Edmunds and Rákosník.** As early as 1992, Edmunds and Rákosník [ER1] gave a sufficient condition for the density of smooth functions in variable exponent Sobolev spaces. Their condition is the following: for every  $x \in \Omega$  there exist numbers  $r_x \in (0, 1]$ ,  $h_x > 0$  and a vector  $\xi_x \in \mathbb{R}^n \setminus \overline{B}(h_x)$  such that  $z, z + y \in \Omega$  and  $p(z) \leq p(z + y)$  for almost every  $x \in \Omega$ ,  $z \in B(x, r_x)$  and  $y$  in the cone

$$\bigcup_{0 < t \leq 1} B(t\xi_x, th_x).$$

Actually, this is a corrected version of the condition in [ER1]. The version in [ER1] only assumes  $p(x) \leq p(x + y)$ . To see that the result does not hold in this case we need only consider the example of Zhikov (see the beginning of this section). (The additional assumption is used on line 12 of page 234 of [ER1]. It is actually clear from the paper that the above condition was what the authors had in mind. However, since the incorrect version has been quoted in the literature, this remark seems in place.)

It is clear that the Edmunds–Rákosník condition is neither implied by log-Hölder continuity, nor implies the same. Therefore an interesting problem is whether it is possible to combine the approaches of Edmunds–Rákosník and Diening/Samko to prove density if the exponent is log-Hölder continuous at saddle points and satisfies the monotony condition elsewhere. In view of the example presented in this article, we could expect this condition to be very near optimal. Another interesting point is that it is absolutely essential for our example that the saddle point occur at the critical value  $n$  (the dimension). For  $p(x) \geq n$ , Sobolev functions are usually continuous (see [HH, Section 3]) so density of continuous functions is not an issue. But for  $p(x) < n$  there is at present no counter-example, even with discontinuous exponent, of the density of smooth functions.

### 3. Lebesgue points

In this section we will show that the example from the previous section implies that Lebesgue points in variable exponent Sobolev spaces are less abundant than we would expect based on the classical case.

We say that  $x \in \Omega$  is a *Lebesgue point* of  $f \in L^{p(\cdot)}(\Omega)$  if

$$(3.1) \quad \lim_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)|^{p(y)} dy = 0.$$

It was shown in [HH2, Theorem 3.1] that almost every point of an  $L^{p(\cdot)}$  function is a Lebesgue point. For Sobolev functions we can make a more precise statement using capacity. We define the capacity of a set  $E \subset \mathbb{R}^n$  by the formula

$$\text{cap}_{p(\cdot)}(E) = \inf \varrho_{p(\cdot)}(f) + \varrho_{p(\cdot)}(\nabla f),$$

where the infimum is taken over all functions  $f \in W^{1,p(\cdot)}(\mathbb{R}^n)$  such that  $f \geq 1$  on an open set containing  $E$ . Although the same symbol  $\text{cap}_{p(\cdot)}$  is used for the capacity and the pseudocapacity, no confusion is possible since the former has one argument and the latter three.

A claim is said to hold *quasieverywhere*, if it holds except in a set of capacity zero. In [HH2, Theorem 4.13] (see also Remark 4.15, [HH2]) it was shown that (3.1) holds quasieverywhere for Sobolev functions, provided we choose a suitable representative for the function  $f$  (which, as a Sobolev function, is defined only up to sets of measure zero). To show this, it was assumed that  $p$  is log-Hölder continuous and that  $1 < p^-$ . Let us have a look at why our example implies that we cannot do much better than this. The notation in the next proposition is same as in Section 2.

**PROPOSITION 3.2.** *Let  $u$  and  $p$  be as before. Then  $\text{cap}_{p(\cdot)}(\{0\}) > 0$ , but 0 is not a Lebesgue point of  $u$ . Thus not quasievery point is a Lebesgue point of  $u$ .*

**PROOF.** Let  $f \in W^{1,p(\cdot)}(D)$  be a function which equals  $u$  almost everywhere. Then

$$\begin{aligned} \lim_{r \rightarrow 0} \int_{B(r)} |f(y) - f(0)| dy &\geq \\ \lim_{r \rightarrow 0} \frac{1}{|B(r)|} \left( \int_{Q_1 \cap B(r)} |1 - f(0)| dy + \int_{Q_3 \cap B(r)} |f(0)| dy \right) &> c > 0, \end{aligned}$$

irrespective of the value of  $f(0)$ . This proves that 0 is not a Lebesgue point of any representative of  $u$ . We next have to estimate the capacity of  $\{0\}$ . It suffices to show that

$$\inf \varrho_{p(\cdot), Q'_1}(f) + \varrho_{p(\cdot), Q'_1}(\nabla f) \geq c > 0$$

with a lower bound independent of  $r$ , where the infimum is taken over functions in  $W^{1,p(\cdot)}(Q'_1)$  which equal 1 on the set  $B(r) \cap Q'_1$ . In the set  $Q'_1 \setminus B(r/2)$  the exponent  $p$  is bounded away from 2, the dimension of the space. Therefore the functions we are minimizing over are continuous in  $Q'_1 \setminus B(r/2)$  [HH2, Theorem 3.1]. We can then enlarge the set of functions we are considering and take the infimum over functions in  $W^{1,p(\cdot)}(Q'_1)$  which are continuous and equal 1 at the origin. But we showed in Section 2 that precisely this infimum is positive, so we are done.  $\square$

#### 4. Discontinuity of the Dirichlet energy integral minimizer

As was already mentioned, one can define and study the Dirichlet problem in variable exponent Sobolev spaces, even when the exponent is not log-Hölder continuous. Let us start by giving the precise definitions, from [HHKV2].

We say that  $w \in W^{1,p(\cdot)}(\mathbb{R}^n)$  is *quasicontinuous* if for every  $\epsilon > 0$  there exists an open set  $G$  such that  $w$  is continuous in  $\mathbb{R}^n \setminus G$  and  $\text{cap}_{p(\cdot)}(G) < \epsilon$ . We say that  $w$  has *zero boundary values*,  $w \in W_0^{1,p(\cdot)}(\Omega)$ , if there exists a quasicontinuous function  $\tilde{w} \in W^{1,p(\cdot)}(\mathbb{R}^n)$  such that  $w = \tilde{w}$  almost everywhere in  $\Omega$  and  $\tilde{w} = 0$  quasieverywhere in  $\mathbb{R}^n \setminus \Omega$ . This definition of zero boundary values is taken from the context of Sobolev functions on metric measure spaces, see [HHKV2] for details.

The Dirichlet problem is the following: given  $w \in W^{1,p(\cdot)}(\Omega)$  we want to minimize

$$(4.1) \quad \int_{\Omega} |\nabla f(x)|^{p(x)} dx,$$

subject to the constraint  $f - w \in W_0^{1,p(\cdot)}(\Omega)$ . This constraint says that  $f$  equals  $w$  on the boundary in the sense of Sobolev functions. Assuming that  $p$  satisfies a certain local jump-condition, it was shown in [HHKV2] that (4.1) has a unique minimizer. We will not define the jump condition here, suffice it to say that uniformly continuous exponents satisfy it.

The next step in the classical paradigm for analyzing variational integrals is to prove that the minimizer is continuous and satisfies Harnack's inequality. Other investigators (e.g., Acerbi & Mingione [AM] and Eleuteri [E]) have proven various regularity results for minimizers, under the assumption that the exponent is log-Hölder continuous. Under the minimal assumptions of [HHKV2], Harjulehto, Hästö, Koskenoja and Varonen were not able to prove even continuity. We will show here that the reason for this is that the minimizer is not in general continuous!

**PROPOSITION 4.2.** *There exists a uniformly continuous exponent and a uniformly continuous boundary value function  $w$  such that the minimizer of the Dirichlet energy integral (4.1) is not continuous.*

**PROOF.** Let us again return to the notation of Section 2. We consider the integral (4.1) in domains  $B(r)$  for  $0 < r < 1/4$  and choose  $u$  as our boundary value function. (Although  $u$  is discontinuous, it is nevertheless continuous near the boundary of  $B(r)$ , hence in any given  $B(r)$  we can modify it to get an equivalent,

continuous boundary value function.) By [HHKV2, Theorem 5.3] there exists a unique minimizer for this problem for every  $r > 0$ .

Fix  $r$  and denote a continuous function with the correct boundary values by  $v_r$ . Then  $v_r$  equals 1 on  $S(r) \cap Q'_1$  and 0 on  $S(r) \cap Q'_3$ . As before we find that  $\varrho_{p(\cdot)}(\nabla v_r) > c > 0$ . Notice also that the lower bound does not depend on  $r$ . On the other hand,

$$\int_{B(r)} |\nabla u(x)|^{p(x)} dx \rightarrow 0$$

as  $r \rightarrow 0$ , since  $u \in W^{1,p(\cdot)}(D)$ . Therefore, there exists an  $r$  small enough that

$$\int_{B(r)} |\nabla v_r(x)|^{p(\cdot)} dx > \int_{B(r)} |\nabla u(x)|^{p(x)} dx$$

for all continuous  $v_r$  satisfying the correct boundary value condition. But this means that  $v_r$  is certainly not a minimizer. Therefore the minimizer is not continuous. (Note that  $u$  may or may not be the minimizer, this is of no relevance for our argument.)  $\square$

We have thus shown that in general the minimizers of this simple variational problem are not continuous in variable exponent Sobolev space. One immediate question is whether density of smooth functions is sufficient to guarantee the regularity of the minimizer? If this were so, then one might hope to introduce a second, weaker "standard" assumption alongside log-Hölder continuity, viz. the density condition. This speculation gets some support from what happens in the one-dimensional case: smooth functions are dense provided only that  $p^+ < \infty$  [HH3] and similarly the Dirichlet energy integral minimizer is continuous provided  $1 < p^- \leq p^+ < \infty$  [HHK]. However, the proofs of these results are not directly related.

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