

The maximal operator in Lebesgue spaces with variable exponent near 1

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This article contains results about the boundedness of the Hardy–Littlewood maximal operator in variable exponent Lebesgue spaces. We study the situation where the exponent approaches one in some parts of the domain. We show that the boundedness depends on how fast the exponent approaches one and give near-optimal bounds for necessary and sufficient growths.

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1 Introduction

In this article we will study the Hardy–Littlewood maximal operator in variable exponent Lebesgue spaces. Loosely speaking, we will prove minimal boundedness results under minimal assumptions. The article is divided into three sections: this section furnishes the background for the results and presents related results both in classical and variable exponent Lebesgue spaces. The second section contains the statements and proofs of the results, along with some comments on how they might be extended. A short third section contains a geometric lemma needed in the proofs.

The variable exponent Lebesgue space is defined in the spirit of Orlicz spaces. The definition is the following: let $\Omega \subset \mathbb{R}^n$ be open and let $p : \Omega \rightarrow [1, \infty)$ be a measurable bounded function, called the *variable exponent* on Ω . We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ for which $\varrho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx < \infty$. We define the *Luxemburg norm* on this space by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leq 1\}.$$

Notice that if p happens to be a constant function the we get exactly the classical Lebesgue space. Variable exponent Lebesgue spaces in \mathbb{R}^n were introduced in 1991 by O. Kováčik and J. Rákosník [21], although the one-dimensional case had been considered already earlier, starting with W. Orlicz [24] in 1931.

Variable exponent Lebesgue and Sobolev spaces have been intensely investigated in the last couple of years, see [12] for an overview and e.g. [4, 20] for some recent results. One of the main motivations for studying these spaces are differential equations with non-standard coercivity and growth assumptions, see e.g. [1, 2]. Another factor giving rise to much interest in these questions was the advance in the study of the Hardy–Littlewood maximal operator. Recall that the maximal operator is defined for a locally integrable function u as

$$\mathcal{M}u(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(x)| dx,$$

where $B(x,r)$ denotes a ball centered at x with radius r . L. Diening [5] was first to prove local boundedness of the maximal operator under the *logarithmic Hölder continuity* condition on the exponent $p(x)$:

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|},$$

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for a constant C and $|x - y| \leq 1/2$. By an example of L. Pick and M. Růžička [25] this condition is essentially optimal.

Subsequently D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer [3] and A. Nekvinda [22] proved global boundedness results for the maximal operator. Although the conditions for global estimates are not superfluous, they are also not strictly speaking necessary, see [23]. Recently, L. Diening has derived some new results on the boundedness of the maximal operator in more general Orlicz–Musielak spaces [6] and P. Harjulehto, P. Hästö and M. Pere have investigated the maximal operator on variable exponent Lebesgue spaces on metric measure spaces [14]. Progress with the maximal operators has in turn led to investigations of potential-type operators and singular integrals (e.g., L. Diening and M. Růžička [7], D. Edmunds, V. Kokilashvili and A. Meskhi [8], A. Karlovich and A. Lerner [18], and V. Kokilashvili and S. Samko [19, 20]).

The purpose of the present note is to establish much weaker conditions under which the maximal operator is still minimally regular. More precisely, we are interested in the question of when

$$\|\mathcal{M}u\|_1 \leq C\|u\|_{p(\cdot)}, \quad (1.1)$$

for all functions in variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$. This weakening can in some sense be seen as an analogue to the study of $(1, q)$ –Poincaré inequalities in fixed-exponent Sobolev spaces on metric measure spaces which do not support ordinary (q, q) –Poincaré inequalities (see e.g., [11] or [17]).

Let us recall some basic facts about the maximal operator in classical (fixed-exponent) Lebesgue spaces. If $q > 1$, then $\|\mathcal{M}u\|_q \leq C\|u\|_q$ for all $u \in L^q(\Omega)$. However, for $q = 1$ this is not true. Instead \mathcal{M} takes L^1 to weak- L^1 , i.e.

$$|\{x \in \Omega: \mathcal{M}u(x) > \lambda\}| \leq \frac{C}{\lambda}\|u\|_1.$$

The space $L \log L(\Omega)$ is defined as those functions for which the norm

$$\|u\|_{L \log L} = \inf \left\{ \lambda > 0: \int_{\Omega} \frac{|u(x)|}{\lambda} \log \left(2 + \frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}$$

is finite. It is known that the maximal operator takes $L \log L$ locally to L^1 , i.e.

$$\|\mathcal{M}u\|_1 \leq \|u\|_{L \log L},$$

provided Ω is bounded. For a proof of these results see e.g. Sections I.1 and I.5 in [26].

With this in mind we take a closer look at (1.1). The first thing to notice about the inequality is that it is trivial if the variable exponent p is bounded away from 1 and Ω is a bounded domain. For in this case we have

$$\|\mathcal{M}u\|_1 \leq C\|\mathcal{M}u\|_{p^-} \leq C'\|u\|_{p^-} \leq C''\|u\|_{p(\cdot)},$$

where p^- is the infimum of p . Here we have used that variable exponent Lebesgue spaces satisfy the usual inclusion relation, viz. $L^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ for bounded Ω if and only if $p(x) \geq q(x)$ almost everywhere. The case where Ω is unbounded leads to problems similar to those of the global versions of the boundedness results of the maximal operator, which were discussed above, and will not be considered here. We can, however, profitably study the case when $p^- = 1$. In this case D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer [3, Theorem 1.7] have shown that the maximal operator is virtually never bounded from $L^{p(\cdot)}$ to itself, so we really need a lower exponent on the target space.

We will give both necessary and sufficient conditions for (1.1) to hold in reasonably regular bounded domains. For instance, we show that (1.1) holds in Ω if the exponent p is locally bounded away from 1 and satisfies

$$p(x) = 1 + (1 + \epsilon) \frac{\log \log(1/d(x, \partial\Omega))}{\log(1/d(x, \partial\Omega))} + o\left(\frac{\log \log(1/d(x, \partial\Omega))}{\log(1/d(x, \partial\Omega))}\right) \quad (1.2)$$

near the boundary, for a fixed $\epsilon > 0$. On the other hand, (1.1) does not hold if this equation is satisfied for $\epsilon < 0$. Unfortunately we are not able to tell what happens when $\epsilon = 0$, although presumably one has to look at lower order terms to determine this. (Continuing the investigation of this paper, T. Futamura and Y. Mizuta have investigated this question, see [9].) Our results, although apparently optimal to the first order, are based on simple point-wise estimates.

Let us state some conventions before moving on to the results.

Notation

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball with center x and radius r . For a ball B and real number s we denote by sB the ball with the same center as B but s times the radius. By Ω we always denote a non-empty open subset of \mathbb{R}^n and by $\delta(x)$ the distance of x to the boundary of Ω . We denote by Ω_r an inner r -neighborhood of the boundary: $\Omega_r = \{x \in \Omega: \delta(x) < r\}$. We use $\mathcal{M}^s(D)$ to denote the s -dimensional Minkowski measure of D , for a definition see Section 3.

By $u \lesssim v$ we mean that there exists a constant C independent of u and v for which $u \leq Cv$. The notation $u \approx v$ means that $u \lesssim v$ and $v \lesssim u$. We use the little- o notation to mean a term going to zero faster than its argument, e.g. $x + o(x)$ means that the second term decreases faster than x as $x \rightarrow 0$, so for instance $x + x^2$ would do.

Since we often need to control the extreme values of the exponent we define for $D \subset \Omega$

$$p_D^- = \operatorname{ess\,inf}_{x \in D} p(x) \quad \text{and} \quad p_D^+ = \operatorname{ess\,sup}_{x \in D} p(x).$$

We also use the shorthand notation $p^- = p_\Omega^-$ and $p^+ = p_\Omega^+$. We say that p locally bounded away from 1 if $p_K^- > 1$ for every compact subset K of Ω .

2 The results

Recall the following simple inequalities relating the norm and the modular in variable exponent spaces. They are easily derived directly from the definitions:

$$\min \left\{ \rho_{p(\cdot)}(u)^{1/p^-}, \rho_{p(\cdot)}(u)^{1/p^+} \right\} \leq \|u\|_{p(\cdot)} \leq \max \left\{ \rho_{p(\cdot)}(u)^{1/p^-}, \rho_{p(\cdot)}(u)^{1/p^+} \right\}.$$

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\mathcal{M}^{n-1}(\partial\Omega) < \infty$. Suppose also that $p^+ < \infty$. If p is locally bounded away from 1 and satisfies the inequality*

$$p(x) \geq 1 + (1 + \epsilon) \frac{\log \log(1/\delta(x))}{\log(1/\delta(x))} + o\left(\frac{\log \log(1/\delta(x))}{\log(1/\delta(x))}\right)$$

for small $\delta(x)$ and some fixed $\epsilon > 0$, then $L^{p(\cdot)}(\Omega) \hookrightarrow L \log L(\Omega)$ and $\mathcal{M}: L^{p(\cdot)}(\Omega) \rightarrow L^1(\Omega)$ is bounded.

Notice that the assumption on the domain in the theorem is satisfied for instance if Ω is a Lipschitz domain.

Proof. By Lemma 3.1 there exists a constant $C_1 > 0$ such that $|\Omega_\delta| \leq C_1 \delta$ for all $\delta > 0$. The claim of the theorem is that

$$\|u\|_{L \log L} \lesssim \|u\|_{p(\cdot)}$$

for every $u \in L^{p(\cdot)}(\Omega)$. Since this inequality is homogeneous, it suffices to consider the case $\|u\|_{p(\cdot)} = 1$. So we need to show that $\|u\|_{L \log L} \lesssim 1$. This is easily seen to be equivalent to showing that

$$\int_{\Omega} |u(x)| \log(2 + |u(x)|) dx \lesssim 1.$$

Let us choose a compact set $K \subset \mathbb{R}^n$ lying in Ω such that

$$p(x) \geq 1 + (1 + \epsilon) \frac{\log_2 \log_2(1/\delta(x))}{\log_2(1/\delta(x))},$$

for $x \in \Omega \setminus K$ for some $\epsilon > 0$ (possibly smaller than in the statement of the theorem). Let $k \geq 2$ be an integer such that

$$\frac{\log_2 k}{k} \leq \frac{\epsilon}{3(1 + \epsilon)}.$$

Set $K' = K \cup (\Omega \setminus \Omega_{2^{1-k}})$. Then K' is also compact, and so we have $p_{K'}^- > 1$ by assumption. Thus we find that

$$\begin{aligned} \int_{K'} |u(x)| \log(2 + |u(x)|) dx &\lesssim \int_{K'} \max\{1, |u(x)|^{p_{K'}^-}\} dx \\ &\leq \int_{K'} \max\{1, |u(x)|^{p(x)}\} dx \\ &\leq |\Omega| + \rho_{p(\cdot)}(u) = |\Omega| + 1. \end{aligned}$$

Therefore the integral over K' is bounded by a constant independent of u , and so it remains to estimate the integral over $\Omega \setminus K'$.

For integers $i \geq k$ we define

$$A_i = \Omega_{2^{1-i}} \setminus \Omega_{2^{-i}} = \{x \in \Omega: 2^{-i} \leq \delta(x) < 2^{1-i}\}$$

so that $|A_i| \leq C_1 2^{1-i}$. Define the exponent q by

$$q(x) = 1 + (1 + \epsilon) \frac{\log_2 i}{i}$$

on A_i for $i = k, k+1, \dots$, and note that $q(x) \leq p(x)$. Then

$$\int_{\Omega \setminus K'} |u(x)|^{q(\cdot)} dx \leq \int_{\Omega \setminus K'} \max\{1, |u(x)|^{p(\cdot)}\} dx \leq |\Omega \setminus K'| + \rho_{p(\cdot)}(u).$$

Thus we see that it suffices to show that

$$\int_{\Omega \setminus K'} |u(x)| \log(2 + |u(x)|) dx \lesssim \int_{\Omega \setminus K'} |u(x)|^{q(\cdot)} dx$$

in order to prove the claim.

The proof of the remaining estimate is based on dividing the function into small and large parts. Let $E_i \subset A_i$ be the set where $|u(x)| \leq 2^i/i^3$. Then

$$\int_{E_i} |u(x)| \log(2 + |u(x)|) dx \leq \int_{E_i} \frac{2^i}{i^3} \log\left(2 + \frac{2^i}{i^3}\right) dx \leq 2C_1 \frac{\log(2 + 2^i/i^3)}{i^3} \leq \frac{2C_1}{i^2}.$$

Let $E = \cup_i E_i$. Using the previous inequality, we find that

$$\int_E |u(x)| \log(2 + |u(x)|) dx \leq \sum_{i=k}^{\infty} \frac{2C_1}{i^2} < \infty.$$

Therefore we need not worry about the points where $|u(x)| \leq 2^i/i^3$.

For the moment we fix $i \geq k$ and denote

$$a = (1 + \epsilon) \log_2(i)/i.$$

Notice that by our assumption on k we have $3a \leq \epsilon$. We will prove that for $z \geq 2^i/i^3$ we have the inequality

$$z^a \geq \log(2 + z).$$

We start by showing that $f(z) = z^a - \log(2 + z)$ is increasing. We find that

$$f'(z) = az^{a-1} - 1/(2+z) \geq az^{a-1} - 1/z.$$

To show that $f'(z) \geq 0$ it suffices to show that $az^a \geq 1$, and, taking the smallest value of z , $a2^{ia} \geq i^{3a}$. Using the expression for a on the left-hand-side, this reduces to

$$(1 + \epsilon) \log_2(i) 2^{(1+\epsilon) \log_2(i)} \geq i^{1+3a}.$$

Throwing away the first two terms from the left-hand-side (which are greater than or equal to 1), we get the sufficient inequality $i^{1+\epsilon} \geq i^{1+3a}$, which is equivalent to our assumption $3a \leq \epsilon$. Therefore f is increasing, and it suffices to check that $f(2^i/i^3) \geq 0$:

$$(2^i/i^3)^{(1+\epsilon)\log_2(i)/i} \geq \log(2 + 2^i/i^3).$$

Since $i \geq 2$, we have $2 + 2^i/i^3 \leq 2^i$. Therefore it is enough to show that $i^{1+\epsilon}i^{-3a} \geq i \log 2$. Again we see that the condition $3a \leq \epsilon$ suffices.

Using the inequality derived in the previous paragraph we easily conclude that

$$\begin{aligned} \int_{\Omega \setminus (K \cup E)} |u(x)| \log(2 + |u(x)|) dx &\leq \int_{\Omega \setminus (K \cup E)} |u(x)| |u(x)|^{(1+\epsilon)\log_2(i)/i} dx \\ &= \int_{\Omega \setminus (K \cup E)} |u(x)|^{q(x)} dx \leq |\Omega| + \rho_{p(\cdot)}(u). \end{aligned}$$

Therefore $u \in L \log L$, which by classical theory implies that $\mathcal{M}: L^{p(\cdot)}(\Omega) \rightarrow L^1(\Omega)$ is bounded for a bounded set Ω . \square

The next theorem uses a standard example to show that the sufficient growth condition of the previous theorem is nearly optimal.

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\mathcal{M}^{n-1}(\partial\Omega) < \infty$. Suppose also that $\epsilon > 0$ is such that*

$$p(x) \leq 1 + (1 - \epsilon) \frac{\log \log(1/\delta(x))}{\log(1/\delta(x))} + o\left(\frac{\log \log(1/\delta(x))}{\log(1/\delta(x))}\right).$$

Then $L^{p(\cdot)}(\Omega) \not\hookrightarrow L \log L(\Omega)$ and $\mathcal{M}: L^{p(\cdot)}(\Omega) \rightarrow L^1(\Omega)$ is not bounded.

Proof. By Lemma 3.1 there exists a constant $C_1 > 0$ such that

$$\frac{1}{C_1} \delta \leq |\Omega_\delta| \leq C_1 \delta$$

for all $\delta > 0$. Since we need to construct a counter-example, we can restrict our attention to a neighborhood of the boundary by setting $u(x) = 0$ outside our neighborhood. Specifically, let $K \subset \Omega$ be compact in \mathbb{R}^n such that

$$p(x) \leq 1 + (1 - \epsilon) \frac{\log_2 \log_2(1/\delta(x))}{\log_2(1/\delta(x))}$$

for some $\epsilon > 0$ and every $x \in \Omega \setminus K$.

As in the previous theorem, we define $A_i = \Omega_{2^{1-i}} \setminus \Omega_{2^{-i}}$. We fix integers $k \geq \log_2(2/\delta_1)$ and $l \geq \log_2(2C_1^2)$. Consider a set of l consecutive indices greater than k , say $j+1, \dots, j+l$. Suppose that $|A_i| \leq C_2 2^{-i}$ for all these indices. Then

$$\frac{1}{C_1} 2^{-j} \leq |\Omega_{2^{-j}}| \leq |A_{j+1}| + \dots + |A_{j+l}| + |\Omega_{2^{-j-l}}| \leq C_2 2^{-j} + C_1 2^{-j-l}.$$

Hence

$$\frac{1}{2C_1} \leq \frac{1}{C_1} - C_1 2^{-l} \leq C_2.$$

This means that in any set of l consecutive indices larger than k there is at least one for which $|A_i| \geq 2^{-i-1}/C_1$. We can thus choose a set J of indices greater than k with the following two properties:

1. for every m there is an element in J between $k + lm$ and $k + l(m + 1)$;
2. $|A_i| \geq 2^{-i-1}/C_1$ for every $i \in J$.

Let us define

$$q(x) = 1 + (1 - \epsilon) \frac{\log_2 i}{i}$$

for $x \in A_i, i = k, k+1, \dots$. By the monotony of the norm [21, Theorem 2.8], we find that $\|u\|_{p(\cdot)} \lesssim \|u\|_{q(\cdot)}$, for every $u \in L^{p(\cdot)}(\Omega)$ which equals 0 on K . Therefore it suffices to construct a function $u \in L^{q(\cdot)}(\Omega)$ which equals 0 on K and which does not belong to $L \log L(\Omega)$. Let χ_i denote the characteristic function of A_i and define the function $u: \Omega \rightarrow \mathbb{R}$ by

$$u(x) = \sum_{i=k}^{\infty} \frac{2^i}{i^2} \chi_i(x).$$

We start by showing that $u \notin L \log L(\Omega)$. We calculate

$$\begin{aligned} \int_{\Omega} u(x) \log(2 + u(x)) dx &= \sum_{i=k}^{\infty} \int_{A_i} \frac{2^i}{i^2} \log\left(2 + \frac{2^i}{i^2}\right) dx \\ &\geq \frac{1}{2C_1} \sum_{i \in J} \frac{\log\left(2 + \frac{2^i}{i^2}\right)}{i^2} \geq \frac{1}{2C_1} \sum_{i \in J} \frac{i/2}{i^2}. \end{aligned}$$

But the last sum is clearly divergent, so $u \notin L \log L$. To show that $u \in L^{q(\cdot)}$ we observe that

$$\begin{aligned} \int_{\Omega} |u(x)|^{q(x)} dx &= \sum_{i=k}^{\infty} \int_{A_i} \left(\frac{2^i}{i^2}\right)^{q(x)} dx = \sum_{i=k}^{\infty} \left(\frac{2^i}{i^2}\right)^{1+(1-\epsilon)\log_2(i)/i} \int_{A_i} dx \\ &\leq 2C_1 \sum_{i=k}^{\infty} \frac{2^i}{i^2} i^{1-\epsilon} i^{-2(1-\epsilon)\log_2(i)/i} 2^{-i} = 2C_1 \sum_{i=k}^{\infty} i^{-1-\epsilon-2(1-\epsilon)\log_2(i)/i} < \infty. \end{aligned}$$

Thus $u \in L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}$ and so we have shown that $L^{p(\cdot)} \not\hookrightarrow L \log L$. In particular this implies that $\mathcal{M}: L^{p(\cdot)} \rightarrow L^1$ is not bounded. \square

We can also use the previous theorems also in the case when $p(x) = 1$ at some points in the domain.

Corollary 2.3 *Let Ω be an open set and $p: \Omega \rightarrow [1, \infty)$ be a variable exponent. Let $E = \{x \in \Omega: p(x) = 1\}$ and $\Omega' = \Omega \setminus E$.*

- If E is closed, $|E| = 0$ and Ω' satisfies the conditions of Theorem 2.1, then $L^{p(\cdot)}(\Omega) \hookrightarrow L \log L(\Omega)$.
- If E is closed, $|E| = 0$ and Ω' satisfies the conditions of Theorem 2.2, then $L^{p(\cdot)}(\Omega) \not\hookrightarrow L \log L(\Omega)$.
- If $|E| > 0$, then $L^{p(\cdot)}(\Omega) \not\hookrightarrow L \log L(\Omega)$.

Proof. Suppose first that $|E| > 0$. Let $x_0 \in E$ be a Lebesgue point of E , i.e. a point such that

$$\lim_{r \rightarrow 0} \frac{|E \cap B(x_0, r)|}{|B(x_0, r)|} = 1.$$

(Almost every point of E is a Lebesgue point (e.g. [26, Proposition I.2.1]), so such an x_0 exists.) Then the standard example times the characteristic function of E ,

$$u(x) = \frac{1}{|x - x_0| [\log(|x - x_0|)]^2} \chi_E$$

belongs to $L^1(\Omega)$ (and hence to $L^{p(\cdot)}(\Omega)$) but not to $L \log L(\Omega)$. The calculations are similar to those in Theorem 2.2 and are thus omitted.

Since $L^{p(\cdot)}(\Omega)$ and $L^{p(\cdot)}(\Omega \setminus E)$ are isometric as Banach spaces (and similarly for $L \log L$), the claims when $|E| = 0$ are clear. \square

Remark 2.4 Suppose that p is bounded away from 1 except in neighborhoods of a single point in Ω (say the origin). Then one can argue similarly to what was done in the proofs of the previous theorems to show that

$$p(x) = 1 + \frac{1 \pm \epsilon \log \log(1/|x|)}{n \log(1/|x|)} + o\left(\frac{\log \log(1/|x|)}{\log(1/|x|)}\right)$$

is a necessary (–) or sufficient (+) condition for the boundedness of the maximal operator. Based on this one might conjecture that the critical growth rate of p with set $E = \{x \in \Omega: p(x) = 1\}$ is

$$p(x) = 1 + \frac{1 \pm \epsilon}{n - \dim(E)} \frac{\log \log(1/d(x, E))}{\log(1/d(x, E))},$$

where $\dim(E)$ denotes the dimension of E (in some appropriate sense).

Remark 2.5 It is of interest to note that the condition (1.2) is almost precisely the same as that given in [13] for the continuity and boundedness of Sobolev functions. More precisely, if we change the first 1 in (1.2) to an n , then we get the condition of [13]. This result is based on estimating the capacity of a variable exponent capacity, and is as such quite different from the approach in this paper. The order of growth

$$\frac{\log \log(1/|x|)}{\log(1/|x|)}$$

also appeared in [15], an article presenting a counter-example to density of smooth functions in variable exponent Sobolev space and in [10] which deals with Riesz potentials. Therefore it seems that this type of growth, which is just slightly greater than allowed by logarithmic Hölder continuity, is also of general importance in variable exponent spaces.

Another remark is that in the study of variable exponent spaces the condition $1 < p < \infty$ from the classical setting usually translates into

$$1 < \operatorname{ess\,inf} p(x) \leq \operatorname{ess\,sup} p(x) < \infty.$$

Such is the case for instance for reflexivity of Lebesgue spaces (see [21]). It is therefore interesting to see that this is not always the case. As far as the author knows, the results in the present paper are the first in which the classical condition $1 < p$ translates into $1 < p(x)$ for almost every x (plus a growth condition).

3 A geometric lemma

The following simple lemma is probably not new, but since it is also not ubiquitous, a proof is included for completeness. Let us recall the definition of the Minkowski measure. Let $G \subset \mathbb{R}^n$ and $\delta, s > 0$. We define

$$\mathcal{M}_\delta^s(G) = \inf_{\mathcal{B}} \sum_{B \in \mathcal{B}} \operatorname{diam}(B)^s = \delta^s \#\mathcal{B},$$

where the infimum is taken over all coverings \mathcal{B} of G by balls of diameter δ . The *Minkowski measure* is defined as the limit

$$\mathcal{M}^s(G) = \lim_{\delta \rightarrow 0} \mathcal{M}_\delta^s(G).$$

Note that the Minkowski measure of a set is always larger than or equal to the Hausdorff measure of the same set.

Lemma 3.1 *Let Ω be bounded. The condition $|\Omega_\delta| \approx \delta$ for sufficiently small δ is equivalent to the condition $\mathcal{M}^{n-1}(\partial\Omega) < \infty$.*

Proof. Fix $x \in \Omega$. Then $B(x, \delta(x)) \subset \Omega$. Denote $r = \delta(x)/5$. Let \mathcal{O} be the component of $\Omega \setminus \overline{\Omega}_r$ which contains x . We easily conclude that $B(x, 3r) \subset \mathcal{O}$. Thus it follows that $\mathcal{M}^{n-1}(\partial\mathcal{O}) \geq \mathcal{M}^{n-1}(\partial B(x, 3r))$.

Notice that Ω_{2r} is an r -neighborhood of $\partial\mathcal{O}$. Let \mathcal{B} be a family of balls of diameter r covering $\partial\mathcal{O}$. By a standard covering theorem [16, Theorem 1.2] we choose a subfamily \mathcal{B}' of disjoint balls intersecting $\partial\mathcal{O}$ such that

$$\partial\mathcal{O} \subset \bigcup_{B \in \mathcal{B}'} 5B.$$

We find that $B \subset \Omega_{2r}$ for $B \in \mathcal{B}'$, because every such B is contained in an r -neighborhood of $\partial\mathcal{O}$. Then, using the definition of the Minkowski measure, and the inclusion $B \subset \Omega_{2r}$ (and the disjointness), we find that

$$\mathcal{M}_{5r}^{n-1}(\partial\mathcal{O}) \leq \sum_{B \in \mathcal{B}'} (5 \operatorname{diam}(B))^{n-1} = \frac{5^{n-1}}{r} \sum_{B \in \mathcal{B}'} \frac{|B|}{|B(1/2)|} \leq \frac{2 \cdot 5^{n-1}}{|B(1/2)|} \frac{|\Omega_{2r}|}{2r}.$$

From this we see that $\mathcal{M}^{n-1}(\partial\Omega) < \infty$, if $\frac{|\Omega_{2r}|}{2r}$ is bounded as $r \rightarrow 0$. If we assume that $\mathcal{M}^{n-1}(\partial\Omega) < \infty$, then this inequality allows us to conclude that $|\Omega_{2r}| \gtrsim 2r$ as $r \rightarrow 0$.

It remains to show that $\mathcal{M}^{n-1}(\partial\Omega) < C_1 < \infty$ implies that $|\Omega_\delta| \lesssim \delta$. The assumption means that for sufficiently small δ we can choose a covering \mathcal{B} of $\partial\Omega$ by balls of diameter δ such that

$$\sum_{B \in \mathcal{B}} \operatorname{diam}(B)^{n-1} < C_1.$$

Define $\mathcal{B}' = \{2B : B \in \mathcal{B}\}$. Then

$$\left| \bigcup_{B \in \mathcal{B}'} B \right| \leq |B(1/2)| 2\delta \sum_{B \in \mathcal{B}'} \operatorname{diam}(B)^{n-1} < 2^n |B(1/2)| C_1 \delta.$$

Let $z \in \partial\Omega$. Then there exists $B \in \mathcal{B}$ such that $z \in B$. But then $B(z, \delta) \subset 2B \in \mathcal{B}'$. This means that $\Omega_\delta \subset \cup_{B \in \mathcal{B}'} B$, so that $|\Omega_\delta| \leq |B(1/2)| C_1 \delta$. \square

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