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# ON THE EXISTANCE OF MINIMIZERS OF THE VARIABLE EXPONENT DIRICHLET ENERGY INTEGRAL

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Abstract. In this note we consider the Dirichlet energy integral in the variable exponent case under minimal assumptions on the exponent. First we show that the Dirichlet energy integral always has a minimizer if the boundary values are in $L^\infty$. Second, we give an example which shows that if the so-called "jump-condition", known to be sufficient, is violated, then a minimizer need not exist for unbounded boundary values.

1. **Introduction.** After decades of sporadic research efforts, the field of variable exponent function spaces entered into a phase of great activity starting in the late 1990's. The researchers were motivated both by inherent interest in developing techniques that work in non-translation invariant, but otherwise very natural, function spaces, and by interesting applications (see, e.g., [3, 24]). For a survey of the present state of the field, the reader is referred to the article [8] which also includes a comprehensive bibliography of over a hundred titles on the subject. Two questions can be singled out as having received most attention: integral operators (Hardy–Littlewood, Riesz, other singular integrals, see, e.g., [6, 7, 22]) and $p(x)$-Laplacian equations with Dirichlet boundary values. The latter is the subject matter of this paper.

In the classical (Laplacian) Dirichlet boundary value problem we are given a domain $\Omega$ in $\mathbb{R}^n$ and a continuous function $w : \partial\Omega \to \mathbb{R}$. The problem is to find a continuous function $u : \overline{\Omega} \to \mathbb{R}$ so that the Laplace equation $-\Delta u = 0$ is satisfied on $\Omega$ and $u = w$ on $\partial\Omega$. By Weyl's lemma, such a $u$ is always $C^2$ in $\Omega$, and hence the problem may be considered in the classical sense.

The $p$-Dirichlet boundary value problem for fixed $p \in (1, \infty)$ is to find a continuous function $u$ on $\overline{\Omega}$ so that the $p$-Laplace equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0 \qquad (1)$$

is satisfied on $\Omega$ and $u = w$ on $\partial\Omega$. When $p \neq 2$, equation (1) is nonlinear and it must be understood in the weak sense, i.e. as the minimization of an energy integral: the

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$p$-Laplace equation (1) is the Euler–Lagrange equation for the variational integral

$$\int_\Omega |\nabla u(x)|^p \, dx, \tag{2}$$

so smooth minimizers of this integral satisfy (1). We call (2) the $p$-Dirichlet energy integral on $\Omega$. In the non-linear case also the boundary values must be understood in a weak sense, i.e. we look for a minimizer $u \in W^{1,p}(\Omega)$ with the property that $u - w \in W_0^{1,p}(\Omega)$, where $w \in W^{1,p}(\Omega)$ gives the boundary values.

The generalization of (2) to the variable exponent case is immediate: we want to minimize

$$\int_\Omega |\nabla u(x)|^{p(x)} \, dx, \tag{3}$$

subject to the appropriate boundary conditions. Early work on this problem was done by Zhikov [25, 26] and Alkhutov [2]. This work was carried on by Acerbi & Mingione and their collaborators, e.g. [1, 4, 5], and, independently, by Fan and collaborators, e.g. [9, 10]. Much of this work was directed at proving regularity results or generalizing other classical results in the case when the variable exponent satisfies certain quite strong regularity assumption (e.g. log-Hölder or $\alpha$-Hölder continuity). Harjulehto, Hästö, Koskenoja and Varonen [16] showed that it is possible to derive existence and uniqueness results under much weaker conditions on the exponent, see Theorem 1. To shed further light on what assumptions are really necessary, Harjulehto, Hästö and Koskenoja [13] studied the Dirichlet problem in one dimension. In that simple case, the minimizer exists, except possibly when the exponent approaches 1. Therefore, it is natural to look for existence results under weaker assumptions also in higher dimensions.

In this note we complement the results of [16] in two ways: first, we show that the Dirichlet energy integral always has a minimizer if the boundary values are in $L^\infty$ (Theorem 2). Note that this in some sense corresponds to the one-dimensional case, as the boundary values are necessarily bounded in one dimension. Second, we give an example which shows that if the boundary values are allowed to be unbounded, and the "jump-condition" from [16] is violated, then a minimizer need not exist (Theorem 3).

**Definitions.** We denote by $\mathbb{R}^n$ the Euclidean space of dimension $n \geqslant 2$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote the open ball with center $x$ and radius $r$ by $B(x, r)$, and for a real number $K > 0$ we denote by $KB(x, r)$ the ball $B(x, Kr)$. By $C$ we denote a generic constant, i.e. a constant whose value may change from appearance to appearance.

Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $p : \Omega \to [1, \infty)$ be a measurable function, called the *variable exponent* on $\Omega$. We write $p_G^+ = \operatorname{ess\,sup}_{x \in G} p(x)$ and $p_G^- = \operatorname{ess\,inf}_{x \in G} p(x)$, where $G \subset \Omega$, and abbreviate $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$.

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\varrho_{p(\cdot)}(\lambda u) = \int_\Omega |\lambda u(x)|^{p(x)} \, dx < \infty$ for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula

$$\|u\|_{p(\cdot)} = \|u\|_{L^{p(\cdot)}(\Omega)} = \inf \big\{ \lambda > 0 \colon \varrho_{p(\cdot)}(u/\lambda) \leqslant 1 \big\}.$$

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of all $u \in L^{p(\cdot)}(\Omega)$ such that the absolute value of the distributional gradient $\nabla u$ is in $L^{p(\cdot)}(\Omega)$. The norm $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. For the basic theory of variable exponent spaces see [23].

2. **Existence.** We start by reviewing the results from [16]. First of all, we need to define zero boundary value spaces. Next we recall the main existence theorem from [16] which states that a certain "jump-condition" is sufficient. Then we show how this result can be improved in the case when the boundary value function is in $L^\infty$.

**Zero boundary values.** If $C^\infty(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$, then it is easy to define the space of functions with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, namely, it is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. However, in general smooth functions are not dense in the variable exponent Sobolev space, see [18, 25]. When this is the case, it is possible to adapt a different definition, inspired by work on Sobolev spaces on metric spaces (as in [19]). This was the path taken in [16].

In this approach we need a suitable capacity, like the Sobolev capacity from [15, Section 3]. The Sobolev $p(\cdot)$-capacity of $E$ is defined by

$$C_{p(\cdot)}(E) = \inf_u \int_{\mathbb{R}^n} |u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\, dx,$$

where the infimum is taken over functions $u$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$ which equal 1 in an open set containing $E$. If $1 < p^- \leqslant p^+ < \infty$, then $C_{p(\cdot)}$ is an outer measure and a Choquet capacity [15, Corollaries 3.3 and 3.4].

Recall that a function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be $p(\cdot)$-*quasicontinuous* if for every $\varepsilon > 0$ there exists an open set $G$ with $C_{p(\cdot)}(G) < \varepsilon$ such that $u|_{\mathbb{R}^n\setminus G}$ is continuous in $\mathbb{R}^n \setminus G$.

**Definition 1.** We say that the function $u$ belongs to the space $W_0^{1,p(\cdot)}(\Omega)$ if there exists a $p(\cdot)$-quasicontinuous function $\tilde{u} \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $u = \tilde{u}$ almost everywhere in $\Omega$ and $\tilde{u} = 0$ in $\mathbb{R}^n \setminus \Omega$. The set $W_0^{1,p(\cdot)}(\Omega)$ is endowed with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\tilde{u}\|_{W^{1,p(\cdot)}(\mathbb{R}^n)}.$$

If smooth functions are dense, then this definition gives the same space as taking the closure of $C_0^\infty$ [16, Theorem 3.3]. If $1 < p^- \leqslant p^+ < \infty$, then $W_0^{1,p(\cdot)}(\Omega)$ is a Banach space [16, Theorem 3.1]. The relationship between different concepts of zero boundary value spaces has been studied by Harjulehto in [11].

**The results.** If $p^+ < \infty$ and if there exists $\delta > 0$ such that for every $x \in \Omega$ either

$$p^-_{B(x,\delta)} \geqslant n \qquad \text{or} \qquad p^+_{B(x,\delta)} \leqslant \frac{n\, p^-_{B(x,\delta)}}{n - p^-_{B(x,\delta)}}$$

holds, then the variable exponent $p$ is said to satisfy the *jump condition* in $\Omega$. Roughly, the jump condition guarantees that $p$ does not jump too much locally in $\Omega$. Note that if $\Omega$ is bounded and if $p$ is uniformly continuous, then $p$ satisfies the jump condition in $\Omega$.

Let us define an "energy integral" operator on the Sobolev space by $I_{p(\cdot)}(u) = \varrho_{p(\cdot)}(|\nabla u|)$. The following is the main theorem from [16]:

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $w \in W^{1,p(\cdot)}(\Omega)$. Assume that $p$ satisfies the jump condition in $\Omega$ and that $1 < p^- \leqslant p^+ < \infty$. Then there exists a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that*

$$I_{p(\cdot)}(u + w) = \inf_{v \in W_0^{1,p(\cdot)}(\Omega)} I_{p(\cdot)}(v + w).$$

We next show that the jump condition is not needed when we have some additional control of the boundary values. As in [16], we rely on the following basic result:

**Lemma 1** (Theorem 2.1, [21]). *Let $\mathcal{B}$ be a reflexive Banach space. If $I : \mathcal{B} \to [0, \infty)$ is a convex, lower semicontinuous and coercive operator, then there is an element in $\mathcal{B}$ which minimizes $I$.*

Recall that the operator $I$ is said to be *convex* if the inequality $I(tu+(1-t)v) \leqslant tI(u)+(1-t)I(v)$ is satisfied for all $t \in [0, 1]$ and every pair $u, v \in \mathcal{B}$. The operator is *lower semicontinuous* if $I(u) \leqslant \liminf_{i\to\infty} I(u_i)$ whenever $u_i$ is a sequence of elements in $\mathcal{B}$ converging to $u$, and *coercive* if $I(u_i) \to \infty$ whenever $\|u_i\|_{\mathcal{B}} \to \infty$.

A glance at the proof of Theorem 1 shows that no assumptions are needed on $p$ when proving convexity and lower semicontinuity. Therefore, we need only worry about coercivity, when extending the result. In our setting coercivity means that $\|u\|_{p(\cdot)} \to \infty$ implies that $\|\nabla u\|_{p(\cdot)} \to \infty$. In other words, we need some sort of a Poincaré inequality. Although we do not get a real Poincaré inequality under the weak assumptions of the following theorem (see [12, Section 2]), we can get what we need by reducing to the $W^{1,1}$-Poincaré inequality, as in [14, Lemma 4.1].

**Theorem 2.** *Let $\Omega$ be bounded and $1 < p^- \leqslant p^+ < \infty$. Suppose that $v \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Then $I_{p(\cdot)}$ has a unique minimizer in the set*

$$U = \{u\colon u - v \in W_0^{1,p(\cdot)}(\Omega)\}.$$

*Proof.* Let $u \in U$ and denote $a = \|v\|_\infty$. Let $u_a$ be the function $u$ which has been cut-off at $-a$ and $a$, i.e.

$$u_a(x) = \min\{a, \max\{-a, u(x)\}\}.$$

Since $a$ is the largest value taken by $|v|$, it is easy to see that also $u_a - v \in W_0^{1,p(\cdot)}(\Omega)$. Moreover, $\nabla u_a(x)$ equals $\nabla u(x)$ or 0 at almost every point in $\Omega$ [15, Section 2], hence $|\nabla u_a| \leqslant |\nabla u|$ and so $I_{p(\cdot)}(u_a) \leqslant I_{p(\cdot)}(u)$. It follows that it suffices to look for minimizers of $I_{p(\cdot)}$ in the set $U_a = \{u_a\colon u \in U\}$.

But we easily conclude by the $W^{1,1}$-Sobolev–Poincaré inequality that

$$\int_\Omega |u(x)|^{p(x)}\,dx \leqslant a^{p^+-1}\int_\Omega |u(x)|\,dx \leqslant C\,\mathrm{diam}(\Omega)\,a^{p^+-1}\int_\Omega |\nabla u(x)|\,dx$$

for $u \in U_a$. Thus $\|u\|_{p(\cdot)} \leqslant C\|\nabla u\|_1 \leqslant C\|\nabla u\|_{p(\cdot)}$, where the second inequality is just the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$, proved in [23]. This means that the operator $I_{p(\cdot)}$ is coercive on $U_a$, which is a convex subset of a reflexive Banach space. As convexity and lower semicontinuity are clear, existence follows by Lemma 1.

The uniqueness follows easily since $I_{p(\cdot)}$ is strictly convex, see [20, Theorem 5.27] or [16, Theorem 5.3]. □

It is perhaps natural to think that the previous result could be combined with a monotonicity argument to remove the assumption that the boundary values be in $L^\infty$. This, however, turns out not to be the case, as we will show in the next section.

3. **Non-existence.** In this section we give an example of a domain and boundary values such that the Dirichlet energy integral does not have a minimizer.

We will need the variational capacity for a fixed exponent $q$:

$$\operatorname{cap}_q(K,\Omega) = \inf_u \varrho_q(|\nabla u|),$$

where $K \subset \Omega$ is compact and the infimum is taken over functions $u$ in $W^{1,q}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ which equal 1 on $K$ and 0 on $\mathbb{R}^n \setminus \Omega$. Recall the formula for the variational capacity of an annulus:

$$\operatorname{cap}_q(\overline{B(0,r)}, B(0,R)) = m_{n-1}(S^{n-1}) \left|\frac{q-n}{q-1}\right|^{q-1} \left|R^{(q-n)/(q-1)} - r^{(q-n)/(q-1)}\right|^{1-q};$$

here $q$ is a fixed exponent not equal to the dimension $n$ [20, Example 2.12, p. 35]. If we set $r = \epsilon R$, then we get the convenient formula

$$\operatorname{cap}_q(\overline{B(0,\epsilon R)}, B(0,R)) = C\left|1 - \epsilon^{(q-n)/(q-1)}\right|^{1-q} R^{n-q}.$$

**The basic construction.** Fix $p \in (1,n)$ and $q \in (p^*, n)$. Here $p^* = np/(n-p)$ is the Sobolev conjugate exponent.

Let $r_i = 2^{-i}$ and let $(\epsilon_i)$ be a sequence of real numbers in $(0,1)$ which will be specified later. We define a sequence of balls by $B_i = B(2r_ie_1, r_i)$, where $e_1$ is a unit vector. Our domain is the union of the annuli $A_i = (2B_i) \setminus \overline{\epsilon_i B_i}$, $\Omega = \cup_i A_i$. We split the annuli into two parts, $O_i = (2B_i) \setminus \overline{B_i}$ where the exponent is set to $p$, and $I_i = B_i \setminus \overline{\epsilon_i B_i}$ where the exponent is set to $q$. We use the following abbreviations for the corresponding capacities:

$$i_i = \operatorname{cap}_q(\overline{\epsilon_i B_i}, B_i) = C|\epsilon_i^{(q-n)/(q-1)} - 1|^{1-q} r_i^{n-q}$$

and

$$o_i = \operatorname{cap}_p(\overline{B_i}, 2B_i) = C r_i^{n-p}.$$

The boundary values are given by a sequence $(v_i)$ of constants on the inner boundary $\partial(\epsilon_i B_i)$ and 0 on all outer boundaries $\partial(2B_i)$.

The idea of the construction is to choose the parameters $\epsilon_i$ and $v_i$ in such a way that minimizing the modular of the gradient leads to a function which is approximately equal to $v_i$ on the inner annulus. Moreover, the values of $v_i$ are chosen so, that such a function has infinite $L^{p(\cdot)}$ norm.

Let us first check that we can construct some function in $W^{1,p(\cdot)}(\Omega)$ which has boundary values $(v_i)$ and 0. Let $u_i \in W^{1,p(\cdot)}(\Omega)$ be the solution of the $q$-Dirichlet problem in the annulus $I_i$, with inner and outer boundary values $v_i$ and 0, respectively, and extend it by these constants to $\epsilon_i B_i$ and $\mathbb{R}^n \setminus B_i$. By definition of $i_i$ as the capacity (and a simple scaling), we have

$$\varrho_{L^{p(\cdot)}(\Omega)}(|\nabla u_i|) = \varrho_{L^q(B_i)}(|\nabla u_i|) = v_i^q i_i.$$

The function $u$ is defined as the sum of all functions $u_i$. So if we choose our sequence so that

$$\varrho_{L^q(\Omega)}(|\nabla u|) = \sum v_i^q i_i < \infty, \tag{4}$$

then $|\nabla u| \in L^{p(\cdot)}(\Omega)$. The function $u$ has zero boundary values in $B_i$. Therefore it follows from the $L^q$-Poincaré inequality that the $L^{p(\cdot)}$-norm of $u$ is dominated by the $L^{p(\cdot)}$-norm of its gradient, so $u \in W^{1,p(\cdot)}(\Omega)$. Thus (4) is sufficient to guarantee that $(v_i)$ and 0 define proper boundary values.

Suppose that there exists a minimizer for our Dirichlet problem, and denote it by $u$. It is clear that $u$ is radially symmetric and radially decreasing in each annulus

$A_i$. Obviously, $u$ is minimizing on both $I_i$ and $O_i$. Supposing that $u$ takes the value $x$ on $\partial B_i$, we see that

$$\varrho_{L^{p(\cdot)}(B_i)}(|\nabla u|) = (v_i - x)^q i_i + x^p o_i =: F(x).$$

Since $u$ is assumed to be minimizing, $x$ has to be a minimum of $F$. Moreover, $F$ is a convex function with a unique minimum in $(0, v_i)$. Let us choose the parameters so that the minimum occurs at $x = v_i/2$, in other words, we require that $F'(v_i/2) = 0$, which is equivalent with

$$q(v_i/2)^q i_i = p(v_i/2)^p o_i. \tag{5}$$

Since $u$ is radially decreasing, this implies that $u \geqslant v_i/2$ on $I_i$, hence

$$\varrho_{L^{p(\cdot)}(I_i)}(u) \geqslant C v_i^q r_i^n.$$

We will choose $(v_i)$ so that

$$\varrho_{p(\cdot)}(u) = \sum \varrho_{L^{p(\cdot)}(I_i)}(u) \geqslant C \sum v_i^q r_i^n = \infty, \tag{6}$$

which contradicts the assumption that $u$ lies in $L^{p(\cdot)}(\Omega)$.

So it remains to show that we can choose sequences $(\epsilon_i)$ and $(v_i)$ such that (4), (5) and (6) hold. Let us simply choose $v_i$ so that $v_i^q r_i^n = 1$. Then certainly (6) holds. Next, we use equation (5), $v_i = r_i^{-n/q}$ and $o_i = C r_i^{n-p}$ which gives us

$$\sum v_i^q i_i = \frac{p 2^q}{q 2^p} \sum v_i^p o_i = C \sum r_i^{n-p-np/q} = C \sum 2^{-i(n-p-np/q)} < \infty.$$

Here the last inequality follows since $q > p^*$, which implies that $n - p - np/q > 0$. Thus (4) holds, so it remains only to choose the numbers $\epsilon_i$ so that (5) holds. Condition (5) can be written as

$$|\epsilon_i^{(q-n)/(q-1)} - 1|^{1-q} r_i^{n-q} r_i^{-n} = C r_i^{n-p} r_i^{-np/q},$$

or, equivalently,

$$|\epsilon_i^{(q-n)/(q-1)} - 1|^{1-q} = C r_i^{n+q-p-np/q}. \tag{7}$$

Since the exponent of $\epsilon_i$ is negative, we see that the left hand side can take any value in $(0, \infty)$, in particular, there exists an appropriate $\epsilon_i$ for any choice of $r_i$. This completes the construction.

**Modifications.** So far we have constructed an open bounded set in which the exponent takes on two values. Let us first note that the exponent can easily be modified to a continuous (or even $C^\infty$) exponent in the domain by letting the exponent decrease smoothly from $q$ to $p$ in the annulus $(\frac{3}{2}B_i) \setminus B_i$. However, obviously, this exponent is not uniformly continuous, and indeed, as was noted before Theorem 1, uniform continuity is sufficient for existence. Thus we see that the gap between the necessary and sufficient conditions is very small in so far as the regularity of the exponent is concerned.

A second issue is that the set constructed so far is not connected. However, this is easy to remedy: just open up a small canal between adjacent balls. Since the original "would-be" minimizer was greater than $u_i/2$ on the inner annulus, it follows that the canal can be chosen so small that the minimizer would be at least $u_i/3$ on the inner annulus, which gives the same contradiction as before.

We summarize this construction in the following theorem.

**Theorem 3.** *Let $n \geq 3$, $q_1 \in (1, n/(n-1))$ and $q_2 \in (q_1^*, n)$. Then there exist a smooth exponent $p$ with $p^- = q_1$ and $p^+ = q_2$, a bounded domain $\Omega$ and a boundary value function $v \in W^{1,p(\cdot)}(\Omega)$ such that $I_{p(\cdot)}$ does not have a minimizer in the set $\{u \colon u - v \in W_0^{1,p(\cdot)}(\Omega)\}$.*

Note that if we had $q_2 \leqslant q_1^*$ in the previous theorem, then a minimizer would always exist, by Theorem 1, so in this sense our example (and Theorem 1) is the best possible.

4. **Discussion and questions.** The example in the previous section (with the modifications) featured a bounded domain with a continuous exponent in which the variable exponent Dirichlet energy integral has no minimizer. But the domain has the nasty feature that the boundary consists of lots of isolated spheres. One may therefore ask whether there exists a domain with connected boundary which features non-existence of minimizers? There does not seem to be any simple way of modifying the example to allow for this.

A second short-coming of our example is the requirement that $q < n$. It seems to be possible to extend the example to the case $q = n$, but beyond that we run into serious problems, essentially because we cannot fully control the capacity of the inner annulus. The reason for the lack of control in the $q > n$ case is that the capacity of a single point is already has positive, so we cannot get small capacities for our annulus (specifically, we cannot satisfy (7), since the exponent of $\epsilon_i$ is positive). In particular, since $1^* = 2$ for $n = 2$, it follows that the example does not work at all in the plane, so an especially interesting question is: does there exists a planar domain such that the Dirichlet energy integral does not have a minimizer?

It is also interesting to note that the exponent in the example was essentially radial. Thus (the proof of) Lemma 4.6 from [18] implies that continuous functions are dense in the Sobolev space we are considering. So it appears that density of continuous functions plays no role in the existence question. In contrast to this, the example given in [17] suggested that the density of continuous functions might be intimately related to the continuity of the minimizer when it exists.

There is one assumption in the existence theory that has so far not gotten the attention it deserves. It is the condition $1 < p^-$. From a technical point of view, this is a very convenient assumption, as it allows us to work in a reflexive space. However, it is not clear that this condition is really needed for the results (when the minimizer is understood in the proper space (whatever that may be)). Moreover, for applications of image restoration (as suggested in [3]), it is critical that the exponent be allowed to equal 1 (so-called total variation smoothing). In [3] this problem is circumvented by having the otherwise smooth exponent jump to 1 once it reaches a certain threshold. However, it would obviously be preferable not to be forced to take such *ad hoc* measures.

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## REFERENCES

[1] E. Acerbi and G. Mingione: *Gradient estimates for the $p(x)$–Laplacean system*, J. Reine Angew. Math., **584** (2005), 117–148.

[2] Yu. A. Alkhutov: *The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition*, (Russian) Differential Equations, **33** (1997), no. 12, 1653–1663 (1998). [Translation of Differ. Uravn., **33** (1997), no. 12, 1651–1660, 1726.]

[3] Y. Chen, S. Levine and R. Rao: *Functionals with $p(x)$-growth in image processing*, Duquesne University, Department of Mathematics and Computer Science Technical Report #04-01. Available at `www.mathcs.duq.edu/~sel/CLR05SIAPfinal.pdf`

[4] V. Chiadò Piat and A. Coscia: *Hölder continuity of minimizers of functionals with variable growth exponent*, Manuscripta Math., **93** (1997), no. 3, 283–299.

[5] A. Coscia and G. Mingione: *Hölder continuity of the gradient of $p(x)$-harmonic mappings*, C. R. Acad. Sci. Paris, **328** (1999), 363–368.

[6] D. Cruz-Uribe, A. Fiorenza, J. M. Martell and C. Pérez: *The boundedness of classical operators in variable $L^p$ spaces*, Ann. Acad. Sci. Fenn. Math., to appear.

[7] L. Diening and M. Růžička: *Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math., **563** (2003), 197–220.

[8] L. Diening, P. Hästö and A. Nekvinda: *Open problems in variable exponent Lebesgue and Sobolev spaces*, FSDONA04 Proceedings (Drabek and Rakosnik (eds.); Milovy, Czech Republic, 2004), 38–58, Academy of Sciences of the Czech Republic, Prague, 2005.

[9] X.-L. Fan and X. Han: *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $R^N$*, Nonlinear Anal., **59** (2004), no. 1–2, 173–188.

[10] X.-L. Fan, Q.-H. Zhang and D. Zhao: *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl., **302** (2005), 306–317.

[11] P. Harjulehto: *Variable exponent Sobolev spaces with zero boundary values*, preprint (2004).

[12] P. Harjulehto and P. Hästö: *A capacity approach to Poincaré inequalities and Sobolev imbedding in variable exponent Sobolev spaces*, Rev. Mat. Complut., **17** (2004), no. 1, 129–146.

[13] P. Harjulehto, P. Hästö and M. Koskenoja: *The Dirichlet energy integral on intervals in variable exponent Sobolev spaces*, Z. Anal. Anwend., **22** (2003), no. 4, 911–923.

[14] P. Harjulehto, P. Hästö and M. Koskenoja: *Properties of capacities in variable exponent Sobolev spaces*, preprint (2005).

[15] P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen: *Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$*, J. Funct. Spaces Appl., **1** (2003), 17–33.

[16] P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen: *The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values*, Potential Anal., (probably) to appear.

[17] P. Hästö: *Counter-examples of regularity in variable exponent Sobolev spaces*, The $p$-Harmonic Equation and Recent Advances in Analysis (Manhattan, KS, 2004), 133–143, Contemp. Math. 367, Amer. Math. Soc., Providence, RI, 2005.

[18] P. Hästö: *On the density of smooth functions in variable exponent Sobolev space*, preprint (2005).

[19] J. Heinonen: *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001.

[20] J. Heinonen, T. Kilpeläinen and O. Martio: *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, Oxford, 1993.

[21] D. Kinderlehrer and G. Stampacchia: *An introduction to variational inequalities and their applications*, Academic Press, London, 1980.

[22] V. Kokilashvili and S. Samko: *Maximal and fractional operators in weighted $L^{p(x)}$ spaces*, Rev. Mat. Iberoamericana, **20** (2004), no. 2, 493–515.

[23] O. Kováčik and J. Rákosník: *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J., **41(116)** (1991), 592–618.

[24] M. Růžička: *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2000.

[25] V. V. Zhikov: *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR-Izv., **29** (1987), no. 1, 33–66. [Translation of Izv. Akad. Nauk SSSR Ser. Mat., **50** (1986), no. 4, 675–710, 877.]

[26] V. V. Zhikov: *On some variational problems*, Russian J. Math. Phys., **5** (1997), no. 1, 105–116 (1998).


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