The Journal of Geometric Analysis
Volume , Number ,

Apollonian isometries of planar domains are Möbius mappings

Peter Hästö*
Department of Mathematics Sciences
Norwegian University of Science and Technology
7491 Trondheim, Norway
`peter.hasto@helsinki.fi`

Zair Ibragimov
Department of Mathematical Sciences
University of Cincinnati
P.O. Box 210025, Cincinnati, OH 45221-0025, USA
`ibragizs@math.uc.edu`

ABSTRACT. The Apollonian metric is a generalization of the hyperbolic metric, defined in a much larger class of open sets. Beardon introduced the metric in 1998, and asked whether its isometries are just the Möbius mappings. In this paper we show that this is the case in all open subsets of the plane with at least three boundary points.

---

* Supported by a Gehring–Finland Fellowship, by the Finnish Academy of Science and Letters and by the Research Council of Norway, Grant 160192/V30
Mathematics Subject Classification (2000): 30F45 (primary), 30C65 (secondary)

ISSN 1050-6926

The Apollonian metric, which is defined in arbitrary open sets of $\overline{\mathbb{R}^n}$, is an extension of the hyperbolic metric. It was introduced by Alan Beardon in 1998 [3], but it later turned out that the same metric had been studied previously by Dan Barbilian [1] (see [20], and, for some further developments, [5, 6, 25]). The Apollonian metric has been studied recently from the perspective of geometric function theory for instance in [9, 11, 12, 14, 15, 17, 18, 19, 23, 24]. As an example we mention the result by Gehring and Hag that the Apollonian metric and the hyperbolic metric are bilipschitz equivalent in a planar domain if and only if the domain is a quasidisc [9]. Some other interesting properties are given below.

There are of course several other generalizations of the hyperbolic metric, like Seittenranta's metric [24], the cosh-metric [13], Ferrand's metric [8] and the Kulkarni–Pinkall metric [16, 21]. Of these the first two are relatively easy to compute, but not geodesic, whereas the situation is the opposite for the latter two metrics. Probably the best-known variant of the hyperbolic metric in geometric function theory is the quasihyperbolic metric of Gehring and Palka [10]. This metric is totally geodesic and somewhat easier to compute than the length-metrics mentioned above, but in contrast to the above metrics, it is only quasi-invariant under Möbius mappings and equals the hyperbolic metric in the half-space but not the ball.

The problem of determining the isometries of such metrics is quite difficult. Partial results for the quasihyperbolic metric were obtained in [22]. For the other metrics mentioned above nothing is known about the isometries, although the obvious guess is that Möbius mappings are the only isometries in every case.

The specific isometry question in the case of the Apollonian metric in planar domains was first posed by Beardon in [3]. The authors were recently able to establish that all the isometries are indeed Möbius mappings in the case of a domain with regular boundary [15]. This proof was based on the use of so-called pseudo-geodesics, which are curves sharing many desirable properties of geodesics, but much more ubiquitous.

In this article we use a different method to show that all Apollonian isometries are Möbius mappings in planar domains, whether or not the boundary is regular. This proof is based on the fact that we have good control of the metric near the boundary. The proof also relies on the topology of the 2-dimensional space.

Notation We denote by $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$ the one point compactification of $\mathbb{R}^2$, so its discs are the (open) discs of $\mathbb{R}^2$, complements of closed discs and half-planes. If $G \subset \overline{\mathbb{R}^2}$ we denote by ∂G and $\overline{G}$ its boundary and closure, respectively, both with respect to $\overline{\mathbb{R}^2}$. For $x \in G \subsetneq \mathbb{R}^2$ we denote $\delta(x) = d(x, \partial G) = \min\{|x-z| \colon z \in \partial G\}$. The cross-ratio $|a,b,c,d|$ is defined by

$$|a,b,c,d| = \frac{|a-c||b-d|}{|a-b||c-d|}$$

for distinct points $a,b,c,d \in \overline{\mathbb{R}^2}$, with the understanding that $|\infty - x|/|\infty - y| = 1$ for all $x, y \in \mathbb{R}^2$. A homeomorphism $f \colon \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a,b,c,d|$$

for every quadruple of distinct points $a,b,c,d \in \overline{\mathbb{R}^2}$. For more information on Möbius mappings see e.g. [2, Section 3].

The Apollonian metric We will be considering open non-empty sets D in $\overline{\mathbb{R}^2}$. The Apollonian metric for $x, y \in D \subsetneq \mathbb{R}^2$ is defined by

$$\alpha_D(x,y) = \sup_{a,b \in \partial D} |a,y,x,b|.$$

It is in fact a metric if and only if the complement of D is not contained in a circle [3, Theorem 1.1]. Clearly α_D is Möbius invariant, as noted in [3, Introduction (2)].

This metric has a very nice interpretation in terms of maximal Apollonian discs in the domains. The reader is referred to [15] for a discussion on this. Some further interesting geometrical notions arise when we look at the regularity of the Apollonian metric. For instance, the second author [17] found a connection between the Apollonian metric and constant width sets, objects which have been studied by geometers for several centuries (see e.g. [7] and the references therein). The first author [14] showed that the Apollonian metric is bilispchitz equivalent to the j_G metric if and only if the complement of the domain of definition is thick in the sense of Väisälä, Vourinen and Wallin (see [26] for a definition of thickness and applications of this concept in other areas of analysis).

Recall that a homeomorphism $f \colon D \to f(D) \subset \overline{\mathbb{R}^2}$ is said to be an *Apollonian isometry* if

$$\alpha_D(x,y) = \alpha_{f(D)}(f(x), f(y))$$

for all $x, y \in D$. We restrict our attention to homeomorphisms because it was shown in [15, Proposition 1] that every isometry is a homeomorphism unless $\overline{\mathbb{R}^2} \setminus D$ is contained in a circle.

The main result We now set out to prove our main result, that all planar Apollonian isometries are Möbius mappings. The proof consists of three steps: first we derive a sharper version of boundary rigidity by considering boundary points at infinity. This implies that level-sets of the $\delta(\cdot)$ function are invariant, which means that the action of the isometry is essentially only to move points along one-dimensional curves. Once we get some initial control over this "vertical" motion by considering geodesic, this allows us to use an iteration argument to show that in effect they cannot move at all.

Theorem 1 (Theorem 1 and Lemma 3, [15]). *Let $D \subset \overline{\mathbb{R}^2}$ be an open set with at least 3 boundary points. Let $f\colon D \to \overline{\mathbb{R}^2}$ be an Apollonian isometry. Then f extends to $\overline{D}$ continuously, and up to composition by Möbius maps, $f(\overline{D}) = \overline{D}$ and $f|_{\partial D} = id_{\partial D}$.*

If f is normalized so that $f(\overline{D}) = \overline{D}$ and $f|_{\partial D} = id_{\partial D}$, then

$$\lim_{i\to\infty} \frac{\delta(f(z_i))}{\delta(z_i)} = 1$$

for any sequence of points $z_i \in D$ tending to a boundary point.

The next proposition extracts some more information of the boundary behavior of an Apollonian isometry by looking at the action of the isometry from infinity. Note that this result is also valid in higher dimensions.

Proposition 2. *Let $D \subset \mathbb{R}^2$ be a domain and suppose $\infty \in \partial D$. If $f\colon D \to D$ is an Apollonian isometry, which extends continuously to the identity map on the boundary, then $\delta(x) = \delta(f(x))$ for all $x \in D$.*

Proof. We start as in the proof of [15, Lemma 1]: using the triangle inequalities $|x-y| - |x-a| \le |y-a| \le |x-y| + |x-a|$ we find that

$$\frac{|x-y|}{\delta(x)} - 1 \le \max_{a\in\partial D} \frac{|y-a|}{|x-a|} \le \frac{|x-y|}{\delta(x)} + 1.$$

From this we easily derive

$$\log\left(\frac{|x-y|}{\delta(x)} - 1\right)\left(\frac{|x-y|}{\delta(y)} - 1\right) \le \alpha_G(x,y) \le \log\left(\frac{|x-y|}{\delta(x)} + 1\right)\left(\frac{|x-y|}{\delta(y)} + 1\right).$$

Using these inequalities and the fact that f is an Apolloian isometry, we find that

$$\log\left(\frac{|x-y|}{\delta(x)}-1\right)\left(\frac{|x-y|}{\delta(y)}-1\right) \le \log\left(\frac{|f(x)-f(y)|}{\delta(f(x))}+1\right)\left(\frac{|f(x)-f(y)|}{\delta(f(y))}+1\right).$$

Taking the exponential function of both sides and multiplying by $\delta(y)$ gives

$$\left(\frac{|x-y|}{\delta(x)}-1\right)\big(|x-y|-\delta(y)\big) \le \frac{\delta(y)}{\delta(f(y))}\left(\frac{|f(x)-f(y)|}{\delta(f(x))}+1\right)\big(|f(x)-f(y)|+\delta(f(y))\big).$$

Suppose then that we let y approach a finite boundary point w. Since the previous inequality holds for all y, it holds in the limit as well. Using that $\delta(y)/\delta(f(y))\to 1$ and $f(w)=w$ we get

$$|x-w|\left(\frac{|x-w|}{\delta(x)}-1\right) \le |f(x)-w|\left(\frac{|f(x)-w|}{\delta(f(x))}+1\right). \tag{1}$$

This inequality holds for all $x\in D$ and all $w\in\partial D\setminus\{\infty\}$. Suppose first that ∞ is not an isolated boundary point. Fix x and let $w\to\infty$. Then $|x-w|/|f(x)-w|\to 1$ and the previous inequality gives $\delta(f(x))\le\delta(x)$. Because of the symmetry of the situation (f^{-1} has the same properties as f), the reverse inequality also follows.

Suppose next that ∞ is an isolated boundary point. Define $K=\partial D\setminus\{\infty\}$ and fix $x\in D$. Let M be the maximal distance from x and $f(x)$ to K. Suppose that $y\in D$ is a point with $\delta(y)>M$ and $\delta(f(y))>M$. Then we see that $\frac{|x-a|}{|y-a|}<1$ for all finite boundary points a, and similarly $\frac{|f(x)-a|}{|f(y)-a|}<1$. This implies that

$$\alpha_D(x,y)=\sup_{a\in\partial D}\log\frac{|x-a|}{|y-a|}+\sup_{b\in\partial D}\log\frac{|y-b|}{|x-b|}=\sup_{b\in\partial D}\log\frac{|y-b|}{|x-b|},$$

and similarly for $\alpha_D(f(x),f(y))$. Let $M'=\operatorname{diam}(K)$. We find that

$$\frac{\delta(y)}{\delta(x)}\le\max_{b\in K}\frac{|y-b|}{|x-b|}=\alpha_G(x,y)=\alpha_G(f(x),f(y))=\max_{b\in K}\frac{|f(y)-b|}{|f(x)-b|}\le\frac{\delta(f(y))+M'}{\delta(f(x))}.$$

By Theorem 1 we have $\delta(f(y))/\delta(y)\to 1$, as $y\to\infty$. This implies that $\delta(x)\ge\delta(f(x))$, so, by symmetry, $\delta(x)=\delta(f(x))$ in this case also. □

Recall that a curve $\gamma\subset D$ is a *geodesic* if

$$\alpha_D(x,y)=\alpha_D(x,z)+\alpha_D(z,y)$$

for every $x,y,z\in\gamma$ appropriately ordered. A *geodesic line* is a geodesic of infinite length in both directions. There are some very natural geodesic lines in the Apollonian metric:

suppose that B is a disc in D so that $\partial B \cap \partial D$ contains at least two points, a and b. Then the circular arc γ in B which is orthogonal to ∂B at a and b is a geodesic line. The converse of this is also true, namely if γ is a geodesic line connecting the boundary points a and b then there is a ball in D whose boundary contains these points [9].

Lemma 3. *Let $D \subset \mathbb{R}^2$ be a domain. Suppose that $e_1, -e_1, \infty \in \partial D$. If γ is a geodesic line from $-e_1$ to e_1, then γ does not cross the e_1-axis outside the segment $[-e_1, e_1]$.*

Proof. Suppose that γ would be a geodesic line from $-e_1$ to e_1 which crosses the e_1-axis at the point $z = te_1$, with $t > 1$. Fix $r > 0$ and let $x, y \in \gamma$ be points with $|x - e_1| = |y + e_1| = r$. Since x and y lie on a geodesic line connecting e_1 and $-e_1$ we can use these points to calculate the Apollonian distance, see [15, Corollary 2], so $\alpha_D(x, y) = \log |e_1, x, y, -e_1| \le 2 \log \frac{2+2r}{r}$. By restricting the supremum we find that

$$\alpha_D(x, z) \ge \log \frac{t-1}{r} \quad \text{and} \quad \alpha_D(z, y) \ge \log \Big(\frac{t+1}{r} \frac{|y - e_1|}{t-1}\Big).$$

Since x, z, y are supposed to lie on a geodesic, we have $\alpha_D(x, y) = \alpha_D(x, z) + \alpha_D(z, y)$ (for small enough r). Thus

$$2 \log \frac{2+2r}{r} \ge \log \frac{t-1}{r} + \log \Big(\frac{t+1}{r} \frac{|y - e_1|}{t-1}\Big) = \log \frac{(t+1)|y - e_1|}{r^2}.$$

As $r \to 0$ we have $|y - e_1| \to 2$, which leads to a contradiction since $t + 1 > 2$. □

Lemma 4. *Let $D \subset \mathbb{R}^2$ be a domain whose complement is not contained in a circle. Suppose that $f \colon D \to \overline{\mathbb{R}^2}$ is an Apollonian isometry, which extends continuously to the boundary as the identity. Let $\gamma \subset D$ be a geodesic connecting two boundary points a and b. Then $f(z) = z$ for all $z \in \gamma$.*

Proof. By an auxilliary Möbius mapping we may assume that $\infty \in \partial D$ and $a, b \in \mathbb{R}^2$ (since the domain has at least three boundary points). For points $x, y \in \gamma$ lying on γ in the order a, y, x, b we have

$$\alpha_D(x, y) = \log \left(\frac{|x - a|}{|y - a|} \frac{|y - b|}{|x - b|} \right),$$

see e.g. [15, Corollary 2]. Since $f(\gamma)$ is also a geodesic connecting a and b, the same formula holds for $f(x)$ and $f(y)$. Taking the exponential function to get rid of the logarithm we have

$$\frac{|x - a|}{|y - a|} \frac{|y - b|}{|x - b|} = \frac{|f(x) - a|}{|f(y) - a|} \frac{|f(y) - b|}{|f(x) - b|}.$$

Expressed differently, this says that the ratio

$$\frac{|y-b|}{|y-a|}\frac{|f(y)-a|}{|f(y)-b|}=c_f$$

does not depend on y (or on γ). Suppose that $c_f>1$. By iteration we find that

$$\frac{|y-b|}{|y-a|}\frac{|f^k(y)-a|}{|f^k(y)-b|}=c_f^k,$$

which means that the points $f^k(y)$ tend to b which is impossible, since the level-sets of $\delta(\cdot)$ are preserved by f. The case $c_f<1$ leads to a similar contradiction, so $c_f=1$ and we have

$$\frac{|y-b|}{|y-a|}=\frac{|f(y)-b|}{|f(y)-a|}.$$

In other words, if x lies on a geodesic (between a and b) and on the circle

$$C_r=\Big\{y\in D\colon \frac{|y-b|}{|y-a|}=r\Big\},$$

then $f(x)$ lies on the circle C_r as well. For simplicity of exposition we normalize the situation so that $a=-e_1$ and $b=e_1$. Consider the set of intersection points of γ with $f(\gamma)$, denoted by F (with the convention that $a,b\in F$). Since γ intersects each C_r only once, it directly follows that F is point-wise fixed. Assume now that there exists a point $x\in\gamma$ which is not fixed. The set F is closed since f is continuous, so we can find closest intersection points to x on both sides, denoted by a' and b'. Consider the loop $L=\gamma'\cup\{a'\}\cup f(\gamma')\cup\{b'\}$, where γ' is the part of γ which lies between a' and b'. By the choice of a' and b' it is clear that γ' and $f(\gamma')$ do not intersect. Also, neither of these curves intersect the rays $(-\infty,a')$ and (b',∞), by Lemma 3 and since γ' does not cross the circle C_r containing a', if $a'\neq a$, and similarly for the other end-point and geodesic. We denote $J_k=(-\infty,a')\cup f^k(\gamma')\cup(b',\infty)$. We can define a "lies above" relation on these curves. Suppose that J_1 lies above J_0. Since f is orientation preserving, we see that the loop $f(L)=f(\gamma')\cup\{a'\}\cup f^2(\gamma')\cup\{b'\}$ has $f^2(\gamma')$ above $f(\gamma')$ so J_2 lies above J_1; similarly J_{k+1} lies above J_k for all k. Let now r be such that $x\in C_r$. If $r\neq 1$, then $f^k(x)$ will just move "upward"along C_r, which is a compact set, so $f^k(x)$ has a limit w, and clearly $f(w)=w$. Then we use the *iteration argument* to find that

$$0<\alpha_D(x,f(x))=\alpha_D(f(x),f^2(x))=\ldots=\lim_{k\to\infty}\alpha_D(f^k(x),f^{k+1}(x))=\alpha_D(w,w)=0,$$

a contradiction. Obviously a similar argument applies if J_1 lies below J_0. Thus, $f(x) = x$ for all $x \in \gamma \setminus C_1$. But the continuity of f implies that $f(x) = x$ also for $x \in \gamma \cap C_1$, so we are done. □

To use the previous results, we still need to find a sufficient number of geodesics, which is what the following lemma does for us.

Lemma 5. *Let $D \subset \overline{\mathbb{R}^2}$ be a domain with at least three boundary points. Then $z \in D$ either lies on a geodesic, or inside an infinite geodesic triangle (i.e. a triangle whose edges are geodesic lines).*

Proof. By Möbius invariance it suffices to prove the claim for the point $\infty \in D$. As ∂D is then bounded, it follows that there is a unique closed disk B of minimal radius containing it [4, Theorem 11.5.8].

Then ∂B intersects ∂D in two or more points. If the intersection contains exactly two points then they have to be diametrically opposite, and then ∞ lies on the geodesic line which is the extension of this diametrical segment. Otherwise the intersection contains at least three points. In this case the circular arcs in D which pass through two of the boundary points and are orthogonal to ∂B at these points are geodesics lines, so ∞ lies in an infinite geodesic triangle. □

Theorem 6. *Let $D \subset \overline{\mathbb{R}^2}$ be open with at least three boundary points. If $f\colon D \to \overline{\mathbb{R}^2}$ is an Apollonian isometry, then f is the restriction to D of a Möbius mapping.*

Proof. Suppose first that D is a domain. If ∂D is a subset of a circle, then we can apply the isometry results from [9] or [18]. So we assume that this is not the case. By Theorem 1 we may assume that $f(D) = D$ and that f extends continuously to the boundary as the identity.

By Lemma 5 every point $z \in D$ either lies on a geodesic, or inside an infinite geodesic triangle. In the former case z is fixed by Lemma 4. So it remains to consider the second case. We map one of the vertices of the geodesic triangle containing z to infinity by an auxiliary inversion. Then we are in the situation shown in Figure 1. Next we note that in the infinite triangle T the level-sets of the δ function are simple curves joining two edges

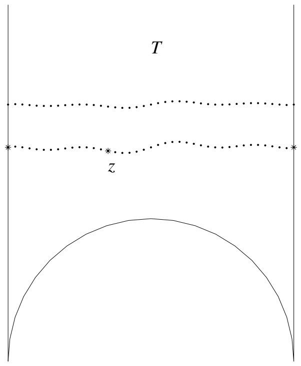

FIGURE 1. The infinite geodesic triangle, with two level-sets drawn.

of the triangle. To see this we need only note that δ is continuously increasing along every vertical ray in T, since the boundary of D is contained in the lower half-plane. Since the edges of T are fixed by Lemma 4, the iteration argument (see the proof of Lemma 4) easily implies that the simple curves joining points on the edges are in fact point-wise fixed, which completes the proof in the domain case.

If D is not connected, then we apply the previous argument to every component of D, so f is a Möbius mapping in every component. Since we know by Theorem 1 that the mapping acts as an Möbius mapping on the boundary, we see that in fact f has to be the restriction of the same Möbius mapping in every component. □

Remark 7. In the case of a domain with two boundary points the conclusion of the previous theorem does not hold. Consider $D = \mathbb{R}^2 \setminus \{0\}$: here $\alpha_D(x,y) = \big|\log\frac{|x|}{|y|}\big|$, so any mapping $f\colon D \to D$ preserving spheres $S^1(0,r)$ is an isometry.

[1] D. Barbilian: Einordnung von Lobatschewsky's Maßbestimmung in gewisse allgemeine Metrik der Jordanschen Bereiche, *Casopsis Mathematiky a Fysiky* **64** (1934–35), 182–183.

[2] A. Beardon: *Geometry of Discrete Groups*, Graduate text in mathematics 91, Springer-Verlag, New York, 1995.

[3] ——: The Apollonian metric of a domain in $\mathbb{R}^n$, pp. 91–108 in *Quasiconformal mappings and analysis* (P. Duren, J. Heinonen, B. Osgood and B. Palka (eds.)), Springer-Verlag, New York, 1998.

[4] M. Berger: *Geometry I, II*, Springer-Verlag, Berlin, 1994.

[5] W.-G. Boskoff: *Hyperbolic geometry and Barbilian spaces*, Istituto per la Ricerca di Base, Series of Monographs in Advanced Mathematics, Hardronic Press, Inc, Palm Harbor, FL, 1996.

[6] W.-G. Boskoff: *Varietăţi cu structură metrică Barbilian* (Romanian) [Manifolds with Barbilian metric structure], Colecţia Biblioteca de Matematică [Mathematics Library Collection], Ex Ponto, Editura, Constanţa, 2002.

[7] G. Chakerian and H. Groemer: Convex bodies of constant width, pp. 49–96 in *Convexity and its applications* (P. Gruber and J. Wills (eds.)), Birkhäuser, Basel, 1983.

[8] J. Ferrand: A characterization of quasiconformal mappings by the behavior of a function of three points, pp. 110–123 in *Proceedings of the 13th Rolf Nevalinna Colloquium* (Joensuu, 1987; I. Laine, S. Rickman and T. Sorvali (eds.)), Lecture Notes in Mathematics Vol. 1351, Springer-Verlag, New York, 1988.

[9] F. Gehring and K. Hag: The Apollonian metric and quasiconformal mappings, pp. 143–163 in *In the tradition of Ahlfors and Bers* (Stony Brook, NY, 1998; I. Kra and B. Maskit (eds.)), Contemp. Math. 256, Amer. Math. Soc., Providence, RI, 2000.

[10] F. Gehring and B. Palka: Quasiconformally homogeneous domains, *J. Anal. Math.* **30** (1976), 172–199.

[11] P. Hästö: The Apollonian metric: uniformity and quasiconvexity, *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 385–414.

[12] ——: The Apollonian inner metric, *Comm. Anal. Geom.* **12** (2004), no. 4, 927–947.

[13] ——: A new weighted metric: the relative metric II, *J. Math. Anal. Appl.*, to appear (2005).

[14] ——: Thick sets and the Apollonian metric, *Ann. Acad. Sci. Fenn. Math.*, submitted (2004).

[15] P. Hästö and Z. Ibragimov: Apollonian isometries of regular domains are Möbius mappings, *J. London Math. Soc.*, submitted (2004).

[16] D. Herron, W. Ma and D. Minda: A Möbius invariant metric for regions on the Riemann sphere, pp. 101–118 in *Future Trends in Geometric Function Theory* (RNC Workshop, Jyväskylä 2003; D. Herron (ed.)), Rep. Univ. Jyväskylä Dept. Math. Stat. **92** (2003).

[17] Z. Ibragimov: The Apollonian metric, sets of constant width and Möbius modulus of ring domains, Ph.D. Thesis, University of Michigan, Ann Arbor, MI, 2002.

[18] ——: On the Apollonian metric of domains in $\overline{\mathbb{R}^n}$, *Complex Var. Theory Appl.* **48** (2003), 837–855.

[19] ——: Conformality of the Apollonian metric, *Comput. Methods Funct. Theory* **3** (2003), 397–411.

[20] P. Kelly: Barbilian geometry and the Poincaré model, *Amer. Math. Monthly* **61** (1954), 311–319.

[21] R. Kulkarni and U. Pinkall: A canonical metric for Möbius structures and its applications, *Math. Z.* **216** (1994), 89–129.

[22] G. Martin and B. Osgood: The quasihyperbolic metric and associated estimates on the hyperbolic metric, *J. Anal. Math.* **47** (1986), 37–53.

[23] A. Rhodes: An upper bound for the hyperbolic metric of a convex domain, *Bull. London Math. Soc.* **29** (1997), 592–594.

[24] P. Seittenranta: Möbius-invariant metrics, *Math. Proc. Cambridge Philos. Soc.* **125** (1999), 511–533.

[25] P. Souza: Barbilian metric spaces and the hyperbolic plane (Spanish), *Miscelánea Mat.* **29** (1999), 25–42.

[26] J. Väisälä, M. Vuorinen and H. Wallin: Thick sets and quasisymmetric maps, *Nagoya Math. J.* **135**, (1994), 121–148.