Isometries of relative metrics

Peter Hästö*
Department of Mathematical Sciences, P.O. Box 3000
90014 University of Oulu, Finland
`peter.hasto@helsinki.fi`

Zair Ibragimov
Department of Mathematical Sciences
University of Cincinnati, P.O. Box 210025
Cincinnati, OH 45221-0025, USA
`ibragizs@math.uc.edu`

Henri Lindén†
Department of Mathematics and Statistics, P.O. Box 68
00014 University of Helsinki, Finland
`henri.linden@helsinki.fi`

December 15, 2005

Abstract

In this paper we consider isometries of relative metrics. We characterize isometries of the j_D metric and of Seittenranta's metric, as well as of their generalizations. We also derive some inequalities and results on the geodesics of these metrics.

Mathematics Subject Classification 2000: 30F45 (primary), 30C65 (secondary)
Keywords: Relative metrics, isometries, distance ratio metric, Seittenranta's metric

---

*PH was supported by the Research Council of Norway, Project 160192/V30
†HL was supported in part by the Finnish Academy of Science and Letters

1 Introduction

In this article we consider isometries of relative metrics. The term "relative metric" implies that the metric is evaluated in a proper subdomain of $\mathbb{R}^n$ relative to its boundary. More precisely, we want the metric to blow up towards the boundary of the domain, i.e., we want the boundary to be at infinity intrinsically.

We will concentrate on two metrics. Let $D \subsetneq \mathbb{R}^n$ be a domain containing the points x and y. The well-known *distance ratio metric* is defined by

$$j_D(x,y) = \log\left(1 + \frac{|x-y|}{\min\{\delta(x), \delta(y)\}}\right),$$

where $\delta(\cdot) = \operatorname{dist}(\,\cdot\,, \partial D)$ denotes distance to the boundary. It was used, for instance, by Gehring and Osgood [6] to characterize uniform domains (namely, in such domains the j_D metric is quasiconvex). Note that this metric has sometimes been called simply "the relative metric", but we will not use this term.

The second metric that we will focus on is not as well-known, but it is a natural, Möbius invariant analogue of the j_D metric. This so-called *Seittenranta's metric* first appeared in [16, 8.39] and [15] and is defined by

$$\delta_D(x,y) = \log\left(1 + \sup_{a,b\in\partial D} \frac{|x-y||a-b|}{|a-x||b-y|}\right).$$

Notice that if a or b in the supremum equals infinity, then we get exactly the j_D metric. This implies that we always have $j_D \le \delta_D$. Also, it is not difficult to prove the upper bound $\delta_D \le 2j_D$.

To see how these metrics fit into a larger framework we recall the concept of an inner metric. Let d be a metric in D and γ be a path in D (i.e. a continuous mapping from an interval I to D). The length of γ is defined as

$$d(\gamma) = \sup \sum_{i=1}^{k-1} d\big(\gamma(t_i), \gamma(t_{i+1})\big),$$

where the supremum is taken over k and all increasing sequences $(t_i)_{i=1}^k$ of points in I. Then the *inner* or *intrinsic metric* of d is defined by

$$\tilde{d}(x,y) = \inf_\gamma d(\gamma),$$

where the infimum is taken over all paths γ connecting x and y in D (note that this need not be finite, unless D is rectifiably connected). It is clear that $d(x,y) \le \tilde{d}(x,y)$ and that $d(\gamma) = \tilde{d}(\gamma)$ for any metric and path. The theory of length-metrics, including in particular intrinsic metrics, is presented e.g. in [3, 4].

Suppose now that $D \subset \mathbb{R}^n$ and d is a metric in D. If

$$\bar{d}(x) = \lim_{y\to x} \frac{d(x,y)}{|x-y|}$$

exists for all $x \in D$ and is continuous, then we can express the inner metric of d by

$$\tilde{d}(x,y) = \inf_\gamma \int_\gamma \bar{d}(z)\,|dz|,$$

where $|dz|$ represents integration with respect to d-arclength, and the infimum is taken over rectifiable curves with end points x and y. In this case $\tilde{d}$ is called a *conformal metric.* We easily see that the inner metric of j_D is the quasihyperbolic metric,

$$\tilde{j}_D(x,y) = k_D(x,y) = \inf_\gamma \int_\gamma \frac{|dz|}{d(z,\partial D)}.$$

For the inner metric of Seittenranta's metric we find that

$$\tilde{\delta}_D(x,y) = \sigma_D(x,y) = \inf_\gamma \int_\gamma \sup_{a,b\in\partial D} \frac{|a-b|}{|a-z||b-z|}\,|dz|,$$

where σ_D is known as Ferrand's metric [5]. Obviously the inequalities $j_D \le \delta_D \le 2j_D$ imply that $k_D \le \sigma_D \le 2k_D$. Length-metrics are interesting from a geometric point of view, but for getting explicit estimates they are often of little use. The role of point-distance functions, like the j_D and δ_D metrics, is that they share features with their inner metrics, but are much more explicit.

In this paper we want to consider not only the j_D and δ_D metrics, but all metrics which resemble them in the very small and very large scale. The small scale equivalence implies that the metrics have the same inner metrics, whereas the large scale equivalence allows us to get a hold of the boundary behavior and thus start unraveling the isometry story.

Remark 1.1. Note that j_D and δ_D are really families of metrics, namely for every domain D we have one metric. We will continue to use this convention when talking about these and other metrics in this paper.

Definition 1.2. We say that d is a *j-type metric* if the following three conditions hold on every domain $D \subsetneq \mathbb{R}^n$:

1. d_D is a metric on D.
2. For each $y \in D$ and for each sequence (x_i) with $j_D(x_i,y) \to 0$ we have
$$\lim_{i\to\infty} \frac{d_D(x_i,y)}{j_D(x_i,y)} = 1.$$
3. For each $y \in D$ and for each sequence (x_i) with $j_D(x_i,y) \to \infty$ we have
$$\lim_{i\to\infty} \big(d_D(x_i,y) - j_D(x_i,y)\big) = 0.$$

The fact that y can be any interior point in (2) means that being a j-type metric is quite a strong condition; for instance, if d and $f \circ d$ are j-type metrics, then $f = \mathrm{id}$ (Corollary 3.8). For δ-type metrics we are able to use a weaker definition, in that we can pose more conditions on when the limit in (2) is supposed to be zero.

Definition 1.3. We say that d is a *δ-type metric* if the following three conditions hold on every domain $D \subsetneq \mathbb{R}^n$:

1. d_D is a metric on D.
2. For each $y \in D$ and for each sequence (x_i) with $\delta_D(x_i, y) \to 0$ we have
$$\lim_{i\to\infty} \frac{d_D(x_i, y)}{\delta_D(x_i, y)} = 1.$$
3. For all sequences (x_i) and (y_i) of points in D tending to ∂D we have
$$\lim_{i\to\infty} \big(d_D(x_i, y_i) - \delta_D(x_i, y_i)\big) = 0.$$

It seems to be quite difficult to construct other natural metrics of j- or δ-type. The cosh-metric, defined by

$$\rho_D(x, y) = \operatorname{arch}\big(1 + \tfrac{1}{2} \sup_{a,b\in\partial D} |a, x, b, y||a, y, b, x|\big)$$

(see [16, Lemma 3.26], [7, Section 5]), is an example of a δ-type metric. The main purpose of our more abstract treatment is to highlight the features that are crucial, which in turn indicates that these techniques might be relevant also for handling the isometries of the corresponding inner metrics.

In this paper we will characterize the isometries of j- and δ-type metrics. Recall that in the setting of domain dependent metrics an isometry is a mapping $f\colon D \to \mathbb{R}^n$ such that $f(D)$ is open and

$$d_{f(D)}(f(x), f(y)) = d_D(x, y)$$

for all $x, y \in D$. We start by collecting some basic properties of these metrics. In Section 3 we solve the isometry problem for j-type metrics using a boundary rigidity result. In Section 4 we solve the isometry problem for δ-type metrics by keeping track of a geodesic line. In Section 5 we take another look at the context of the questions in the article and suggest some directions for further research.

2 Notation and observations

We denote by $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ the one point compactification of $\mathbb{R}^n$, so its open balls are the open balls of $\mathbb{R}^n$, complements of closed balls in $\mathbb{R}^n$ and half-spaces. If $D \subset \overline{\mathbb{R}^n}$ we denote by ∂D and $\overline{D}$ its boundary and closure, respectively, all with respect to $\overline{\mathbb{R}^n}$. By $B^n(x,r)$ and $S^{n-1}(x,r)$ we denote the open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$, and its boundary, respectively. For $x \in D \subsetneq \mathbb{R}^n$ we denote $\delta(x) = d(x, \partial D) = \min\{|x-z| \colon z \in \partial D\}$.

The cross-ratio $|a,b,c,d|$ is defined by

$$|a,b,c,d| = \frac{|a-c||b-d|}{|a-b||c-d|}$$

for distinct $a,b,c,d \in \overline{\mathbb{R}^n}$, with the understanding that $|\infty - x|/|\infty - y| = 1$ for all $x,y \in \mathbb{R}^n$. A homeomorphism $f \colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a,b,c,d|$$

for every quadruple of distinct $a,b,c,d \in \overline{\mathbb{R}^n}$. For more information on Möbius mappings see e.g. [1, Section 3]. Since δ_D is Möbius invariant it is natural to consider questions related to δ-type metrics in the Möbius space.

The following two results are more or less restatements of the definitions. However, they directly imply that we may restrict our focus very much without losing any isometries.

Proposition 2.1. *If d is a j-type, then*

$$\lim_{y\to\partial D\setminus\{\infty\}} \left(d_D(x,y) - \log\frac{|x-y|}{\delta(y)} \right) = 0 \qquad \textit{and} \qquad \lim_{y\to x} \frac{d_D(x,y)}{|x-y|} = \frac{1}{\delta(x)}$$

for every $x \in D$. The inner metric of a j-type metrics is the quasihyperbolic metric k_D. In particular, every isometry of a j-type metric is an isometry of k_D.

Proposition 2.2. *If d is a δ-type, then*

$$\lim_{z,y\to\partial D\setminus\{\infty\}} \left(d_D(z,y) - \log\frac{|z-y|^2}{\delta(z)\delta(y)} \right) = 0$$

and

$$\lim_{y\to x} \frac{d_D(x,y)}{|x-y|} = \sup_{a,b\in\partial D} \frac{|a-b|}{|a-x||b-x|}$$

for every $x \in D$. The inner metric of a δ-type metric is Ferrand's metric σ_D. In particular, every isometry of a δ-type metric is an isometry of σ_D.

Every isometry of k_D is a conformal mapping [14, Theorem 2.6], and a similar proof gives the same result for Ferrand's metric. Hence we conclude:

Corollary 2.3. *Every isometry of a j or δ-type metric is conformal. In particular, if $n \geq 3$, then such an isometry is Möbius.*

3 On j-type metrics

We plunge right into the main result of this section, a characterization of the isometries of j-type metrics. In Section 3.2 we derive some miscellaneous results, which give a clearer picture of j-type metrics.

3.1 Isometries of j-type metrics

The proof of the following theorem is partly based on ideas from [10].

Theorem 3.1. *Let d be a j-type metric, $D \subsetneq \mathbb{R}^n$ and $f\colon D \to \mathbb{R}^n$ be a d-isometry. Then either*

1. *f is a similarity, or*
2. *$D = \mathbb{R}^n \setminus \{a\}$ and, up to similarity, f is an inversion in a sphere centered at a.*

Proof. Denote $D' = f(D)$ and $\delta'(x) = d(x, \partial f(D))$. Fix $z \in \partial D \setminus \{\infty\}$ and let (z_i) be a sequence of points in D tending to z. We first assume that there exists a subsequence, which we also denote by (z_i), such that $\big(f(z_i)\big)$ converges to $w_1 \in \mathbb{R}^n$. Since d is a j-type metric we see, using Proposition 2.1 for the third equality, that for every $x \in D$ we have that

$$\begin{aligned}
0 &= \lim_{i\to\infty} \big(d_{D'}(f(x), f(z_i)) - d_D(x, z_i)\big) \\
&= \lim_{i\to\infty} \Big(d_{D'}(f(x), f(z_i)) - \log \frac{1}{\delta'(f(z_i))}\Big) - \lim_{i\to\infty} \Big(d_D(x, z_i) - \log \frac{1}{\delta(z_i)}\Big) + \\
&\qquad\qquad + \lim_{i\to\infty} \log \frac{\delta(z_i)}{\delta'(f(z_i))} \\
&= \log \frac{|f(x) - w_1|}{|x - z|} + \lim_{i\to\infty} \log \frac{\delta(z_i)}{\delta'(f(z_i))}.
\end{aligned}$$

Taking exponentials gives

$$\lim_{i\to\infty} \frac{\delta'(f(z_i))}{\delta(z_i)} = \frac{|f(x) - w_1|}{|x - z|} < \infty. \tag{3.2}$$

Suppose now that $(\hat{z}_i)$ is a second sequence of points in D tending to z, but that this time $f(\hat{z}_i) \to w_2 \in \overline{\mathbb{R}^n} \setminus \{w_1\}$. Using $x = \hat{z}_j$ for every $j = 1, 2, \ldots$ in (3.2) gives

$$\lim_{i\to\infty} \frac{\delta'(f(z_i))}{\delta(z_i)} = \frac{|f(\hat{z}_j) - w_1|}{|\hat{z}_j - z|} \to \infty$$

as $j \to \infty$, which is a contradiction. In other words, $f(\hat{z}_i) \to w_1$ for every sequence of points $(\hat{z}_i) \to z$, so we may extend f continuously to $\overline{D}$ by defining

$f(z) = \lim_{i\to\infty} f(z_i)$. Therefore we conclude from (3.2), since the left-hand side of this equation does not depend on x, that

$$|f(x) - f(z)| = h_f(z)|x - z|$$

for some function $h_f \colon \partial D \to (0, \infty)$. This means that for $z, w \in \partial D$ we have

$$h_f(z)|w - z| = |f(w) - f(z)| = h_f(w)|w - z|,$$

so h_f is in fact a constant. Therefore f acts as a similarity, say g, on the boundary. We then extend f to all of $\mathbb{R}^n$ by setting $f(x) = g(x)$ outside the original domain of definition. Then it is clear that

$$|f(x) - f(z)| = h_f|x - z| \tag{3.3}$$

for every point $x \in \mathbb{R}^n$, i.e. the sphere $S^{n-1}(z, r)$ maps to $S^{n-1}(f(z), h_f r)$. This clearly implies that the conformal mapping is Möbius, and a Möbius mapping satisfying (3.3) is a similarity.

We still have one assumption to consider. In the beginning of the proof we assumed that we can find a boundary point z and a sequence (z_i) of points in D tending to z such that $f(z_i)$ tends to a finite limit. So we suppose now that no such sequence can be found, i.e. that for every sequence (z_i) of points in D tending to a boundary point z the sequence $\big(f(z_i)\big)$ tends to ∞. As before we conclude that

$$\begin{aligned} 0 &= \lim_{i\to\infty} \Big(d_{D'}(f(x), f(z_i)) - d_D(x, z_i) \Big) \\ &= \lim_{i\to\infty} \Big(j_{D'}(f(x), f(z_i)) - j_D(x, z_i) \Big) \\ &= \lim_{i\to\infty} \log \left(\frac{|f(x) - f(z_i)|}{\min\{\delta'(f(z_i)), \delta'(f(x))\}} \frac{\delta(z_i)}{|x - z|} \right). \end{aligned}$$

So it follows that

$$\lim_{i\to\infty} \frac{|f(x) - f(z_i)|\delta(z_i)}{\min\{\delta'(f(z_i)), \delta'(f(x))\}} = |x - z|.$$

Since $f(z_i) \to \infty$, we see that we can replace $|f(x) - f(z_i)|$ by $|f(z_i)|$ in the above forumla. Since the right-hand-side depends on x (which lies in an open set) we see that the left-hand-side must do so, too, hence we have to choose the second term in the minimum. Taking this into account we have

$$g_f(z) = \lim_{i\to\infty} |f(z_i)|\, \delta(z_i) = |x - z|\, \delta'(f(x)),$$

where $g_f \colon \partial D \to (0, \infty)$. Suppose that D has at least two finite boundary points, and let $a, b \in \partial D$ be such that the open segment (a, b) is contained in D. Now if we first consider x (in the previous equation) to be the mid-point x of (a, b), then we conclude that

$$g_f(a) = |x-a|\,\delta'(f(x)) = |x-b|\,\delta'(f(x)) = g_f(b).$$

But if we take some other point on the segment, then we get $g_f(a) \neq g_f(b)$, a contradiction. So only the case when D has a single boundary point remains to consider. Then we have

$$\lim_{i\to\infty} |f(z_i)|\,|z_i - a| = |x-a|\,|f(x)-b|$$

(for $D = \mathbb{R}^n \setminus \{a\}$ and $D' = \mathbb{R}^n \setminus \{b\}$) and we directly see that $x \mapsto f(x) + b - a$ is an inversion, which concludes the proof. □

Corollary 3.4. *Let d be a similarity invariant j-type metric and let $D \subsetneq \mathbb{R}^n$. Then $f\colon D \to \mathbb{R}^n$ is a d-isometry if and only if*

1. *f is a similarity, or*
2. *$D = \mathbb{R}^n \setminus \{a\}$ and, up to similarity, f is the inversion in a sphere centered at a.*

Proof. The previous proposition established that every d-isometry is of the given kind. If f is a similarity, then it is an isometry by assumption. So it remains (after normalization) to consider the case $D = \mathbb{R}^n \setminus \{0\}$. In this case we see that similarity invariance implies that $d_D(x, y)$ depends only on $\max\left\{\frac{|x|}{|y|}, \frac{|y|}{|x|}\right\}$ and the angle $\widehat{x0y}$. On the other hand, an inversion in a sphere about the origin swaps $|x|/|y|$ and $|y|/|x|$ and leaves $\widehat{x0y}$ invariant, so we see that it is an isometry. □

3.2 Other properties of j-type metrics

We said before that every j-type metric has an upper bound in terms of the quasihyperbolic metric. Surprisingly, it is also possible to get a universal lower bound by a metric, the so-called *half-apollonian metric* [11]. For a domain $D \subsetneq \mathbb{R}^n$ this metric is defined by

$$\eta_D(x, y) = \sup_{z\in\partial D} \left| \log \frac{|x-z|}{|y-z|} \right|.$$

The metric η_D is similarity invariant, and every Möbius mapping is bilipschitz.

Proposition 3.5. *For every j-type metric d and every $D \subsetneq \mathbb{R}^n$ we have $d_D \geq \eta_D$.*

Proof. Fix a boundary point w and $x, y \in D$. Using the triangle inequality for d and Proposition 2.1 we find that

$$\begin{aligned} d_D(x,y) &\geq \lim_{w' \to w} \big(d_D(x,w') - d(y,w')\big) \\ &= \lim_{w' \to w} \big(d_D(x,w') + \log\delta(w') - [d_D(y,w') + \log\delta(w')]\big) \\ &= \log|w-x|/|w-y|. \end{aligned}$$

Since this holds for every boundary point, we get

$$d_D(x,y) \geq \sup_{w \in \partial D} \log|w-x|/|w-y| = \eta_D(x,y). \qquad \square$$

Using the previous proposition and the quasihyperbolic upper bound we can squeeze in j-type metrics to get the exact value on some subset of the domain:

Corollary 3.6. *Let $w \in D$ and $z \in \partial D \cap S^{n-1}(w, \delta(w))$. Then $d_D(x,y) = j_D(x,y)$ for every $x, y \in [w, z)$.*

Proof. Since $\eta_D \leq d_D \leq k_D$ for every j-type metric, it suffices to prove that $\eta_D(x,y) = k_D(x,y)$ for $x, y \in [w,z)$. By the assumptions, for all points x on the line segment $[w,z]$ we have that $\operatorname{dist}(x, \partial D) = |x-z|$. Thus it is clear that

$$\eta_D(x,y) = \log\frac{|x-z|}{|y-z|}$$

It is easy to see that the segment $[x,y]$ is the shortest quasihyperbolic path connecting x and y, so

$$k_D(x,y) = \int_{|y-z|}^{|x-z|} \frac{dt}{t} = \log\frac{|x-z|}{|y-z|} = \eta_D(x,y). \qquad \square$$

Note, that though $d_D = \eta_D$ "on short straight lines", in a general situation we only have inequality. In fact the half-apollonian metric is not a j-type metric; it fails to satisfy the infinitesimal condition.

The following is easily checked by a direct computation using the definition of the j-metric, but also follows from Corollary 3.6 and the fact that line segments are also geodesic rays for the η_D-metric ([11, Example 3.4]).

Corollary 3.7. *Let $w \in D$ and $z \in \partial D \cap S^{n-1}(w, \delta(w))$. Then $[w,z)$ is a geodesic ray for the j-type metric d, i.e. for every $x, \xi, y \in [w,z)$ in this order we have*

$$d_D(x,y) = d_D(x,\xi) + d_D(\xi,y).$$

In general, if we have a metric d and a subadditive function $f\colon [0,\infty) \to [0,\infty)$ for which $f(x) = 0$ if and only if $x = 0$, then $f \circ d$ is also a metric. It turns out that the conditions for begin a j-type metric are so rigid, that this transformation is never possible in this context:

Corollary 3.8. *Let d be a j-type metric and $f_D\colon [0,\infty) \to [0,\infty)$ be a family of arbitrary functions. If $f \circ d$ is a j-type metric, then $f_D = \mathrm{id}$ for all relevant D.*

Proof. Suppose that $f \circ d$ is a j-type metric and fix a domain $D \subsetneq \mathbb{R}^n$. Let $w \in D$ and $z \in \partial D \cap S^{n-1}(w, \delta(w))$. Applying Corollary 3.6 to $f_D \circ d_D$ and d_D we find that

$$j_D(x,w) = f_D(d_D(x,w)) = f_D(j_D(x,w))$$

for every $x \in [w,z)$. But clearly $j_D(x,w)$ increases continuously from 0 to ∞ as x varies from w to z along $[w,z)$, so $f_D = \mathrm{id}$. $\square$

4 Isometries and geodesics of δ-type metrics

In this section we prove that every isometry of a δ-type metric is a Möbius mapping. Recall that a curve $\gamma \subset D$ is a *geodesic* of the metric d if

$$d(x,y) = d(x,z) + d(z,y)$$

for all points $x, z, y \in \gamma$ correctly ordered. A geodesic γ is a *geodesic line* if it is isometric to $\mathbb{R}$. Since $d(\gamma) = \tilde{d}(\gamma)$ for every path γ, we easily see that if γ is a geodesic of d, then it is a geodesic of $\tilde{d}$, as well. Also, if x and y lie on a geodesic of d_D, then $d_D(x,y) = \tilde{d}_D(x,y)$.

Proposition 4.1. *Let $D \subset \overline{\mathbb{R}^n}$ be a domain and γ be a path in D connecting two boundary points a and b. Then γ is a geodesic line of a δ-type metric d if and only if it is a geodesic line of Seittenranta's metric.*

Proof. Suppose first that γ is a geodesic line in the δ_D metric and suppose for a contradiction that γ is not a geodesic of d_D. Then we can find three points x, z, y in this order on γ such that $d_D(x,y) < d_D(x,z) + d_D(z,y)$. Let $x', y' \in \gamma$ be close to the end-points of γ. Then using the triangle inequality and the inequality $d_D \le \sigma_D$ we find that

$$d_D(x',y') \le \sigma_D(x',x) + d_D(x,y) + \sigma_D(y,y') = \delta_D(x',x) + d_D(x,y) + \delta_D(y,y'),$$

since γ is a geodesic of δ_D. Again, by the triangle equality for δ_D on γ, this implies that

$$d_D(x',y') - \delta_D(x',y') \le d_D(x,y) - \delta_D(x,y).$$

Now taking the limit $x' \to a$, $y' \to b$ (or the opposite) gives a contradiction to d_D being a δ-type metric, since

$$d_D(x,y) - \delta_D(x,y) < d_D(x,z) + d_D(z,y) - \big(\sigma_D(x,z) + \sigma_D(z,y)\big) \le 0,$$

where we again used that γ is a geodesic line of the δ_D-metric.

The proof of the converse implication is exactly analogous, all we need to do is exchange δ_D and d_D. $\square$

Now that we know that geodesic lines are the same for all δ-type metrics, it suffices to derive some properties of geodesics for Seittenranta's metric.

Lemma 4.2. *Let $D = \mathbb{R}^n \setminus \{0\}$. The geodesics of j_D are the rays from the origin.*

Proof. In the domain $\mathbb{R}^n \setminus \{0\}$ we have the following formula for the inner metric of j_D, the quasihyperbolic metric:

$$k_D(x,y) = \sqrt{\big(\widehat{x0y}\big)^2 + \big(\log \tfrac{|x|}{|y|}\big)^2},$$

[14, Section 2]. So if x and y lie on a geodesic line of the j_D metric, then

$$\sqrt{\big(\widehat{x0y}\big)^2 + \big(\log \tfrac{|x|}{|y|}\big)^2} = k_D(x,y) = j_D(x,y) = \big|\log \tfrac{|x|}{|y|}\big|,$$

which implies that $x = ry$ for some $r > 0$. □

Theorem 4.3. *Every geodesic line of a δ-type metric is an arc of a circle.*

Proof. By Proposition 4.1, it suffices to prove the claim for Seittenranta's metric. Let D be a domain and $x, y \in D$. Suppose that $a, b \in \partial D$ are such that

$$\delta_D(x,y) = \log(1 + |x,a,y,b|).$$

For simplicity we use a Möbius transformation to normalize to the situation $a = 0$, $b = \infty$. Let $G = \mathbb{R}^n \setminus \{0\}$. Then $\delta_D(z,w) \ge \delta_G(z,w)$ for all $z, w \in D$ and $\delta_D(x,y) = \delta_G(x,y)$. Suppose $z \in D$ lies on a geodesic connecting x and y. Then

$$\delta_G(x,y) = \delta_D(x,y) = \delta_D(x,z) + \delta_D(z,y) \ge \delta_G(x,z) + \delta_G(z,y),$$

So x, z, y lie on the same geodesic of δ_G as well. But now $\delta_G = j_G$, so, by the previous lemma, x, z, y have to lie on a ray from the origin, which means, after we consider the normalizing Möbius mapping, that a, x, z, y and b lie on the same circle. □

Remark 4.4. It is also true that every geodesic *segment* of Seittenranta's metric is an arc of a circle. However, geodesic segments are not necessarily the same for δ_D and d_D, and we do not know whether the latter are always circular.

Now we know that geodesics lines of δ-type metrics are well-behaving. In order to use this knowledge, we still need to show that geodesic lines exist, which is our next aim. Note that the kind of ball described in the next proposition always exists, see e.g. [10, Theorem 2]

Proposition 4.5. *Let $D \subset \overline{\mathbb{R}^n}$ be a domain and $B \subset D$ be a ball such that $\partial D \cap \partial B$ contains at least two points, a and b. Then the circular arc γ in D through a and b, perpendicular to ∂B at those points, is a geodesic line of δ_D. Moreover, if a, x, y, b lie on γ in this order, then*

$$\delta_D(x,y) = \log\left(\frac{|a-y|}{|a-x|}\frac{|b-x|}{|b-y|}\right).$$

Proof. We normalize the situation so that $a = 0$ and $b = \infty$ and $\gamma = \{te_1\colon t > 0\}$. Since the right half-space is contained in D, it is easy to see that

$$\sup_{a,b\in\partial D} \frac{|a-b|}{|a-x|\,|b-x|} = \frac{1}{|x|}.$$

Thus

$$\sigma_D(\gamma|_{[x,y]}) = \int_{|x|}^{|y|} \frac{dt}{t} = \log\frac{|y|}{|x|},$$

where $\gamma|_{[x,y]}$ is the subcurve of γ joining $x, y \in \gamma$. We also easily see that $\delta_D(x,y) \geq \log\frac{|y|}{|x|}$, which, since $\delta_D \leq \sigma_D$, implies that

$$\sigma_D(\gamma|_{[x,y]}) = \delta_D(\gamma|_{[x,y]}) = \log\tfrac{|y|}{|x|}.$$

This implies that γ is a geodesic line and proves the formula for $\delta_D(x,y)$. $\square$

We are now ready for the main theorem.

Theorem 4.6. *Let $D \subset \overline{\mathbb{R}^n}$ be a domain with at least two boundary points. Then every isometry $f\colon D \to \overline{\mathbb{R}^n}$ of a δ-type metric d is a Möbius mapping.*

Proof. Corollary 2.3 implies that every isometry is conformal, so the claim follows directly from Liouville's conformality theorem in dimension 3 or higher. In the case $n = 2$, we let γ be a geodesic line joining the points $a, b \in \partial D$ and fix $z \in \gamma$. Since $f(\gamma)$ is also a geodesic line, it is the arc of a circle, by Theorem 4.3. By Propositions 4.1 and 4.5 we conclude that

$$\begin{aligned}\log\left(\frac{|a-x|}{|a-y|}\frac{|b-y|}{|b-x|}\right) &= d_D(x,y)\\ &= d_{f(D)}(f(x),f(y)) = \log\left(\frac{|fa-f(x)|}{|fa-f(y)|}\frac{|fb-f(y)|}{|fb-f(x)|}\right),\end{aligned}$$

where fa and fb denote the end-points of $f(\gamma)$. Then we have shown that f acts as a Möbius mapping on an arc of a circle. The following well-known argument shows that such a mapping is actually a Möbius mapping.

Let m be a Möbius mapping which is the inverse of f on γ. Define $g = m \circ f$, and note that g acts as the identity on γ. Let F be the set of fixed points of g. Since g is continuous, we see that F is closed. Let $x \in F$ be an accumulation point. Since g is conformal there exists a neighborhood U of x such that g has a power series expansion in U. Since x is an accumulation point of F, there exists infinitely many points $y \in F \cap U$ for which $g(y) = y$. A coefficient comparison of g with the identity map then shows that $g|_U = \mathrm{id}_U$. This argument shows that g equals the identity in an open and closed set plus possibly in a discrete set. However, since D is a domain, the open and closed set is already all of D, which means that $m \circ f = \mathrm{id}$, so $f = m^{-1}$, a Möbius mapping. $\square$

Remark 4.7. Notice that we do not know whether every Möbius mapping is an isometry of a δ-type metric. Since there is no connection between d_D and $d_{D'}$ in the definition of δ-type metrics, we could only hope to prove that if $f\colon D \to D$ is a homeomorphism which also Möbius, then f is an isometry of d_D. Unfortunately, we were not able to do so.

5 Hyperbolic geometry

The philosophy behind relative metrics is that they are, to a greater or lesser degree, toy-models of hyperbolic geometry. They allow us to derive explicit estimates of quantities, starting with the hyperbolic distance. There are of course several generalizations of the hyperbolic metric. We divide these Möbius invariant metrics into two classes:

- Point-distance functions, such as the Apollonian [2] and cosh-metrics [7] and Seittenranta's metric, are simple to compute, but are not totally geodesic except in a ball;
- Length-metrics, such as Ferrand's metric [5] and the Kulkarni–Pinkall metric [12, 13], are totally geodesic, but very difficult to compute or even estimate.

All of the above metrics have the following interesting property, pointed out by D. Minda: suppose that $B \subset D$ is a ball such that $\partial B \cap \partial D$ contains at least two points a and b. Let γ be the circular arc through a and b which is orthogonal to ∂B. Then restricted to γ all of the metrics look exactly the same as the hyperbolic metric in the ball B.

Although hardly anything is known about the isometries of the length-metrics mentioned above, it is believed that the isometry group in each case is very small, consisting only of Möbius mappings. For the Apollonian metric this was shown in [9, 10]. In particular, the automorphism (isomorphisms preserving the domain) groups will then essentially always be finitely generated. This is clearly more rigidity than we would want. One way to get around this is to forget about the small scale altogether, and to study near-isometries, i.e. mappings f satisfying

$$d(x,y) - L \le d(f(x),f(y)) \le d(x,y) + L,$$

instead. This kind of thinking is related to the concept of Gromov hyperbolicity, which in some sense generalizes hyperbolic spaces to a more abstract setting. It is interesting to note that the Apollonian, cosh-, $\tilde{\jmath}_D$ and δ_D metrics are always Gromov hyperbolic, whereas the j_D metric is Gromov hyperbolic if and only if $D = \mathbb{R}^n \setminus \{a\}$, see [8]. It would be interesting to see whether the techniques in this paper could be modified to deal also with the near-isometries case.

Another loose end is the $\tilde{\jmath}$ metric. Recall the definition of this metric:

$$\tilde{\jmath}_D(x,y) = \log\left[\left(1+\frac{|x-y|}{\delta(x)}\right)\left(1+\frac{|x-y|}{\delta(y)}\right)\right]$$

(note that, despite the notation, this has nothing to do with the inner metric of j). The $\tilde{\jmath}_D$ metric has many of the properties of the j_D metric, but its boundary behavior is more like that of the δ_D metric. Unfortunately, this means that it is just beyond the reach of both the approaches featured in this paper. However, it might be more natural to investigate this metric in the approximate setting. For instance, suppose that γ is a geodesic of Seittenranta's metric; is it then a near-geodesic of the $\tilde{\jmath}_D$ metric?

References

[1] A. Beardon: *Geometry of Discrete Groups*, Graduate Texts in Mathematics 91, Springer-Verlag, New York, 1995.

[2] ——: The Apollonian metric of a domain in $\mathbb{R}^n$, pp. 91–108 in *Quasiconformal mappings and analysis* (P. Duren, J. Heinonen, B. Osgood and B. Palka (eds.)), Springer-Verlag, New York, 1998.

[3] L. Blumenthal: *Distance Geometry. A study of the development of abstract metrics. With an introduction by Karl Menger*, Univ. of Missouri Studies Vol. 13, No. 2, Univ. of Missouri, Columbia, 1938.

[4] D. Burago, Yu. Burago and S. Ivanov: *A course in metric geometry*, Graduate Studies in Mathematics, 33, Amer. Math. Soc., Providence, RI, 2001.

[5] J. Ferrand: A characterization of quasiconformal mappings by the behavior of a function of three points, pp. 110–123 in *Proceedings of the 13th Rolf Nevalinna Colloquium* (Joensuu, 1987; I. Laine, S. Rickman and T. Sorvali (eds.)), Lecture Notes in Mathematics Vol. 1351, Springer-Verlag, New York, 1988.

[6] F. Gehring and B. Osgood: Uniform domains and the quasihyperbolic metric, *J. Anal. Math.* **36** (1979), 50–74.

[7] P. Hästö: A new weighted metric: the relative metric II, *J. Math. Anal. Appl.* **301** (2005), no. 2, 336–353.

[8] ——: Gromov hyperbolicity of the j_D and $\tilde{\jmath}_D$ metrics, *Proc. Amer. Math. Soc.* **134** (2006), 1137–1142..

[9] P. Hästö and Z. Ibragimov: Apollonian isometries of planar domains are Möbius mappings, *J. Geom. Anal.* **15** (2005), no. 2, 229–237.

[10] ———: Apollonian isometries of regular domains are Möbius mappings, submitted (2004).

[11] P. Hästö and H. Lindén: Isometries of the half-apollonian metric, *Complex Var. Theory Appl.* **49** (2004), 405–415.

[12] D. Herron, W. Ma and D. Minda: A Möbius invariant metric for regions on the Riemann sphere, pp. 101–118 in *Future Trends in Geometric Function Theory* (RNC Workshop, Jyväskylä 2003; D. Herron (ed.)), Rep. Univ. Jyväskylä Dept. Math. Stat. **92** (2003).

[13] R. Kulkarni and U. Pinkall: A canonical metric for Möbius structures and its applications, *Math. Z.* **216** (1994), 89–129.

[14] G. Martin and B. Osgood: The quasihyperbolic metric and associated estimates on the hyperbolic metric, *J. Anal. Math.* **47** (1986), 37–53.

[15] P. Seittenranta: Möbius-invariant metrics, *Math. Proc. Cambridge Philos. Soc.* **125** (1999), 511–533.

[16] M. Vuorinen: *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., Vol. 1319, Springer, Berlin, 1988.