INEQUALITIES AND GEOMETRY OF THE APOLLONIAN AND RELATED METRICS

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ABSTRACT. In this paper we consider inequalities between the Apollonian metric, the Apollonian inner metric, the j_G metric and the quasihyperbolic metric. We will show that many of these inequalities have nice geometric interpretations in terms of the domain G in which the metrics are defined.

1. INTRODUCTION

In this paper we consider the Apollonian metric which was introduced in [1, 5]. Some basic features of this metric are that it is Möbius invariant and equals the hyperbolic metric in balls and halfspaces. We also consider the inner metric of the Apollonian metric, the j_G metric and its inner metric, the quasihyperbolic metric. We look at inequalities between these metrics and are especially interested in the geometric meaning of these inequalities. We start by defining the metrics and stating our main results. The notation used conforms largely to that of [4] and [34], the reader can consult Section 1.1 of this paper, if necessary.

We will be considering domains (open connected non-empty sets) G in the Möbius space $\overline{\mathbb{R}^n}$. The Apollonian metric is defined for $x, y \in G \subsetneq \overline{\mathbb{R}^n}$ by

$$\alpha_G(x,y) := \sup_{a,b\in\partial G} \log \frac{|a-x|}{|a-y|}\frac{|b-y|}{|b-x|}$$

(with the understanding that $|\infty - x|/|\infty - y| = 1$). This formula has a very nice geometric interpretation (indeed, this is one of the main reasons for the interest in the metric), see Section 1.2. It is in fact a metric if and only if the complement of G is not contained in a hyperplane and a pseudometric otherwise, as was noted in [5, Theorem 1.1]. This metric was introduced in [5] and has also been considered in [7, 12, 28, 30] and [15]–[24]. It should also be noted that the same metric has been studied from a different perspective under the name of the Barbilian metric for instance in [1, 2, 3, 6, 8, 26], cf. [9] for a historical overview and more references. One interesting historical point, made in [9], is that Barbilian himself suggested the name "Apollonian metric", which was later independently coined by Beardon.

Let $\gamma\colon [0,1] \to G \subset \mathbb{R}^n$ be a path, i.e. a continuous function. If d is a metric in G then the d-length of γ is defined by

$$d(\gamma) := \sup \sum_{i=0}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

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where the supremum is taken over $k < \infty$ and all sequences $\{t_i\}$ satisfying $0 = t_0 < t_1 < \cdots < t_k = 1$. All the paths in this paper are assumed to be rectifiable, that is, to have finite Euclidean length. The inner metric of d is defined by the formula

$$\tilde{d}(x, y) := \inf_{\gamma} d(\gamma),$$

where the infimum is taken over all paths connecting x and y in G. We denote the inner metric of the Apollonian metric by $\tilde{\alpha}_G$ and call it the Apollonian inner metric. Strictly speaking, the Apollonian inner metric is only a pseudometric in a general domain $G \subsetneq \mathbb{R}^n$; it is a metric if and only if the complement of G is not contained in an $(n-2)$-dimensional plane [18, Theorem 1.2].

Let $G \subsetneq \mathbb{R}^n$ be a domain and $x, y \in G$. The j_G metric, which is a modification from [33] of a metric from [13], is defined by

$$j_G(x, y) := \log\left(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}}\right).$$

The quasihyperbolic metric from [14] is defined by

$$k_G(x, y) := \inf_{\gamma} \int \frac{|dz|}{d(z, \partial G)},$$

where the infimum is taken over all paths γ joining x and y in G. Note that the quasihyperbolic metric is the inner metric of the j_G metric.

In order to state the results in a succinct manner we define some relations on the set of metrics in G.

DEFINITION 1.1. Let d and d' be metrics on G.

(1) We write $d \lesssim d'$ if there exists a constant $K > 0$ such that $d \le K d'$.
(2) We write $d \approx d'$ if $d \lesssim d'$ and $d \gtrsim d'$.
(3) We write $d \ll d'$ if $d \lesssim d'$ and $d \not\gtrsim d'$.
(4) We write $d \lessgtr d'$ if $d \not\lesssim d'$ and $d \not\gtrsim d'$.

Let us first of all note that the following inequalities hold in every domain $G \subsetneq \mathbb{R}^n$:

$$\alpha_G \lesssim j_G \lesssim k_G \quad \text{and} \quad \alpha_G \lesssim \tilde{\alpha}_G \lesssim k_G. \tag{1}$$

The first two are from [5, Theorem 3.2] and the second two from [15, Remark 5.2 and Corollary 5.4]. We see that of the four metrics to be considered, the Apollonian is the smallest and the quasihyperbolic is the largest.

We will undertake a systematic study of which of the inequalities in (1) can hold in the strong form with $\ll$ and which of the relations $j_G \ll \tilde{\alpha}_G$, $j_G \approx \tilde{\alpha}_G$ and $j_G \gg \tilde{\alpha}_G$ can hold. Thus we are led to twelve inequalities, which are given along with the results in Table 1, where we have indicated in column A whether the inequality can hold in simply connected planar domains and in column B whether it can hold in an arbitrary proper subdomains of $\mathbb{R}^n$. From the table we see that most of the cases cannot occur, which means that there are many restrictions on which inequalities can occur together. For instance, we deduce from items 1–4 that $j_G \approx \tilde{\alpha}_G$ implies that $\alpha_G \approx k_G$ and from items 9–12 that the inequality $\tilde{\alpha}_G \ll j_G$ cannot occur in simply connected planar domains.

Since $\lesssim$ is not a linear order, it is also possible that two metrics are not comparable. Therefore we consider separately the case when $j \lessgtr \tilde{\alpha}$ in Section 6. Since the table does not

#	Inequality	A	B	#	Inequality	A	B
1.	$\alpha \approx j \approx \tilde\alpha \approx k$	+	+	7.	$\alpha \approx j \ll \tilde\alpha \ll k$	-	-
2.	$\alpha \ll j \approx \tilde\alpha \approx k$	-	-	8.	$\alpha \ll j \ll \tilde\alpha \ll k$	-	-
3.	$\alpha \approx j \approx \tilde\alpha \ll k$	-	-	9.	$\alpha \approx \tilde\alpha \ll j \approx k$	-	+
4.	$\alpha \ll j \approx \tilde\alpha \ll k$	-	-	10.	$\alpha \ll \tilde\alpha \ll j \approx k$	-	+
5.	$\alpha \approx j \ll \tilde\alpha \approx k$	+	+	11.	$\alpha \approx \tilde\alpha \ll j \ll k$	-	?
6.	$\alpha \ll j \ll \tilde\alpha \approx k$	+	+	12.	$\alpha \ll \tilde\alpha \ll j \ll k$	-	?

TABLE 1. Inequalities between the metrics α_G, j_G, $\tilde\alpha_G$ and k_G. The subscripts are omitted for clarity with the understanding that every metric is defined in the same domain. The A-column refers to whether the inequality can occur in simply connected planar domains, the B-column to whether it can occur in proper subdomains of $\mathbb{R}^n$.

list this case, one should be careful with the interpretations; for instance, it is not true that the inequality $\tilde\alpha_G \ll k_G$ cannot occur in simply connected planar domains, contrary to what might be thought by considering entries 3, 4, 7, 8, 11 and 12.

As a motivation for this study we mention that many of the inequalities have been previously studied and some have geometrical characterizations. A domain in which $j_G \approx k_G$ holds is known as uniform and has found applications in many areas of analysis (see e.g. [11]). The condition $\alpha_G \approx j_G$ was shown in [19, Theorem 1.3] to be equivalent with the complement of G being thick in the sense of [32], which means that this inequality is also connected to various interesting properties, see [32, Section 1.4] for details. Additionally, the inequalities $\tilde\alpha_G \approx k_G$, $\alpha_G \approx \tilde\alpha_G$ and $\alpha_G \approx k_G$, which have been called quasi-isotropy, Apollonian quasiconvexity and A-uniformity, respectively, have some nice geometric interpretations and have been considered in [15, 16, 17].

The structure of the rest of this paper is as follows. We start by reviewing the notation and terminology used. The bulk of the paper consists of five sections which are organized along the different methods used to prove the inequalities in Table 1. Specifically, in Section 2 we consider the comparison property and uniformity and in Section 3 quasi-isotropy. The main problem in Sections 4 and 5 is the inequality $\alpha_G \gtrsim \tilde\alpha_G$. Since we do not have a good geometric understanding of this inequality the proofs in these sections are sometimes quite long. In Section 6 we consider the case when the metrics j_G and $\tilde\alpha_G$ are not comparable.

1.1. **NOTATION.** The notation used conforms largely to that in [4] and [34], as was mentioned in the introduction.

We denote by $\{e_1, e_2, \ldots, e_n\}$ the standard basis of $\mathbb{R}^n$ and by n the dimension of the Euclidean space under consideration and assume that $n \geq 2$. For $x \in \mathbb{R}^n$ we denote by x_i its i^{th} coordinate. The following notation is used for Euclidean balls and spheres:

$$B^n(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\},\ S^{n-1}(x,r) := \{y \in \mathbb{R}^n : |x-y| = r\},$$

$$B^n := B^n(0,1),\ S^{n-1} := S^{n-1}(0,1),$$

We denote by $[x,y]$ the closed segment between x and y.

We use the notation $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$ for the one point compactification of $\mathbb{R}^n$, equipped with the spherical chordal metric. Thus an open ball of $\overline{\mathbb{R}^n}$ as an open Euclidean ball, an open half-space or the complement of a closed Euclidean ball. We denote by ∂G, G^c and $\overline{G}$ the boundary, complement and closure of G, all with respect to $\overline{\mathbb{R}^n}$.

We also need some notation for quantities depending on the underlying Euclidean metric. For $x \in G \subsetneq \mathbb{R}^n$ we write $\delta(x) := d(x, \partial G) := \min\{|x-z| \colon z \in \partial G\}$. For a path γ in $\mathbb{R}^n$ we denote by $\ell(\gamma)$ its Euclidean length.

1.2. **THE APOLLONIAN BALLS APPROACH.** In this subsection we present the Apollonian balls approach which gives a geometric interpretation of the Apollonian metric.

For $x, y \in G \subsetneq \overline{\mathbb{R}^n}$ we define

$$q_x := \sup_{a \in \partial G} \frac{|a-x|}{|a-y|}, \; q_y := \sup_{b \in \partial G} \frac{|b-y|}{|b-x|}.$$

The numbers q_x and q_y are called the Apollonian parameters of x and y (with respect to G) and by definition $\alpha_G(x,y) = \log(q_x q_y)$. The balls (in $\overline{\mathbb{R}^n}$!)

$$B_x := \Big\{ z \in \overline{\mathbb{R}^n} \colon \frac{|z-x|}{|z-y|} < \frac{1}{q_x} \Big\} \quad \text{and} \quad B_y := \Big\{ z \in \overline{\mathbb{R}^n} \colon \frac{|z-y|}{|z-x|} < \frac{1}{q_y} \Big\},$$

are called the *Apollonian balls* about x and y, respectively. We collect some immediate results regarding these balls, similar results obviously hold with x and y interchanged.

(1) $x \in B_x \subset G$ and $\overline{B_x} \cap \partial G \neq \emptyset$.
(2) If i_x and i_y denote the inversions in the spheres ∂B_x and ∂B_y, then $y = i_x(x) = i_y(x)$.
(3) If $\infty \notin G$, we have $q_x \geq 1$. If, moreover, $\infty \notin \overline{G}$, then $q_x > 1$.

2. BASIC INEQUALITIES

In this section we define the comparison property and uniformity which are the relations from the introduction that have been most thoroughly studied in the past.

2.1. **THE COMPARISON PROPERTY.** In the paper [19] the term *comparison property* was introduced for the relation $\alpha_G \approx j_G$. In [19, Theorem 1.3] it was shown that this property is equivalent to the complement of G being thick in the sense of [32]. There is a discussion of implications of thickness in [32, Section 1.4], as an example we mention that a bilipschitz mapping with sufficiently small bilipschitz constant from a thick set may be continued to a bilipschitz mapping of the whole space, by [31, Theorem 6.2].

Recall that the inner metric is defined as a supremum over a sum of the base distance. From this it directly follows that if d_1 and d_2 are metrics in the same domain, then $d_1 \approx d_2$ implies that $\tilde{d}_1 \approx \tilde{d}_2$. Therefore Inequalities 3, $\alpha_G \approx j_G \approx \tilde{\alpha}_G \ll k_G$, and 7, $\alpha_G \approx j_G \ll \tilde{\alpha}_G \ll k_G$, cannot occur, since in both we have assumed the comparison property but not the equivalence of the inner metrics, $\tilde{\alpha}_G$ and k_G.

Recall that $\alpha_G \leq 2 j_G$ in every domain $G \subsetneq \mathbb{R}^n$ by [5, Theorem 3.2]. Also, it was shown in [30, Theorem 4.2] that if $G \subsetneq \mathbb{R}^n$ is convex, then $j_G \leq \alpha_G$. So $\alpha_G \approx j_G$ in convex domains.

LEMMA 2.1. *Inequality 5, $\alpha_G \approx j_G \ll \tilde{\alpha}_G \approx k_G$, holds in the domain $G := \{x \in \mathbb{R}^n \colon |x_n| < 1\}$.*

Proof. The domain G is clearly convex, hence it has the comparison property by [30, Theorem 4.2]. From this it follows that $\alpha_G \approx j_G$ and $\tilde{\alpha}_G \approx k_G$. Consider then the points Re_1 and $-Re_1$, where $R > 0$. We have $j_G(Re_1, -Re_1) = \log(1 + 2R)$ and $k_G(Re_1, -Re_1) = 2R$, hence $j_G \ll k_G$, which concludes the proof. □

2.2. **UNIFORMITY.** Uniform domains were introduced by O. Martio and J. Sarvas in [27, 2.12], but the following definition is an equivalent form from [13, (1.1)]. In the paper [11] there is a survey of characterizations and implications of uniformity, as an example we mention that a Sobolev mapping can be extended from G to the whole space if G is uniform, see [25].

DEFINITION 2.2. A domain $G \subsetneq \mathbb{R}^n$ is said to be *uniform* with constant K if for every $x, y \in G$ there exists a path γ, parameterized by arc-length, connecting x and y in G, such that

(1) $\ell(\gamma) \le K|x-y|$; and
(2) $K\delta(\gamma(t)) \ge \min\{t, \ell(\gamma)-t\}$.

The relevance of uniformity to our investigation comes from Corollary 1 of [13] which states that a domain is uniform if and only if $k_G \approx j_G$. Thus we have a geometric characterization of domains satisfying this inequality as well.

EXAMPLE 2.3. The unit ball is uniform and has the comparison property. Hence $\alpha_{B^n} \approx j_{B^n} \approx \tilde{\alpha}_{B^n} \approx k_{B^n}$ and so Inequality 1 can occur.

In fact, Inequality 1 holds in every quasiball, by [15, Corollary 6.9].

LEMMA 2.4. *Inequalities 9, $\alpha_G \approx \tilde{\alpha}_G \ll j_G \approx k_G$, and 10, $\alpha_G \ll \tilde{\alpha}_G \ll j_G \approx k_G$ cannot occur in simply connected planar domains.*

Proof. We note that in both these inequalities we have $j_G \approx k_G$ among the assumptions. But a simply connected planar domain is uniform if and only if it is a quasidisk, by [27, Theorem 2.24], and we know that quasidisks have the comparison property, by [19, Corollary 6.3]. Therefore $j_G \approx k_G$ implies that $\alpha_G \approx j_G$ which contradicts $\alpha_G \ll j_G$ in both inequalities. □

3. QUASI-ISOTROPY

We start by introducing some concepts which allow us to calculate the Apollonian inner metric. The concept of quasi-isotropy was introduced in [15] and was studied in depth in [17]. A very similar notion used by Zair Ibragimov is conformality, see [22, 23, 24].

DEFINITION 3.1. We say that a metric space (G, d) with $G \subset \mathbb{R}^n$ is K*–quasi-isotropic* if

$$\limsup_{r\to 0} \frac{\sup\{d(x,z)\colon |x-z|=r\}}{\inf\{d(x,y)\colon |x-y|=r\}} \le K$$

for every $x \in G$. A 1-quasi-isotropic metric space is called *isotropic.*

We say that a domain $G \subsetneq \mathbb{R}^n$ is quasi-isotropic if (G, α_G) is, similarly for isotropic. We define the function qi on the set of proper subdomains of $\mathbb{R}^n$ so that $qi(G)$ is the least constant for which G is quasi-isotropic or $qi(G) = \infty$ if G is not quasi-isotropic for any K. The notion of quasi-isotropy is extended to domains in $\overline{\mathbb{R}^n}$ by Möbius invariance.

Although the Apollonian metric is not isotropic in general, it is possible to define a directed density as follows:

$$\bar{\alpha}_G(x;r) = \lim_{t\to 0} \tfrac{1}{t}\alpha_G(x, x+t\tfrac{r}{|r|}),$$

where $r \in \mathbb{R}^n \setminus \{0\}$. If $\bar{\alpha}_G(x;r)$ is independent of r in every point of G, then the Apollonian metric is isotropic and we may denote $\bar{\alpha}_G(x) := \bar{\alpha}_G(x;e_1)$ and call this function the density of α_G at x. With this concept we can give the following alternative characterization of quasi-isotropy.

LEMMA 3.2 (Lemma 3.5, [17]). *For $G \subsetneq \mathbb{R}^n$ we have*

$$qi(G) = \sup_{x\in G} \frac{\sup_{r\in S^{n-1}} \bar{\alpha}_G(x;r)}{\inf_{r\in S^{n-1}} \bar{\alpha}_G(x;r)},$$

with the understanding that if $\bar{\alpha}_G(x;r) = 0$ for some $x \in G$ and $r \in S^{n-1}$, then $qi(G) = \infty$.

When we do not need the exact value of the quasi-isotropy constant the following lemma is often more convenient to use.

LEMMA 3.3 (Corollary 5.11, [15]). *Let $G \subsetneq \mathbb{R}^n$ be L–quasi-isotropic. Then $\bar{\alpha}_G(x;r)\delta(x) \geq 1/L$ for every $x \in G$ and $r \in S^{n-1}$. If conversely $1/L \leq \bar{\alpha}_G(x;r)\delta(x)$ for every $x \in G$ and $r \in S^{n-1}$, then G is $2L$–quasi-isotropic.*

In order to present an integral formula for the Apollonian inner metric we need to relate the density of the Apollonian metric with the limiting concept of the Apollonian balls, which we call the Apollonian spheres.

DEFINITION 3.4. Let $G \subsetneq \overline{\mathbb{R}^n}$, $x \in G$ and $\theta \in S^{n-1}$.

- If $B^n(x+s\theta, s) \subset G$ for every $s > 0$ and $\infty \notin G$, then let $r_+ = \infty$.
- If $B^n(x+s\theta, s) \subset G$ for every $s > 0$ and $\infty \in G$, then let r_+ be the largest negative real number such that $G \subset B^n(x+r_+\theta, |r_+|)$.
- Otherwise let $r_+ > 0$ be the largest real number such that $B^n(x+r_+\theta, r_+) \subset G$.

Define r_- in the same way but using the vector $-\theta$ instead of θ. We define the *Apollonian spheres through x in direction θ* by $S_+ := S^{n-1}(x+r_+\theta, r_+)$ and $S_- := S^{n-1}(x-r_-\theta, r_-)$ for finite radii and by the limiting half-space for infinite radii.

Using these spheres we can present a useful result from [15].

LEMMA 3.5 (Lemma 5.8, [15]). *Let $G \subsetneq \overline{\mathbb{R}^n}$ be open, $x \in G \setminus \{\infty\}$ and $\theta \in S^{n-1}$. Let $r_\pm$ be the radii of the Apollonian spheres $S_\pm$ at x in direction θ. Then*

$$\bar{\alpha}_G(x;\theta) = \frac{1}{2r_+} + \frac{1}{2r_-},$$

where we understand $1/\infty = 0$.

REMARK 3.6. The previous lemma was proven in [15] only for the case $G \subsetneq \mathbb{R}^n$. The general case is proved in exactly the same manner.

The following result shows that we can find the Apollonian inner metric by integrating over the directed density, as should be expected. Piecewise continuously differentiable means continuously differentiable except in a finite number of points.

LEMMA 3.7 (Theorem 1.4, [18]). *If $x, y \in G \subsetneq \mathbb{R}^n$, then*

$$\tilde{\alpha}_G(x,y) = \inf_\gamma \int \bar{\alpha}_G(\gamma(t);\gamma'(t))|\gamma'(t)|dt,$$

where the infimum is taken over all paths connecting x and y in G that are piecewise continuously differentiable (with the understanding that $\bar{\alpha}_G(z;0)0 = 0$ for all $z \in G$, even though $\bar{\alpha}_G(z;0)$ is not defined).

The relevance of quasi-isotropy to the study of inequalities comes from the following lemma.

LEMMA 3.8 (Corollary 5.4, [18]). *For $G \subsetneq \mathbb{R}^n$ the following conditions are equivalent:*

(1) *G is quasi-isotropic;*
(2) *$\tilde{\alpha}_G \approx k_G$; and*
(3) *$j_G \lesssim \tilde{\alpha}_G$.*

COROLLARY 3.9. *Inequalities 4, $\alpha_G \ll j \approx \tilde{\alpha}_G \ll k_G$, and 8, $\alpha_G \ll j \ll \tilde{\alpha}_G \ll k_G$, cannot occur.*

Proof. In both cases the assumption $\tilde{\alpha}_G \ll k_G$ implies that $j_G \not\lesssim \tilde{\alpha}_G$, by the previous lemma. This contradicts $j_G \approx \tilde{\alpha}_G$ (in 4) and $j_G \ll \tilde{\alpha}_G$ (in 8). □

In [15] an exterior ball condition of G was defined as follows: for every $z \in \partial G$ there exists a ball of radius r in the set $G^c \cap B(z, Lr)$. This condition was shown to be sufficient for the comparison property. The following theorem features a local version of this property.

THEOREM 3.10. *Let $G \subsetneq \mathbb{R}^n$ be arbitrary and $L > 1$. For every $x \in G$, let $z \in \partial G$ be such that $|x - z| = \delta(x)$ and suppose there exists a ball B with radius $r_0 = \delta(x)/\sqrt{L^2-1}$ such that*

(1) *$d := d(z, \partial B) \le r_0(L-1)$; and*
(2) *for any $y \in B$ the line segment $[x, y]$ connecting x and y intersects ∂G.*

Then the inequality $\tilde{\alpha}_G \approx k_G$ holds.

Proof. It follows from Lemma 3.8 that $\tilde{\alpha}_G \approx k_G$ if and only if G is quasi-isotropic. By Lemma 3.3 it suffices to check that there exists a constant K such that $\bar{\alpha}_G(x; r)\delta(x) \ge 1/K$ for every $x \in G$ and $r \in S^1$ in order to show that G is quasi-isotropic.

Let $x \in G$ and $r \in S^1$, and fix a ball B as in the statement of the theorem. By (2), we see that the Apollonian spheres with respect to G are smaller in size than with respect to $\mathbb{R}^n \setminus B$ and since $\mathbb{R}^n \setminus B$ is isotropic (as the Apollonian metric equals the hyperbolic metric in a ball) we get

$$\bar{\alpha}_G(x; r) \ge \bar{\alpha}_{\mathbb{R}^n \setminus B}(x; r) = \bar{\alpha}_{\mathbb{R}^n \setminus B}(x) = \frac{1}{\delta(x) + d} - \frac{1}{\delta(x) + d + 2r_0},$$

(the second term is negative, as the corresponding ball contains the point ∞). Now if we use (1) and $r_0 = \delta(x)/\sqrt{L^2-1}$, from the hypothesis, then it is easy to estimate that

$$\bar{\alpha}_G(x; r)\delta(x) \ge \frac{2r_0\delta(x)}{(\delta(x) + r_0(L-1))(\delta(x) + r_0(L+1))} = \frac{1}{L + \sqrt{L^2-1}},$$

and we have a lower bound for $\bar{\alpha}_G(x; r)\delta(x)$. □

The next result gives some concrete examples of when the conditions of the previous theorem are satisfied. Although it is intuitively obvious that the examples satisfy the conditions of the theorem, verifying this requires some lengthy calculations and several different cases. Details are provided in [29].

EXAMPLE 3.11. Let $D \subsetneq \mathbb{R}^2$ be convex and D' be a subset of D which is compact and convex. Let F be a line segment connecting ∂D to $\partial D'$. Then Inequality 6, $\alpha_G \ll j_G \ll \tilde{\alpha}_G \approx k_G$, holds in the domain $G := D \setminus (D' \cup F)$.

4. APOLLONIAN QUASICONVEXITY, BASED ON THE COMPARISON PROPERTY

In this section we consider Inequalities 2, 11 and 12. We prove that none of them can occur in simply connected planar domains and that the first one cannot occur in more general domains, either. Whether the latter two can occur in this case is unclear, although it seems improbable.

We say that a metric space (G,d) is K–*quasiconvex* if for every $x,y\in G$ there exists a path γ connecting x and y in G such that $d(\gamma)\le Kd(x,y)$. We note that the metric d is quasiconvex if and only if $d\approx\tilde d$. In [15, Proposition 7.3] it was shown that if α_G is quasiconvex in a simply connected planar domain, then G has the comparison property. Thus $\alpha_G\approx\tilde\alpha_G$ implies $\alpha_G\approx j_G$ and so Inequality 11, $\alpha_G\approx\tilde\alpha_G\ll j_G\ll k_G$, cannot occur in this case. Let us move on to the other two inequalities.

4.1. THE TWELFTH INEQUALITY.

In this subsection we prove that the inequalities $\alpha_G\ll\tilde\alpha_G\ll j_G\ll k_G$ cannot occur in simply connected planar domains. We are not be able to establish whether or not it can occur in domains in general. Let us first quote two lemmas from [15].

LEMMA 4.1. [15, Lemma 7.1] *Let $G\subset\mathbb{R}^n$ be a domain such that $G\cap B^n=H^n\cap B^n$. Then for every $0<s<1$ and every path γ connecting se_n with S^{n-1} we have*

$$\alpha_G(\gamma)\ge\tfrac12(\operatorname{arsinh}s^{-1}-\operatorname{arsinh}1).$$

LEMMA 4.2. [15, Lemma 7.2] *Let $G\subsetneq\mathbb{R}^2$ be a simply connected domain and $x,y\in G$ be such that $N\alpha_G(x,y)<j_G(x,y)$ for some $N>40$. Then there exists a disk $B:=B^2(b,r)$ and a unit vector $e\in S^1$ such that*

1. *for all $z\in G^c\cap B$ we have $\langle z-b,e\rangle\le 4N^{-1/2}r$; and*
2. *the points $b\pm 0.9re$ belong to different path components of $B\cap G$.*

(Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product.)

The proof of the next result is similar to that of Proposition 7.3 in [15].

PROPOSITION 4.3. *If $G\subsetneq\mathbb{R}^2$ is a simply connected domain which does not have the comparison property, then $\tilde\alpha_G\not\lesssim j_G$.*

Proof. Let us assume that G is simply connected but does not have the comparison property. Let $x,y\in G$ be such that $N\alpha_G(x,y)\le j_G(x,y)$ for some $N>300$ and define $\epsilon:=2N^{-1/4}$.

Let B be the disk from Lemma 4.2 and assume without loss of generality that $B=B^2$ and $e=e_2$. Let γ be a path connecting ϵe_2 and $-\epsilon e_2$ in G. Every such path passes through S^1, since it is easy to see that ϵe_2 and $-\epsilon e_2$ are in different components of $B^2\cap G$.

Let γ_1 be the part of γ in the component of $G\cap B^2$ which contains ϵe_2. In order to derive a lower bound for the density of the Apollonian metric in γ_1 it suffices to consider the subset $B^2\cap\partial G$ of the boundary of G. The lower bound gets even smaller if we assume that $B^2\cap\partial G=\{x\in B^2\colon x_2=-4/\sqrt N\}$. We can then apply Lemma 4.1 to γ_1 after using an auxiliary translation $(x\mapsto x+4e_2/\sqrt N)$ and scaling $(x\mapsto\sqrt N x/\sqrt{N-16})$. Under these operations the point ϵe_2 is mapped to $(\epsilon\sqrt N+4)e_2/\sqrt{N-16}$ and so the lemma applies with

$s = (\epsilon\sqrt{N}+4)/\sqrt{N-16} = (2N^{1/4}+4)/\sqrt{N-16}$. Thus we find that

$$\tilde{\alpha}_G(\epsilon e_2, -\epsilon e_2) \geq \frac{1}{2}\operatorname{arsinh}\left(\frac{\sqrt{N-16}}{2N^{1/4}+4}\right) - \frac{1}{2}\operatorname{arsinh} 1.$$

On the other hand, we have

$$j_G(\epsilon e_2, -\epsilon e_2) \leq \log(1+2\epsilon/(\epsilon - 4/\sqrt{N})) = \log(1+2N^{1/4}/(N^{1/4}-2)).$$

Hence we see that $\tilde{\alpha}_G(\epsilon e_2, -\epsilon e_2)/j_G(\epsilon e_2, -\epsilon e_2) \to \infty$ as $N \to \infty$ which means that $\tilde{\alpha}_G \not\lesssim j_G$. □

The following corollary is immediate.

COROLLARY 4.4. *Inequalities 11 and 12 cannot occur in simply connected planar domains.*

Recall that a quasidisk is the image of a disk under a quasiconformal mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. Using the previous result we get yet another characterization of quasidisks (for characterizations in terms of the Apollonian metric see [15], for lots of other characterizations see [10]).

COROLLARY 4.5. *A simply connected planar domain G is a quasidisk if and only if $\tilde{\alpha}_G \lesssim j_G$.*

Proof. If G is a quasidisk, then G is uniform by [27, Theorem 2.24] and [27, Corollary 2.33], hence $\tilde{\alpha}_G \lesssim k_G \approx j_G$. Assume conversely that $\tilde{\alpha}_G \lesssim j_G$. It follows from Proposition 4.3 that G has the comparison property and hence also $\tilde{\alpha}_G \approx k_G$ (as in Section 2.1). We thus have $k_G \approx \tilde{\alpha}_G \lesssim j_G$, which means that G is uniform and hence a quasidisk by [27, Theorem 2.24]. □

4.2. **THE SECOND INEQUALITY.** In this subsection we prove that Inequality 2, $\alpha_G \ll j_G \approx \tilde{\alpha}_G \approx k_G$, cannot occur in any domain.

Let us quote a lemma from [19] that was used in the proof of Lemma 4.2 which we use to derive a variant of that lemma which is valid in $\mathbb{R}^n$.

LEMMA 4.6. [19, Lemma 3.1] *Let $G \subsetneq \mathbb{R}^n$ be a domain and $x, y \in G$ be points such that $\alpha_G(x,y) \leq j_G(x,y)/N$, for $N \geq 16$. Then there exist balls B, B_1 and B_2 with radii r and $r_1 = r_2 \geq (1-3N^{-1/2})r/2$ such that $B_1, B_2 \subset G \cap B$, $d(B_1, B_2) = 2(r-2r_1)$ and that the segment connecting the centers of B_1 and B_2 intersects ∂G.*

The following corollary is proved from this lemma by considering a sufficiently small ball centered at a boundary point on the segment connecting the centers of B_1 and B_2.

COROLLARY 4.7. *If $G \subsetneq \mathbb{R}^n$ does not have the approximation property, then for every $\epsilon > 0$ there exists a point $z \in \partial G$, a real number $r > 0$ and a unit vector $\theta \in S^{n-1}$ such that for every $w \in G^c \cap B^n(z,r)$ we have $\langle w, \theta \rangle \leq \epsilon r$.*

It follows directly from the next theorem that Inequality 2 cannot occur.

THEOREM 4.8. *If $G \subsetneq \mathbb{R}^n$ is quasi-isotropic and uniform, then G has the comparison property.*

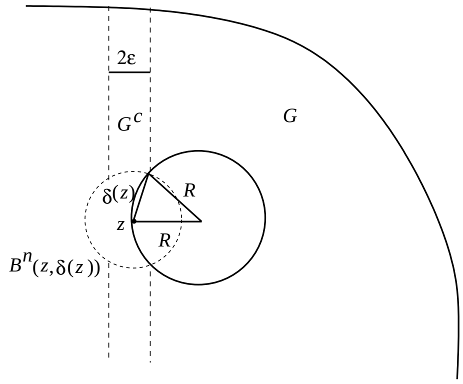

FIGURE 1. The Apollonian sphere at z.

Proof. Assume that G is L–quasi-isotropic but does not have the comparison property. We will show that this implies that G is not uniform.

Let $0 < \epsilon < 1/(256L^4)$ and choose $u \in \partial G$, $r > 0$ and $e \in S^{n-1}$ such that $\langle v, e\rangle \le \epsilon r$ for all $v \in G^c \cap B^n(u,r)$ (possible by Corollary 4.7). We assume without loss of generality that $u = 0$, $r = 1$ and $e = e_1$. Consider the points $x := \sqrt{\epsilon}e_1$ and $y := -\sqrt{\epsilon}e_1$ and paths connecting them in G. Let us denote $D := \{z \in B^n(0, \sqrt[4]{\epsilon})\colon z_1 = 0\}$ and define A to be the set of paths joining x and y in G which intersect D, and B to be the set of paths joining x and y in G which do not intersect D.

Let us consider first a path $\gamma \in A$ parameterized by arclength. Let $z \in \gamma \cap D$, $w \in S^{n-1}(z, \delta(z)) \cap \partial G$ and denote $\theta := (w - z)/|z - w|$. Then $\bar\alpha_G(z;\theta) \ge 1/|z - w| = 1/\delta(z)$. Since $B^n(z, \delta(z)) \subset G$ and $G^c \cap B^n$ is contained within ϵ distance from the plane $P := \{\xi \in \mathbb{R}^n : \xi_1 = 0\}$, we find that the Apollonian spheres through z in direction e_1 have radii at least $\min\{\delta(z)^2/(2\epsilon), 1/4\}$, see Figure 1. Here the first term comes from spheres limited by the boundary component near the plane P (the case shown in the figure) and the second one comes from spheres limited by S^{n-1}. It follows from this estimate that

$$\frac{\bar\alpha_G(z;e_1)}{\bar\alpha_G(z;\theta)} \le \max\{2\epsilon/\delta(z), 4\delta(z)\}.$$

Since G is L–quasi-isotropic, this implies by Lemma 3.2 that $\delta(z) \le 2L\epsilon$ or $4\delta(z) \ge 1/L$. Since $z \in D$ and $0 \in \partial G$, we have $\delta(z) < \sqrt[4]{\epsilon} < 1/(4L)$ and so we see that the second condition does not hold. This means that $\delta(z) \le 2L\epsilon$. If t_0 is such that $z := \gamma(t_0) \in D$, then it is clear that $\min\{t_0, \ell(\gamma) - t_0\} \ge d(x, D) = \sqrt{\epsilon}$. Therefore the inequality $K\delta(\gamma(t_0)) \ge \min\{t_0, \ell(\gamma) - t_0\}$, which is the second condition from the definition of uniformity, implies that $K \ge \sqrt{\epsilon}/(2L\epsilon) = 1/(2L\sqrt{\epsilon})$.

On the other hand, for $\gamma \in B$ we have $\ell(\gamma)/|x - y| \ge 1/\sqrt[4]{\epsilon}$. Recall that $\ell(\gamma) \le K|x - y|$ is the first condition in the definition of uniformity. Thus we see that as $\epsilon \to 0$ a path γ will violate either the first (if $\gamma \in B$) or the second condition (if $\gamma \in A$) of uniformity, which means that G is not uniform, as was to be shown. □

Using Theorem 4.8 we can prove the following improvement of Proposition 6.6 from [15] which assumed the comparison property instead of quasi-isotropy in item (2).

THEOREM 4.9. *Let $G \subsetneq \mathbb{R}^n$ be a domain. The following conditions are equivalent:*

(1) *G is A-uniform (i.e. $k_G \lesssim \alpha_G$);*
(2) *G is uniform and quasi-isotropic; and*
(3) *G is Apollonian quasiconvex and quasi-isotropic.*

Proof. The three conditions can be written as (1) $\alpha_G \approx k_G$, (2) $j_G \approx k_G$ and $k_G \approx \tilde{\alpha}_G$, and (3) $\tilde{\alpha}_G \approx \alpha_G$ and $k_G \approx \tilde{\alpha}_G$, respectively. If (1) holds, then $\alpha_G \approx j_G \approx \tilde{\alpha}_G \approx k_G$ and it is clear that (2) and (3) hold. Assuming (3) and combining the two inequalities we again get $\alpha_G \approx k_G$, i.e. (1). Finally, if (2) holds, then G has the comparison property by Theorem 4.8, which means that $\alpha_G \approx j_G$ and so $\alpha_G \approx k_G$ and all the metrics are again equivalent. □

5. APOLLONIAN QUASICONVEXITY, OTHER CONSTRUCTIONS

In this section we show that Inequalities 9, $\alpha_G \approx \tilde{\alpha}_G \ll j_G \approx k_G$, and 10, $\alpha_G \ll \tilde{\alpha}_G \ll j_G \approx k_G$, can occur in general domains. Recall that we saw in Lemma 2.4 that these inequalities cannot occur in simply connected planar domains.

5.1. **THE NINTH INEQUALITY.** We will next prove the inequality $\alpha_G \approx \tilde{\alpha}_G$ in some set of domains. Unfortunately, we do not have a simple geometric interpretation of this inequality, which means that the proof is somewhat long. However, the structure is simple: first we deal with the "trivial" cases, where the extra boundary point p has no bearing on the claim. In the other cases we construct a near-geodesic path and estimate its length.

The general idea with the following theorem and its corollary is that the inequality $\alpha_G \approx \tilde{\alpha}_G$ is not disturbed by the addition of some boundary components of codimension at least two, but does not hold for the addition of lower codimension boundary components.

THEOREM 5.1. *Let $D \subsetneq \mathbb{R}^n$ be a bounded domain. Suppose p is a point in D and define $G := D \setminus \{p\}$. If $\alpha_D \approx \tilde{\alpha}_D$, then $\alpha_G \approx \tilde{\alpha}_G$ as well.*

Proof. In this proof we denote by δ the distance to the boundary of D, not of G. We prove $\alpha_G \gtrsim \tilde{\alpha}_G$ since $\alpha_G \le \tilde{\alpha}_G$ always holds. Let $x, y \in G$ and denote $B := B^n(p, \delta(p)/2)$. Let γ_{xy} be a path connecting x and y such that $\alpha_G(\gamma_{xy}) = \tilde{\alpha}_G(x, y)$.

First consider the case $x, y \in D \setminus B$. If $\gamma_{xy} \cap B = \emptyset$, we proceed as follows. Let $z \in \partial G$ be such that $\delta(p) = |p - z|$. Denote $R_D := \operatorname{diam} D/\delta(p)$. For $w \in D \setminus B$ and $r \in S^1$, we have

$$\bar{\alpha}_D(w; r) = \frac{1}{2r_-} + \frac{1}{2r_+} \ge \frac{2}{\operatorname{diam} D} \ge \frac{R_D}{|w - p|},$$

where the last inequality holds since $|w - p| \ge \delta(p)/2$. We also see that if the Apollonian spheres are affected by the boundary point p, then

$$\bar{\alpha}_G(w; r) \le \frac{1}{|w - p|} + \frac{1}{2r_+} \le \frac{1}{|w - p|} + \bar{\alpha}_D(w; r)$$

holds, where r_+ denotes the radius of the Apollonian sphere which touches ∂D. Otherwise, we have $\bar{\alpha}_G(w; r) = \bar{\alpha}_D(w; r)$. So for all $w \in D \setminus B$, the inequalities

$$\bar{\alpha}_G(w; r) \le \bar{\alpha}_D(w; r) + \frac{1}{|w - p|} \le \left(1 + \frac{1}{R_D}\right) \bar{\alpha}_D(w; r) \tag{2}$$

hold. By Lemma 3.7 we get $\alpha_G(\gamma_{xy}) \le C\alpha_D(\gamma_{xy})$, for some constant C, which gives

$$\tilde{\alpha}_G(x, y) \lesssim \tilde{\alpha}_D(x, y) \approx \alpha_D(x, y) \le \alpha_G(x, y), \tag{3}$$

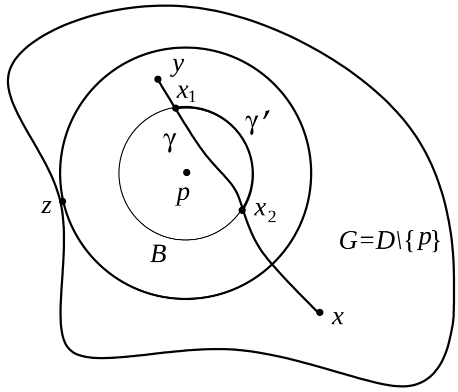

FIGURE 2. The geodesic path γ_{xy} connecting x and y intersects B.

where the second inequality holds by assumption and the third holds trivially, as G is a subdomain of D.

If γ_{xy} intersects B, let γ be an intersecting part of γ_{xy} from x_1 to x_2 (if there are more intersecting parts, we proceed similarly). Let γ' be the shortest circular arc on ∂B from x_1 to x_2 as shown in the Figure 2. Using the density bounds $2/\operatorname{diam} D \le \bar{\alpha}_D(u;r) \le 2/\delta(u)$, we see that $\alpha_D(\gamma) \ge 2\ell(\gamma)/\operatorname{diam} D$ and $\alpha_D(\gamma') \le 4\ell(\gamma')/\delta(p)$ hold. But since $\ell(\gamma) \ge |x_1 - x_2|$ and $\ell(\gamma') \le \frac{\pi}{2}|x_1 - x_2|$, we have $\ell(\gamma') \lesssim \ell(\gamma)$. This shows that $\alpha_D(\gamma'_{xy}) \lesssim \alpha_D(\gamma_{xy})$ holds. Since $\gamma'_{xy} \subset G \setminus B$, (2) implies that $\alpha_G(\gamma'_{xy}) \lesssim \alpha_D(\gamma'_{xy})$. So we get

$$\alpha_G(x,y) \ge \alpha_D(x,y) \approx \tilde{\alpha}_D(x,y) = \alpha_D(\gamma_{xy}) \gtrsim \alpha_D(\gamma'_{xy}) \gtrsim \alpha_G(\gamma'_{xy}) \ge \tilde{\alpha}_G(x,y).$$

Thus we have shown that $\alpha_G(x,y) \gtrsim \tilde{\alpha}_G(x,y)$ holds for all $x,y \in D \setminus B$.

We now consider the case $x,y \in B^n(p, \frac{3}{4}\delta(p))$. Without loss of generality we assume that $|y-p| \le |x-p|$. Since $\partial G = \partial D \cup \{p\}$, it is clear that

$$\alpha_G(x,y) \ge \max\Big\{ \log\frac{|x-p|}{|y-p|}, \alpha_D(x,y)\Big\}.$$

Let $\gamma := \gamma_1 \cup \gamma_2$, where γ_1 is the path which is circular about the point p from y to $|y-p|\frac{x-p}{|x-p|}+p$ and γ_2 is the radial part from $|y-p|\frac{x-p}{|x-p|}+p$ to x, as shown in the Figure 3. Since the Apollonian spheres are not affected by the boundary point p in the circular part, we have

$$\begin{aligned}\bar{\alpha}_G(\gamma_1(t);\gamma_1'(t)) \le \bar{\alpha}_{B^n(p,\delta(p))}(\gamma_1(t);\gamma_1'(t)) &= \frac{1}{\delta(p)-|y-p|} + \frac{1}{\delta(p)+|y-p|} \\ &= \frac{2\delta(p)}{\delta(p)^2-|y-p|^2},\end{aligned}$$

where the first equality holds since the Apollonian metric is isotropic in balls (since it equals the hyperbolic metric). For $\gamma_2(t)$, by monotony in the domain of definition, we see that

$$\begin{aligned}\bar{\alpha}_G(\gamma_2(t);\gamma_2'(t)) &\le \bar{\alpha}_{B^n(p,\delta(p))\setminus\{p\}}(\gamma_2(t);\gamma_2'(t)) \\ &= \frac{1}{|p-\gamma_2(t)|} + \frac{1}{\delta(p)-|p-\gamma_2(t)|}.\end{aligned}$$

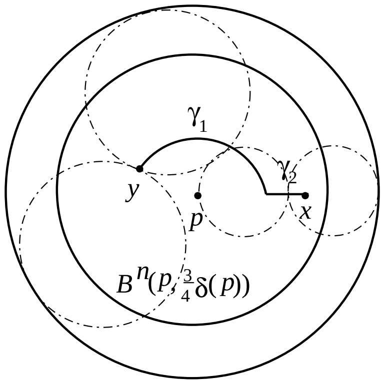

FIGURE 3. A short geodesic path connecting x and y in $B^n(p,\frac{3}{4}\delta(p))$.

Hence we have

$$\begin{aligned}\tilde\alpha_G(x,y)\le\alpha_G(\gamma)&\le\int_{\gamma_1}\frac{2\delta(p)}{\delta(p)^2-|y-p|^2}\,|dy|+\int_{|y-p|}^{|x-p|}\frac{1}{t}+\frac{1}{\delta(p)-t}\,dt\\&=\frac{2\delta(p)\ell(\gamma_1)}{\delta(p)^2-|y-p|^2}+\log\left(\frac{|x-p|}{|y-p|}\frac{\delta(p)-|y-p|}{\delta(p)-|x-p|}\right)\\&\le\frac{32}{7}\frac{\ell(\gamma_1)}{\delta(p)}+\log\left(\frac{|x-p|}{|y-p|}\frac{\delta(p)-|y-p|}{\delta(p)-|x-p|}\right).\end{aligned}$$

Since $u\mapsto u^3(\delta(p)-u)$ is increasing for $0<u<3\delta(p)/4$ and we have $|y-p|\le|x-p|\le 3\delta(p)/4$, the inequality $|x-p|^3(\delta(p)-|x-p|)\ge|y-p|^3(\delta(p)-|y-p|)$ holds. This inequality is equivalent to

$$\log\left(\frac{|x-p|}{|y-p|}\frac{\delta(p)-|y-p|}{\delta(p)-|x-p|}\right)\le 4\log\frac{|x-p|}{|y-p|}.$$

Using $\alpha_D\approx\tilde\alpha_D$ we easily get $\alpha_D(x,y)\gtrsim\ell(\gamma_1)/\delta(p)$. We have thus shown that

$$\tilde\alpha_G(x,y)\le K\alpha_D(x,y)+4\log\{|x-p|/|y-p|\}\le(K+4)\alpha_G(x,y),$$

for some constant K.

It remains to consider the case $x\notin B^n(p,3\delta(p)/4)$ and $y\in B$. Let $w\in S^{n-1}(p,\frac{3}{4}\delta(p))$ be such that $|y-w|=d(y,S^{n-1}(p,\frac{3}{4}\delta(p)))$. Let $\gamma:=\gamma_1\cup\gamma_2$, where $\gamma_1=[y,w]$ and γ_2 is a path connecting w and x such that $\alpha_G(\gamma_2)=\tilde\alpha_G(x,w)$. As we discussed in the previous case, we have

$$\alpha_G(\gamma_1)\le 4\log\frac{3\delta(p)}{4|y-p|}\le 4\log\frac{|x-p|}{|y-p|}\le 4\alpha_G(x,y).$$

Since $x,w\notin B$, it follows by the previous cases that

$$\alpha_G(\gamma_2)=\tilde\alpha_G(w,x)\lesssim\alpha_D(w,x)\le 2j_D(w,x).$$

It is not difficult to see that $j_D(x,w)\lesssim\alpha_D(x,y)$, details are again provided in [29].

We have now verified the inequality in all the possible cases, so the proof is complete. □

COROLLARY 5.2. *Let $D\subsetneq\mathbb{R}^n$ be bounded. Suppose $(p_i)_{i=1}^k$ is a finite non-empty sequence of points in D and define $G:=D\setminus\{p_1,p_2,\dots,p_k\}$. Assume that $\alpha_D\approx\tilde\alpha_D$ and $j_D\approx k_D$. Then Inequality 9, $\alpha_G\approx\tilde\alpha_G\ll j_G\approx k_G$, holds.*

Proof. Since $j_D \approx k_D$, D is uniform and thus so G is, as can be seen from the definition, which implies that $j_G \approx k_G$ holds. Let $\epsilon_0 > 0$ be such that the sphere $S^{n-1}(p_1, \epsilon) \subset D$ for all $\epsilon \in (0, \epsilon_0)$. Let $x, y \in S^{n-1}(p_1, \epsilon)$ be diametrically opposite. Then we see that $\alpha_G(x, y) \to 0$ as $\epsilon \to 0$, but on the other hand $j_G(x, y) = \log 3$. Hence $\alpha_G(x, y)/j_G(x, y) \to 0$ as $\epsilon \to 0$, which implies that $\alpha_G \ll j_G$. We have thus proved that $\alpha_G \ll j_G \approx k_G$. So, it remains to prove $\alpha_G \gtrsim \tilde{\alpha}_G$, since $\alpha_G \le \tilde{\alpha}_G$ always holds. For $1 \le i \le k$, define $G_i = G_{i-1} \setminus \{p_i\}$, where $G_0 = D$. Since D is bounded and $\alpha_D \approx \tilde{\alpha}_D$, we conclude by Theorem 5.1 that $\alpha_{G_1} \approx \tilde{\alpha}_{G_1}$. Inductively, we get $\alpha_{G_i} \approx \tilde{\alpha}_{G_i}$ for all i, $1 \le i \le k$. Since $G_k = D \setminus \{p_1, \dots, p_k\} = G$, we have shown that $\alpha_G \approx \tilde{\alpha}_G$. □

5.2. **THE TENTH INEQUALITY.** In this subsection we construct a domain in $\mathbb{R}^3$ which is topologically equivalent (in $\overline{\mathbb{R}^3}$) to a ball in which the inequalities $\alpha_G \ll \tilde{\alpha}_G \ll j_G \approx k_G$ hold.

PROPOSITION 5.3. *Define* $R_1 := \{e_1 + te_3 \colon t \in [0, \infty)\}$ *and* $R_2 := \{-e_1 + te_3 \colon t \in [0, \infty)\}$. *In the domain*

$$G := \mathbb{R}^3 \setminus \Big(R_1 \cup R_2 \cup \overline{B^3}(e_1 - e_3, 1) \Big)$$

Inequality 10, $\alpha_G \ll \tilde{\alpha}_G \ll j_G \approx k_G$*, holds.*

Proof. It is easy to see that G is uniform, if we handle the cases when $|x - y|$ is small and when it is large separately. In the former case, both x and y are near a single boundary component of G and hence we need only consider the boundary components separately. If $|x - y|$ is large, then we may choose the path to curve away from all boundary components. Since G contains a boundary component which is a ray, it is clear that G is not quasi-isotropic, hence $\tilde{\alpha}_G \ll k_G$ by Lemma 3.8 and it remains only to prove that $\alpha_G \ll \tilde{\alpha}_G$.

Set $x := te_3 + 2e_2$ and $y := te_3 - 2e_2$ for $t \in [3/2, \infty)$. It is clear that $\alpha_G(x, y) \to 0$ as $t \to \infty$, since the rays R_1 and R_2 do not affect this distance. We next derive a lower bound for $\tilde{\alpha}_G(x, y)$ which is independent of t. In so doing we may forget about the boundary points in the sphere $S^2(e_1 - e_3, 1)$ since this only makes the bound smaller. Denote $B^x := B^3(x, 1)$ and $B^y := B^3(y, 1)$. Let $u \in B^x$, $\theta \in S^1$ and denote $d := d(u, \partial B^y)$. We see that any ball which intersects B^x and B^y will also intersect R_1 or R_2. Thus the Apollonian spheres through u in direction θ with respect to G are smaller in size than with respect to $\mathbb{R}^3 \setminus B^y$. Since $\mathbb{R}^3 \setminus B^y$ is isotropic, we get

$$\bar{\alpha}_G(u; \theta) \ge \bar{\alpha}_{\mathbb{R}^3 \setminus B^y}(u; \theta) = \bar{\alpha}_{\mathbb{R}^3 \setminus B^y}(u) = \frac{1}{d} - \frac{1}{d+2} \ge \frac{1}{12},$$

where we used $d \le 4$ for the last inequality (the minus sign occurs because the corresponding Apollonian sphere contains ∞). Let γ be a path connecting x and y. Then it certainly connects x to ∂B^x; denote this part by γ'. By Theorem 3.7 we get $\tilde{\alpha}_G(x, y) \ge \frac{1}{12} \inf_{\gamma'} \ell(\gamma') = \frac{1}{12}$. Since $\alpha_G(x, y) \to 0$ as $t \to 0$, we see that $\alpha_G \ll \tilde{\alpha}_G$. □

6. NONCOMPARABILITY

In this short section we sort out the possible inequalities when j_G and $\tilde{\alpha}_G$ are not comparable. It turns out that there is just one possibility in this case. For if $j_G \lesseqgtr \tilde{\alpha}_G$, then it follows without any geometrical considerations that none of the inequalities $\alpha_G \approx j_G$, $\alpha_G \approx \tilde{\alpha}_G$, $j_G \approx k_G$ or $\tilde{\alpha}_G \approx k_G$ can hold, since if for instance $\alpha_G \approx j_G$, then $j_G \approx \alpha_G \lesssim \tilde{\alpha}_G$, contrary to assumption. Hence only the possibility $\alpha_G \ll j_G \ll k_G$ and $\alpha_G \ll \tilde{\alpha}_G \ll k_G$ remains, which is the case of least possible comparability among the metrics. Unfortunately, this occurs in quite many domains.

PROPOSITION 6.1. *Let G be a simply connected planar domain which is not quasi-isotropic. Then $\alpha_G \ll j_G \ll k_G$, $\alpha_G \ll \tilde{\alpha}_G \ll k_G$ and $j_G \lessgtr \tilde{\alpha}_G$.*

Proof. Since G is not quasi-isotropic, we have $j_G \not\gtrsim \tilde{\alpha}_G$ by Corollary 3.8. Since G is simply connected and does not have the comparison property, the inequality $j_G \not\lesssim \tilde{\alpha}_G$ follows from Proposition 4.3. Hence $j_G \lessgtr \tilde{\alpha}_G$ which implies the inequalities $\alpha_G \ll j_G \ll k_G$ and $\alpha_G \ll \tilde{\alpha}_G \ll k_G$, as was shown above. □

EXAMPLE 6.2. The domain $H^2 \setminus [0, e_2]$ satisfies the assumptions of the previous lemma.

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The following sections are included for the referee's conveniance, and are not intended to be part of the final article.

The proof of Example 3.11, from [29]

Let $z \in F$ and $\epsilon_0 > 0$ be such that $B^n(z,\epsilon) \subset D \setminus D'$ for all $\epsilon \in (0,\epsilon_0)$. Let $x, y \in S^{n-1}(z,\epsilon)$ be diametrically opposite such that $[x,y]$ is perpendicular to F. Then it is easy to see that $\alpha_G(x,y) \to 0$ as $\epsilon \to 0$, but on the other hand $j_G(x,y) = \log 3$. Hence $\alpha_G(x,y)/j_G(x,y) \to 0$ as $\epsilon \to 0$, which means that $\alpha_G \ll j_G$ holds. Also we note that G is not uniform as it is not possible to connect the same x and y with a path of length comparable to ϵ as $\epsilon \to 0$, which violates the first condition in the definition of uniformity. Thus we get $j_G \ll k_G$, since G is uniform if and only if $k_G \approx j_G$ and $j_G \le k_G$ always holds. We have thus proved that $\alpha_G \ll j_G \ll k_G$. So it remains to prove the last inequality, $\tilde{\alpha}_G \approx k_G$.

Denote $d' := d(D, D')$. Let $B^n(p,r)$ be largest ball contained in D' and $B^n(p,R)$ be the smallest ball with center p containing D'. Since D' is convex and compact, r and R are finite. We define

$$L = \max\left\{\sqrt{2}, \frac{R}{r}, 1 + \frac{d' + \operatorname{diam} D'}{r}\right\}$$

and check Lemma 3.10 for G with this constant L. For $x \in G$ choose $z \in \partial G$ such that $\delta(x) = |x - z|$. Now, if $z \in \partial D$, take any ball $B \subset D^c$ so that $z \in \partial B$. Then for any $y \in B$ the line segment $[x,y]$ connecting x and y intersects $\partial D \subset \partial G$. Since D is convex we can choose any $L > 1$ in Lemma 3.10 for this x.

Next if $z \in \partial F$, take a line L' perpendicular to F through x and z. Consider the balls with radius $r_0 = \delta(x)/\sqrt{L^2-1}$ tangent to both F and L' but on the other side of F than x. Of the two balls satisfying this condition, denote by B the one closer to $F \cap \partial D$. This gives $d(z, \partial B) = r_0(\sqrt{2} - 1)$. Since $L \ge \sqrt{2}$, the hypotheses of Lemma 3.10 are satisfied for this case.

Finally, suppose $z \in \partial D'$. If $\delta(x) \le r\sqrt{L^2-1}$, construct rays L_1 and L_2 starting from z and tangent to $B^n(p,r)$. Choose a ball $B := B^n(w, r_0)$ centered at w and radius $r_0 = \delta(x)/\sqrt{L^2-1}$ to which L_1 and L_2 are tangent. Since $r_0 \le r$, D' is convex and $B \subset D'$, for any $y \in B$ the line segment $[x,y]$ intersects $\partial D' \subset \partial G$. Let a and b be points where L_1 is tangent to $B^n(p,r)$ and B, respectively. Now it is easy to see that the triangles $\triangle\, apz$ and $\triangle\, bwz$ are similar, which gives $d(z, \partial B) \le r_0(R/r - 1)$, since $|z - p| \le R$. Because $L \ge R/r$, the hypotheses of Lemma 3.10 are satisfied. If $\delta(x) > r\sqrt{L^2-1}$, choose a ball $B \subset D^c$ with radius $r_0 = \delta(x)/\sqrt{L^2-1}$ at a distance $d(z, \partial D)$ from z. We see that for any $y \in B$ the line segment $[x,y]$ intersects $\partial D \subset \partial G$. By the triangle inequality, it is clear that $d(z, \partial B) = d(z, \partial D) \le d' + \operatorname{diam} D'$. Since $L \ge 1 + (d' + \operatorname{diam} D')/r$ and $\delta(x) > r\sqrt{L^2-1}$ we get $\delta(x) > \sqrt{(L+1)/(L-1)}(d' + \operatorname{diam} D')$. This gives $d < r_0(L-1)$. Thus for any $z \in \partial G$ with $|x - z| = \delta(x)$, we get all conditions of Lemma 3.10, which gives the conclusion.

The proof of Theorem 5.1, from [29]

If $\delta(w) \le \delta(x)$, using the first inequality of (3) and the triangle inequality we see that the inequalities

$$\alpha_G(\gamma_2) = \tilde{\alpha}_G(w,x) \lesssim \tilde{\alpha}_D(w,x) \approx \alpha_D(w,x) \le 2 j_D(w,x) \le 2 \log\left(4 + \frac{4|x-p|}{\delta(p)}\right)$$

hold. But for $s \geq 3/2$, we have $\log(4+2s) \leq 5 \log s$. Since $|y-p| \leq \delta(p)/2$, we obtain

$$\alpha_G(\gamma_2) \lesssim \log\left(4 + \frac{4|x-p|}{\delta(p)}\right) \leq 5 \log \frac{|x-p|}{|y-p|} \leq 5\alpha_G(x,y).$$

We then move on to the case $\delta(w) \geq \delta(x)$. If $|x-y| \geq 3\delta(x)$, we see (by the triangle inequality) that

$$\alpha_G(x,y) \geq \sup_{b\in\partial D} \log \frac{|x-y|-|b-x|}{|b-x|} = \log\left(\frac{|x-y|}{\delta(x)} - 1\right)$$

holds. Using this and the fact that $\frac{|x-p|}{|y-p|} \geq \log(3/2)$ we get

$$\alpha_G(x,y) \geq \begin{cases} \log\left(\dfrac{|x-y|}{\delta(x)} - 1\right) & \text{for } \dfrac{|x-y|}{\delta(x)} \geq 3, \\ \log \dfrac{3}{2} & \text{otherwise.} \end{cases}$$

Since $|x-y| \geq \delta(p)/4$, we get the following upper bound for the length of the curve (the first inequality follows as before):

$$\alpha_G(\gamma_2) \lesssim j_D(x,w) \lesssim \log\left(1 + \frac{|x-y|+\delta(p)}{\delta(x)}\right) \leq \log\left(1 + \frac{5|x-y|}{\delta(x)}\right).$$

The function $f(s) = (s-1)^4 - (1+5s)$ is increasing for $s \geq 3$, so $f(s) \geq f(3) = 0$. Thus for $|x-y|/\delta(x) \geq 3$, we get

$$\alpha_G(\gamma_2) \lesssim \log\left(1 + \frac{5|x-y|}{\delta(x)}\right) \leq 4 \log\left(\frac{|x-y|}{\delta(x)} - 1\right) \leq 4\alpha_G(x,y).$$

On the other hand, if $|x-y|/\delta(x) < 3$, then $\alpha_G(\gamma_2)$ is bounded above by $4\log 2$ and $\alpha_G(x,y)$ is bounded bellow by $\log(3/2)$, so the inequality $\alpha_G(\gamma_2) \lesssim \alpha_G(x,y)$ is clear.