THE APOLLONIAN METRIC: THE COMPARISON PROPERTY, BILIPSCHITZ MAPPINGS AND THICK SETS

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ABSTRACT. The Apollonian metric is a generalization of the hyperbolic metric to arbitrary open sets in Euclidean spaces. In this article we show that the Apollonian metric is comparable to the j_G metric in the set G if and only if its complement is unbounded and thick in the sense of Väisälä, Vuorinen and Wallin [Thick sets and quasisymmetric maps, *Nagoya Math. J.* **135** (1994), 121–148]. These conditions are also equivalent to the following: there exists $L > 1$ such that all Euclidean L–bilipschitz mappings are Apollonian bilipschitz with uniformly bounded constant.

1. INTRODUCTION

The Apollonian metric is a generalization of the hyperbolic metric. In contrast to the hyperbolic metric, it is defined in arbitrary open sets of $\overline{\mathbb{R}^n}$. It was introduced by Alan Beardon in 1998 [5], but it later turned out that the same metric had been studied previously by Barbilian [3] (see [27], and for some further developments, [6, 7]). The Apollonian metric has been studied recently from a perspective of geometric function theory in [10, 14, 15, 16, 17, 20, 19, 22, 23, 24, 30, 32].

The regularity of the Apollonian metric has some interesting connections to the geometry of the domain of definition. Zair Ibragimov [22] found a connection between the Apollonian metric and constant-width sets (see e.g. [8] and the references therein). More precisely, he showed that the Apollonian metric is infinitessimally equivalent to the Euclidean metric at a single, distinguished point if and only if the domain of definition is the Möbius image of the complement of a constant-width set.

In this article we will provide another connection between the regularity of the Apollonian metric and the geometry of the domain of definition, namely, we will show that the Apollonian metric in the domain G is comparable to the j_G metric if and only if the complement of G is thick. The comparison property and thickness are also related to regularity of Euclidean bilipschitz mappings in the Apollonian metric. To make these statements precise we need some definitions.

We will consider domains (open connected non-empty sets) G in $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$. The Apollonian metric for $x, y \in G \subsetneq \overline{\mathbb{R}^n}$ is defined by

$$\alpha_G(x,y) := \sup_{a,b\in\partial G} \log \frac{|a-x|}{|a-y|}\frac{|b-y|}{|b-x|}.$$

It is in fact a metric if and only if the complement of G is not contained in a sphere, as was noted in [5, Theorem 1.1]. Beardon [5] proved various inequalities relating the Apollonian

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metric to other well-known metrics, such as the quasihyperbolic metric (see [12]), the Klein-Hilbert metric (see [21]) and the hyperbolic metric (see [10]). We will focus on the j_G metric, from [11, 39], which is defined for $x, y \in G \subsetneq \mathbb{R}^n$ by

$$j_G(x,y) := \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right).$$

We always have $\alpha_G \le 2 j_G$ [5, Theorem 3.2].

Definition 1.1. We say that a domain $G \subsetneq \mathbb{R}^n$ has the *comparison property* if there exists a constant K such that $j_G/K \le \alpha_G$.

Of course, it trivially follows from the comparison property that α_G is a metric, which, as mentioned, implies that the complement of G is not contained in a sphere. Intuitively, the geometric characterizations of this property says that the complement of the domain (uniformly) does not look approximately like a sphere, even locally. One way to make this precise is the following definition, from [38].

Definition 1.2. A set $A \subset \mathbb{R}^n$ is *thick* if there exists $q > 0$ such that for every $x \in A$ and $r > 0$ with $A \setminus B^n(x,r) \neq \emptyset$ there exists an n-simplex S with vertices in $A \cap B^n(x,r)$ such that $m_n(S) \ge q r^n$, where m_n denotes the n-dimensional Lebesgue measure.

Let us mention some features of thick sets, by way of motivation. Jussi Väisälä proved that thick sets have the bilipschitz extension property: if $A \subset \mathbb{R}^n$ is compact and thick, then there exists $M_0 > 1$ such that every M–bilipschitz mapping $f\colon A \to \mathbb{R}^n$ with $M < M_0$ extends to an $M'(M,n)$–bilipschitz mapping $g\colon \mathbb{R}^n \to \mathbb{R}^n$ [35]. Hans Wallin and Peter Wingren proved that $A \subset \mathbb{R}^n$ is thick if and only if for every natural number k there exists a constant C such that

$$\sup_{y \in B^n(x,r)\cap A} |\nabla p(y)| \le \frac{C}{r} \sup_{y \in B^n(x,r)\cap A} |p(y)|,$$

for every $x \in A$ and every algebraic polynomial $p\colon \mathbb{R}^n \to \mathbb{R}^n$ of total degree at most k [40]. This inequality is important in polynomial approximation in function spaces defined on A [25].

The question about the Apollonian metric that has received the widest attention is that of its isometries. Beardon [5] asked if these were just the Möbius mappings. Partial answers were provided by Beardon [5], Gehring and Hag [10] and Ibragimov [23]. Recently the author and Zair Ibragimov were able to settle this question in the case of regular [20] and planar domains [19].

A natural extension of this would be to study rough quasi-isometries in the spirit of Gromov hyperbolic spaces (the Apollonian metric is Gromov hyperbolic, by [18]). In this article we will approach this question from the opposite direction: we try to obtain an understanding of what Euclidean bilipschitz mappings look like in the geometry of the Apollonian metric.

For this purpose we define a bilipschitz modulus of the Apollonian metric, α_G, by

$$\alpha_L(G) := \sup_f \sup_{x,y\in G} \max\left\{\frac{\alpha_{f(G)}(f(x),f(y))}{\alpha_G(x,y)}, \frac{\alpha_G(x,y)}{\alpha_{f(G)}(f(x),f(y))}\right\},$$

where the first supremum is over all Euclidean L–bilipschitz mappings f mapping G into $\mathbb{R}^n$ (with the understanding that terms with zero denominators are ignored).

The main result of this paper provides a connection between domains in which the Apollonian metric can be estimated by the j_G metric, domains for which every bilipschitz mapping is Apollonian bilipschitz and domains with thick complements:

Theorem 1.3. *Let $G \subsetneq \mathbb{R}^n$ be a domain. The following are equivalent:*

(1) *The domain G has the comparison property;*
(2) *The set $G^c \setminus \{\infty\}$ is unbounded and thick;*
(3) *There exists $L > 1$ such that $\alpha_L(G) < \infty$;*
(4) *There exists $L > 1$ for which G does not satisfy the L-IDB condition (see Definition 3.3).*

The structure of the rest of this article is as follows: in the next section we summarize the notation used and review the concept of Apollonian balls, which is a central tool for working with the metric. In Section 3 we derive some auxiliary results in terms of a ball condition and prove the equivalence of (1) and (4) of the main theorem. In Section 4 we prove the equivalence of (1) and (2). As applications of these results we use methods from [38] to show that bilipschitz mappings do not in general preserve the comparison property, but that there exists a critical constant such that bilipschitz mappings with smaller bilipschitz constants do preserve the comparison property, see Corollaries 3.8 and 4.1. In Section 5 we introduce the exterior ball condition and show that it is preserved under bilipschitz mappings. This condition has been shown to imply the comparison property, see Theorem 6.2. In Section 6 we show that it also implies the finiteness of $\alpha_L(G)$ for all $L \geq 1$, which, in particular, implies that the Apollonian metric is well-behaved in quasiballs. In Section 7 we consider the connection between the comparison and bilipschitz properties and complete the proof of Theorem 1.3 by showing that (1) and (3) are equivalent.

2. Notation and terminology

2.1. **Common notation.** We denote by $\{e_1, e_2, \ldots, e_n\}$ the standard basis of $\mathbb{R}^n$ and by n the dimension of the Euclidean space under consideration and assume that $n \geq 2$. For $x \in \mathbb{R}^n$ we denote by x_i its i^{th} co-ordinate. For $x \in \mathbb{R}^n$ and $0 < r < \infty$ we denote by $B^n(x,r)$ the open ball with center x and radius r. By B^n we denote the unit ball centered at the origin. We use the notation $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$ for the one point compactification of $\mathbb{R}^n$, so its open balls are the open balls of $\mathbb{R}^n$, open half-spaces and complements of closed balls. If $G \subset \overline{\mathbb{R}^n}$ we denote by ∂G, G^c and $\overline{G}$ its boundary, complement and closure, respectively, all with respect to $\overline{\mathbb{R}^n}$. For $x \in G \subsetneq \mathbb{R}^n$ we denote $\delta(x) := d(x, \partial G) := \min\{|x - z| \colon z \in \partial G\}$. For $x, y \in \mathbb{R}^n$ we denote by xy the line through x and y and by $[x, y]$ the closed segment between x and y.

The cross-ratio $|a, b, c, d|$ is defined by

$$|a, b, c, d| := \frac{|a - c||b - d|}{|a - b||c - d|}$$

for $a \neq b$, $c \neq d$ and $a, b, c, d \in \overline{\mathbb{R}^n}$, with the understanding that $|\infty - x|/|\infty - y| = 1$ for all $x, y \in \mathbb{R}^n$. A homeomorphism $f \colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|$$

for every quadruple $a, b, c, d \in \overline{\mathbb{R}^n}$ with $a \neq b$ and $c \neq d$. For more information on Möbius mappings see e.g. [4, Section 3]. Using the cross-ratio we can express the Apollonian metric as

$$\alpha_G(x,y) = \sup_{a,b\in\partial G} \log|a,y,x,b|,$$

for $x,y \in G \subset \overline{\mathbb{R}^n}$. This means, in particular that α_G is Möbius invariant, as noted in [5, Introduction (2)].

2.2. **The Apollonian balls approach.** One interesting feature of the Apollonian metric are the Apollonian balls, which allow us to get a geometric understanding of the metric. The concepts from this section will be used extensively in what follows, and the reader unfamiliar with them might want to refer back to this section occasionally.

For $x,y \in G \subsetneq \overline{\mathbb{R}^n}$ we define

$$q_x := \sup_{b\in\partial G} \frac{|b-y|}{|b-x|} \quad \text{and} \quad q_y := \sup_{a\in\partial G} \frac{|a-x|}{|a-y|}.$$

The numbers q_x and q_y are called the *Apollonian ball parameters* of x and y (with respect to G). By definition, $\alpha_G(x,y) = \log(q_x q_y)$. Moreover, the balls (in $\overline{\mathbb{R}^n}$)

$$B_x := \{z \in \overline{\mathbb{R}^n} \colon |z-x|/|z-y| < 1/q_x\} \quad \text{and} \quad B_y := \{z \in \overline{\mathbb{R}^n} \colon |z-y|/|z-x| < 1/q_y\}$$

lie completely in G and have centers on the line xy. We collect some immediate results regarding these balls.

(A1) $B_x \subsetneq G$ and $\overline{B_x} \cap \partial G \neq \emptyset$, similarly for B_y.
(A2) If i_x and i_y denote the inversions in the spheres ∂B_x and ∂B_y, then $y = i_x(x) = i_y(x)$.
(A3) If $\infty \notin G$, then $q_x, q_y \geq 1$; if $\infty \notin \overline{G}$, then $q_x, q_y > 1$.
(A4) If $q_x > 1$ we let x_0 and r_x denote the center and radius of B_x. Then

$$|x - x_0| = \frac{|x-y|}{q_x^2 - 1} = \frac{r_x}{q_x}.$$

(A5) We have $q_x - 1 \leq |x-y|/\delta(x) \leq q_x + 1$.

3. The interior double ball condition

In this section we introduce an auxiliary ball condition which will be used in the proof of the equivalence of (1) and (4) in the main theorem.

We start with a technical lemma. It says, intuitively speaking, that if G does not have the comparison property, then the complement of G has thin parts which stick into G. Deriving an explicit estimate of the width of this part turns out to require some calculations, though.

Lemma 3.1. *Let $G \subsetneq \mathbb{R}^n$ be a domain and $x,y \in G$ be points such that $\alpha_G(x,y) \leq j_G(x,y)/N$ for some $N > 1$. Then $\alpha_G(x,y) \leq 2\log(1 + 2/(N-1))$ and $\max\{q_x, q_y\} \leq (N+1)/(N-1)$.*

Furthermore, assume that $N \geq 25$. Then there exists a point $z \in \partial G \setminus \{\infty\}$, a unit vector $e \in S^{n-1}$ and $r > 0$ such that for every $w \in G^c \cap B^n(z,r)$ we have $\langle z-w, e\rangle < 2r/\sqrt{N}$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product.

Moreover, there exist a ball B with radius r and balls B_1 and B_2 both with radii $r' := (1 - N^{-1/2})r/2$, such that $B_1, B_2 \subseteq G \cap B$, $d(B_1, B_2) = 2(r - 2r')$ and such that the segment connecting the centers of B_1 and B_2 intersects ∂G.

Proof. Fix $x, y \in G$. Since both α_G and j_G are invariant under Euclidean similarity mappings (that is, translations, rotations and scalings) we may assume without loss of generality that $x = -e_1/2$ and $y = e_1/2$. Let q_x, q_y be the Apollonian ball parameters, as described in Section 2.2. Then

$$\log(q_x q_y) = \alpha_G(x,y) \le \frac{j_G(x,y)}{N} \le \log\left(1 + \frac{1}{N\min\{\delta(x),\delta(y)\}}\right). \tag{3.2}$$

Since $B_x \cup B_y$ is contained in G we get using (A5) that

$$(q_x q_y - 1)\min\{1/(q_x+1), 1/(q_y+1)\} \le (q_x q_y - 1)\min\{\delta(x),\delta(y)\} \le 1/N,$$

where the second inequality is just (3.2) in another form.

Assume without loss of generality that $q_x \ge q_y$. If $q_x = 1$, then $\alpha_G(x,y) = 0$ and there is nothing to prove, so we assume that $q_x > 1$. Since $q_y \ge 1$ (by (A3)) this implies that

$$(q_x - 1)/(q_x + 1) \le (q_x q_y - 1)\min\{1/(q_x+1), 1/(q_y+1)\} \le 1/N,$$

and hence $q_y \le q_x \le (N+1)/(N-1)$. The estimate $\alpha_G(x,y) \le 2\log(1 + 2/(N-1))$ follows directly from this.

On the other hand, we can derive an estimate for $\min\{\delta(x),\delta(y)\}$ from inequality (3.2):

$$\min\{\delta(x),\delta(y)\} \le (N(q_x q_y - 1))^{-1} \le (N(q_x - 1))^{-1}.$$

It is easy to see that G contains the Apollonian balls

$$B'_x = \{z \in \mathbb{R}^n : |z-x|/|z-y| < 1/q\} \quad \text{and} \quad B'_y = \{z \in \mathbb{R}^n : |z-y|/|z-x| < 1/q\}.$$

corresponding to the larger parameter $q := q_x \ge q_y$. We continue the previous estimate by

$$\min\{\delta(x),\delta(y)\} \le (N(q-1))^{-1} \le 2R/N,$$

where $R = q/(q^2-1)$ is the radius of B'_x and B'_y.

Let $z \in \partial G$ be a point such that $|x - z| = \delta(x)$ or $|y - z| = \delta(y)$, according as $\delta(x) \le \delta(y)$ or not. We may assume without loss of generality that z lies in the plane spanned by e_1 and e_2 and also that $z_2 \ge 0$, where z_2 denotes the second coordinate of z. It follows that $z_2 \le \min\{\delta(x),\delta(y)\} \le 2R/N$. Since $N \ge 25$, this obviously implies that z lies in between the balls B'_x and B'_y (i.e. in the convex hull of $B'_x \cup B'_y$, but not in either ball).

Now one can compute the distance between the balls B'_x and B'_y at a height $h := R/\sqrt{N}$ from the e_1–axis. Let us call this distance $2s$. The distance between the balls, that is $2u$ in Figure 1, equals

$$|x-y| - d(x, \partial B'_x) - d(y, \partial B'_y) = 1 - 2/(q+1) = (q-1)/(q+1).$$

It is also easily calculated that $v = R - \sqrt{R^2 - R^2/N} = R(1 - \sqrt{1-1/N})$, where v is as in the figure. Thus we get

$$s = u + v = (q-1)/(2(q+1)) + R\big(1 - \sqrt{1-1/N}\big) = R\big((q^2+1)/(2q) - \sqrt{1-1/N}\big).$$

Let us set $r := h - z_2$ and $e := e_1$. Since every point of G^c in $B^n(z,r)$ has e_2 coordinate less than or equal to h, and z lies between B'_x and B'_y, we see that every point in $G^c \cap B^n(z,r)$ has e_1 coordinate between $-s$ and s. Thus $\langle z - w, e\rangle \le 2s$ for every $w \in G^c \cap B^n(z,r)$. We see that it suffices to show that $2s \le 2r/\sqrt{N}$ in order to prove the second claim of the lemma.

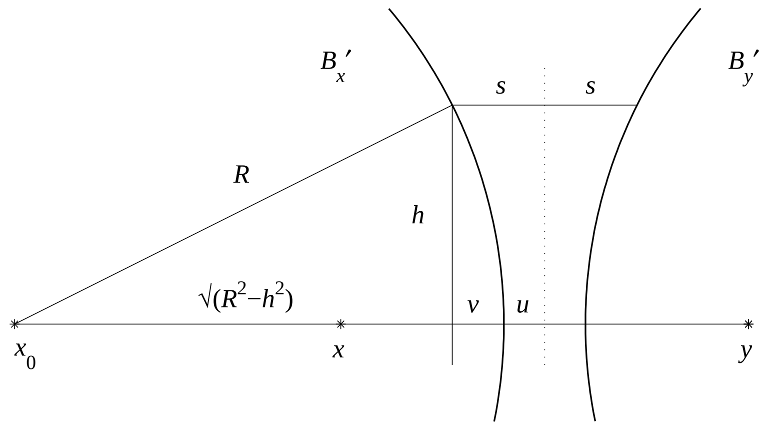

FIGURE 1. Computing the distance between B'_x and B'_y.

Using the values for h and s, the estimate for z_2 and the inequality $(q^2+1)/(2q) \le (N^2+1)/(N^2-1)$ (which follows from $q \le (N+1)/(N-1)$) we get

$$\frac{2s}{2r} = \frac{s}{h-z_2} \le \frac{R((q^2+1)/(2q) - \sqrt{1-1/N})}{R/\sqrt{N} - 2R/N} \le \frac{N(N^2+1)/(N^2-1) - \sqrt{N^2-N}}{\sqrt{N}-2}.$$

We then combine this estimate with the inequality

$$\begin{aligned} N(N^2+1)/(N^2-1) - \sqrt{N^2-N} &= 2N/(N^2-1) + N/(N+\sqrt{N^2-N}) \\ &\le 2N/(N^2-1) + N/(2N-1) \\ &= 2/N + 2/(N(N^2-1)) + 1/2 + 1/(2(2N-1)) \\ &\le 1/2 + 5/(2N). \end{aligned}$$

This gives

$$\frac{2s}{2r} \le \frac{1/2+5/(2N)}{\sqrt{N}-2} \le \frac{1}{\sqrt{N}},$$

where the last step follows by solving a second degree inequality in $\sqrt{N}$.

To prove the last claim of the lemma we simply choose $B := B^n(z_2e_2, r)$,

$$B_1 := B^n\big(-\tfrac{r+s}{2}e_1 + z_2e_2, \tfrac{r-s}{2}\big) \quad \text{and} \quad B_2 := B^n\big(\tfrac{r+s}{2}e_1 + z_2e_2, \tfrac{r-s}{2}\big),$$

as shown in Figure 2. It is clear that $B_1, B_2 \subseteq G \cap B$ and that the segment connecting the centers of B_1 and B_2 intersects ∂G at z. Using the previous estimate for s/r we get

$$2\mathrm{diam}(B_1) \ge \frac{r-s}{r}\mathrm{diam}(B) = (1-s/r)\mathrm{diam}(B) \ge (1-N^{-1/2})\mathrm{diam}(B),$$

which completes the proof. □

We next introduce an auxiliary notion, which is in fact equivalent to the thickness condition, as we will see later.

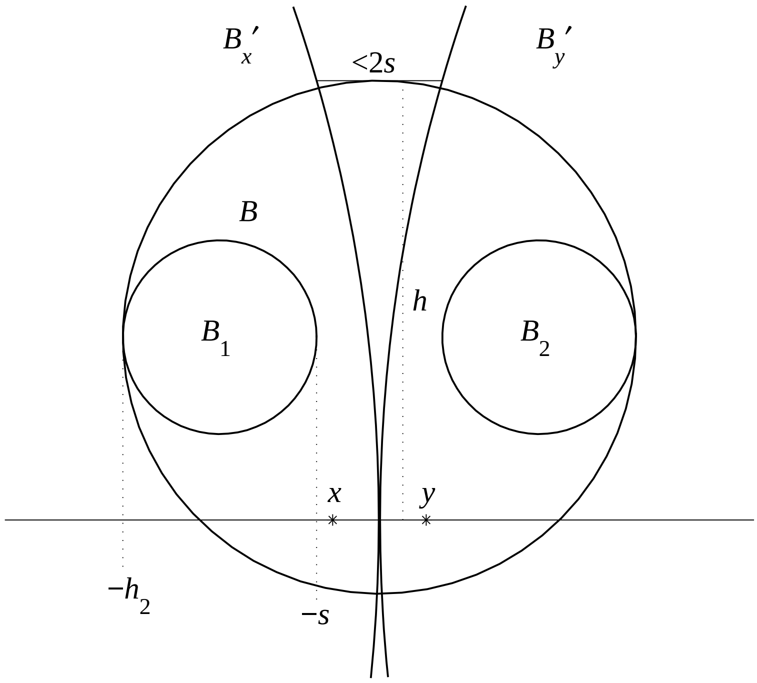

FIGURE 2. Constructing B_1 and B_2. Note that B'_x and B'_y only appear tangent.

Definition 3.3. We say that a domain $G \subsetneq \mathbb{R}^n$ satisfies an *interior double ball condition* with constant L (abbreviated L–IDB condition) if there exists a boundary point $z \in \partial G \setminus \{\infty\}$ and a real number $r > 0$ such that $B^n(z, 2r) \cap G$ contains two disjoint balls with radii r/L.

Example 3.4. The domain H^n satisfies the L–IDB if and only if $L \geq (1 + \sqrt{2})/2$. The domain $H^n \setminus [0, e_n]$ satisfies the L–IDB condition for every $L \geq 1$.

With this terminology, we see that Lemma 3.1 is a quantitative version of the claim that not (1) implies not (4), hence the implication (4)⇒(1) of Theorem 1.3 follows (an argument similar to that in Lemma 3.6 can be used here).

Lemma 3.5. *Let $G \subsetneq \mathbb{R}^n$ satisfy an L–IDB condition and $f \colon G \to \mathbb{R}^n$ be K–bilipschitz. Then $f(G)$ satisfies a K^2L–IDB condition.*

Proof. It is clear that f can be extended to a K–bilipschitz mapping of $\overline{G}$. After this the proof is trivial. □

The next trivial lemma will allow us to simplify the proof of the implication (1)⇒(4) in Theorem 1.3. Notice that $L' \to 1$ as $L \to 1$.

Lemma 3.6. *Let $G \subsetneq \mathbb{R}^n$ be a domain which has the L–IDB property for some $L < 2$. Then there exists a point $z \in \partial G \setminus \{\infty\}$ and a real number $s > 0$ such that $B^n(z, s) \cap G$ contains two disjoint balls B'_1 and B'_2 with radii $s/(2L')$ and $d(z, B'_1) = d(z, B'_2)$, where $L' = L/(2 - L)$.*

Proof. Let $B^n(z, 2r)$ be as in the definition of the IDB condition and set $s := 2r$. Let B_1 and B_2 be the balls from the definition of the IDB condition and let b_1 and b_2 be their centers. Set $a_1 := (b_1 - z)/|b_1 - z|$ and $a_2 := (b_2 - z)/|b_2 - z|$. Assume without loss of generality that $a := d(z, B_1) \geq b := d(z, B_2)$ and set $c := (a + b)/2 + r/L$ and $d := r/L + (b - a)/2$. Then we may choose $B_1' := B^n(ca_1 + z, d) \subseteq B_1$ and $B_2' := B^n(ca_2 + z, d) \subseteq B_2$. We easily calculate for the ratio of the radii of the balls that $d/s \geq (r/L - a/2)/(2r) \geq 1/L - 1/2$. ☐

We are now ready to complete the proof of the equivalence of the conditions in the main theorem involving the comparison property and the IDB condition.

Proof of Theorem 1.3, (1)⇒(4). Suppose that condition (4) is not true, i.e. that G has the L–IDB property for every $L > 1$. Let B be the large ball $B^n(z, s)$ from Lemma 3.6 and let B_1 and B_2 be the disjoint balls in $B \cap G$ with radii $r' := r/(2L)$ and $d(z, B_1) = d(z, B_2)$, where $L \leq 1.1$ corresponds to L' in the lemma. Let $x \in B_1$ and $y \in B_2$ be symmetric with respect to ∂B_1 and ∂B_2 (i.e. satisfy condition (A2) from Section 2.2).

As the Apollonian balls about x and y are at least as large as B_1 and B_2, it follows by monotony that

$$\alpha_G(x, y) \leq \sup_{a,b \in \partial(B_1 \cup B_2)} \log \frac{|a - x|}{|a - y|} \frac{|b - y|}{|b - x|} = 2\log(1 + 2c/s),$$

where $s := d(x, \partial B_1) = d(y, \partial B_2)$ and $c := d(B_1, B_2)/2$. Let h denote the distance from the line xy to the center z of B. Then

$$\min\{\delta(x), \delta(y)\} \leq \min\{|x - z|, |y - z|\} \leq \sqrt{(s + c)^2 + h^2},$$

since z is a boundary point. It follows that

$$j_G(x, y) \geq \log \left(1 + 2\frac{s + c}{\sqrt{(s + c)^2 + h^2}}\right).$$

Since $(1 + 1/u)\log(1 + u)$ is increasing, the inequality

$$\frac{\log(1 + u)}{\log(1 + v)} \leq \frac{u}{v}\frac{1 + v}{1 + u},$$

is valid for $0 < u \leq v$. Thus we find that

$$\frac{\alpha_G(x, y)}{j_G(x, y)} \leq 2\frac{c\sqrt{(s + c)^2 + h^2}}{s(s + c)} \frac{1 + 2(s + c)/\sqrt{(s + c)^2 + h^2}}{1 + 2c/s}. \tag{3.7}$$

(We will show that the right-hand-side of this inequality tends to 0. This implies that $u \leq v$, so that the inequality is applicable.) Since $1 + 2(s + c)/\sqrt{(s + c)^2 + h^2} \leq 3$ and $1 + 2c/s \geq 1$, we see that the third term on the right hand side of (3.7) is less than 3. It remains to estimate

$$\frac{c\sqrt{(s + c)^2 + h^2}}{s(s + c)} \leq \frac{\sqrt{c^2 + 2cr' + r(r - 2r')}}{c + 2r' - \sqrt{c^2 + 2cr'}},$$

where we used the expression for s and the estimate $h^2 \leq (r - r')^2 - r'^2$. We next multiply the numerator and denominator by $c + 2r' + \sqrt{c^2 + 2cr'}$ to find that

$$\frac{\sqrt{c^2 + 2cr' + r(r - 2r')}}{c + 2r' - \sqrt{c^2 + 2cr'}} \leq \frac{1}{4r'^2}\sqrt{c^2 + 2cr' + r(r - 2r')}\left(c + 2r' + \sqrt{c^2 + 2cr'}\right).$$

The last bound is clearly increasing in c, hence we use $c \leq r - 2r'$ to derive

$$\begin{aligned}\frac{1}{4r'^2}\sqrt{c^2 + 2cr' + r(r - 2r')} \cdot \left(c + 2r' + \sqrt{c^2 + 2cr'}\right) &\leq L\sqrt{2(L - 1)}(\sqrt{L} + \sqrt{L - 1}) \\ &\leq 2.13 \cdot \sqrt{L - 1},\end{aligned}$$

where we used $r/r' = 2L$ for the first and $L \le 1.1$ for the second inequality.

Combining the estimates for the last two terms in (3.7) we see that

$$\frac{\alpha_G(x,y)}{j_G(x,y)} \le 2 \cdot 3 \cdot 2.13\sqrt{L-1} \le 13\sqrt{L-1}$$

and so $\alpha_G/j_G \to 0$ as $L \to 1$, which means that G does not have the comparison property, as was to be shown. □

Corollary 3.8. *Let $G \subsetneq \mathbb{R}^n$ have the comparison property with constant K. Then there exists a constant $L_0 > 1$ depending only on K such that for every L–bilipschitz mapping $f\colon G \to \mathbb{R}^n$, $1 \le L \le L_0$, the domain $f(G)$ has the comparison property with constant K' depending only on K and L.*

Proof. This follows directly from Theorem 1.3, (1)⇔(4), and Lemma 3.5. □

4. Thick sets and the comparison property

In this section we show that a domain has the comparison property if and only if its complement is thick. This allows us to use a mapping constructed in [38] to show that the comparison property is not in general perserved by bilipschitz mappings.

Proof of Theorem 1.3, (1)⇔(2). Assume first that G does not have the comparison property. If $G^c \setminus \{\infty\}$ is bounded, then there is nothing to prove, since the first condition in (2) does not hold. Assume therefore that $G^c \setminus \{\infty\}$ is unbounded. We may then check the volume condition in the thickness property at any point in G^c and any $r > 0$. Let $N > 100$, $x, y \in G$ be points with $\alpha_G(x,y) < j_G(x,y)/N$ and let B, B_1, B_2, r and r_1 be as in Lemma 3.1. Assume without loss of generality that $r = 1$. Let b_1 and b_2 be the centers of B_1 and B_2 and let $z \in [b_1, b_2] \cap \partial G$. We estimate the width of the set $B^n(z,\rho) \setminus \big(B_1 \cup B_2\big)$ in direction $b_1 - b_2$, where $\rho := N^{-1/4}$. This can be done by calculating the distance between the balls B_1 and B_2 at distance ρ from the axis $b_1 b_2$, as in Figure 1. This distance equals $|b_1 - b_2| - \sqrt{r_1^2 - \rho^2} - \sqrt{r_2^2 - \rho^2}$. Since $|b_1 - b_2| = 1 + 3N^{-1/2}$ and $r_1 = r_2 = (1 - 3/\sqrt{N})/2$ this means that the width of $B^n(z,\rho) \setminus \big(B_1 \cup B_2\big)$ is less than or equal to

$$1 + \frac{3}{\sqrt{N}} - \sqrt{1 - \frac{10}{\sqrt{N}} + \frac{9}{N}} = \frac{3}{\sqrt{N}} + \frac{10/\sqrt{N} - 9/N}{1 + \sqrt{1 - 10/\sqrt{N} + 9/N}} \le \frac{13}{\sqrt{N}}.$$

Hence we see that

$$m_n\Big(B^n(z,\rho) \setminus \big(B_1 \cup B_2\big)\Big) \le \frac{13}{\sqrt{N}} \Omega_{n-1}\rho^{n-1} = 13\Omega_{n-1}\rho^{n+1},$$

where Ω_{n-1} is the volume of the $n-1$ dimensional unit ball and we used $\rho^2 = 1/\sqrt{N}$ in the equality. But from this it follows that no simplex in $B^n(z,\rho) \cap G^c$ can have volume proportional to ρ^n as $\rho \to 0$, which proves that G^c is not thick.

We have thus proved that (2)⇒(1). The converse implication contains a conjunction in its consequent, and so we prove the two implications separately, the first one by counter-assumption and the second one directly.

Suppose that $G^c \setminus \{\infty\}$ is bounded. Note that by assumption this set is nonempty. We may assume that $G^c \setminus \{\infty\} \subset B^n$. We then find that

$$\alpha_G(-Re_1, Re_1) \le 2\log\left(\frac{R+1}{R-1}\right) = 2\log\left(1+\frac{2}{R-1}\right) \sim \frac{4}{R},$$

as $R \to \infty$. On the other hand $j_G(-Re_1, Re_1) \sim \log(1+2R/R) = \log 3$, since there is a boundary point close to the midpoint of $[-Re_1, Re_1]$. It follows that G does not have the comparison property. Hence the comparison property implies that $G^c \setminus \{\infty\}$ is unbounded.

Assume next that G has the comparison property. Using the equivalence (1)⇔(4) proved previously, we see that there exists an $L > 1$ such that G does not have the L–IDB property. Let $z \in \partial G$ and $r > 0$ and consider the ball $B := B^n(z, 2r)$. Denote $r_1 := r/L$ and $r_2 = 2r - r_1$ and assume without loss of generality that $z = 0$. Then at least one of the balls $B_{1,+} := B^n(r_2 e_1, r_1)$ and $B_{1,-} := B^n(-r_2 e_1, r_1)$ contains a point z_1 of G^c. We define a set of points inductively. Suppose we have already defined $z_1, \dots z_{i-1}$ and $i \le n$. We choose $\hat{e}_i \in S^{n-1}$ to be orthogonal to $[0, z_1]$, …, $[0, z_{i-1}]$ and we consider $B_{i,+} := B^n(r_2\hat{e}_i, r_1)$ and $B_{i,-} := B^n(-r_2\hat{e}_i, r_1)$. Again one of these balls contains a point from ∂G and we choose z_i to be one of these.

Let S_i be the simplex with vertices $z, z_1, \dots, z_i$ for $1 \le i \le n$. Denote $q' = 2(r - r_1)$. We have $m_1(S_1) \ge q'$ since $|z_1 - z| \ge d(z, B_{1,+}) = d(z, B_{1,-}) = q'$. We can calculate $m_2(S_2)$ from the 1-volume of S_1 and the height z_2. This gives $m_2(S_2) = m_1(S_1)|\langle z_2, \hat{e}_2\rangle| \ge q'^2$. Because of the way the point z_i were selected, it is clear that we can continue like this to get $m_n(S_n) \ge q'^n$. We then see that the thickness condition is fulfilled at z with constant $q = (q'/(2r))^n = (1 - 1/L)^n$.

We still have to consider the thickness condition at exterior points of G. For $z \in \overline{G}^c$ we choose $w \in \partial G$ such that $|z - w| = d(z, \partial G) =: s$. Let $r > 0$. If $r < 2s$, then we choose a simplex of volume $s^n > r^n 2^{-n}$ in $B^n(z, s) \subset G^c$. For $r \ge 2s$ we choose a simplex S with $m_n(S) \ge (1-1/L)^n (r-s)^n \ge (1-1/L)^n r^n 2^{-n}$ in $B^n(w, r-s) \subset B^n(z, s)$, which is possible since $w \in \partial G$, by what was proved in the previous paragraph. Hence we see that G^c has the thickness property at this point with constant $(1-1/L)^n 2^{-n}$. We have thus proved that the comparison property implies that G^c is thick, which completes the proof. □

Using the previous theorem, we get a new proof for Corollary 3.8 by [38, Theorem 6.5]. As a converse of this result, Väisälä, Vuorinen and Wallin constructed a bilipschitz mapping which maps a thick set $A \subset B^2$ onto a set which is not thick [38, Example 6.2]. Considering the set $B^2(2) \setminus A$ this implies that bilipschitz mappings do not conserve the comparison property, either.

Corollary 4.1. *For every $L > 1$ there exists a domain $G \subset \mathbb{R}^2$ which has the comparison property and is mapped by an L–bilipchitz mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ onto a domain $f(G)$ which does not have the comparison property.*

The example $B^2(2) \setminus A$ above is infinitely connected. It remains an open problem whether there the comparison property of the Apollonian metric defined in a simply connected domain is preserved under bilipschitz mappings.

5. The exterior ball condition

In this section we introduce a very simple geometrical condition that is sufficient for the Apollonian metric to be "well behaved."

Definition 5.1. Let $L \geq 1$. We say that a domain $G \subsetneq \mathbb{R}^n$ satisfies an L–*exterior ball condition* (EB-condition) if for every $x \in \partial G \backslash \{\infty\}$ and $r > 0$ there exists a point $z \in \overline{B^n}(x,r)$ such that $B^n(z, r/L) \subseteq G^c$.

Balls and half-spaces are examples of domains satisfying a 1–exterior ball condition. Indeed it holds more generally that a convex domain is 1–EB.

The EB condition has appeared in the past in several similar guises. For instance, [2] and [28] used the term exterior R–ball condition and uniform exterior ball condition, respectively, for a 1–EB condition where we restrict the r in the definition to be less than R. The $A(c)$ condition of [33] and the plumpness condition from [37] correspond to the interior ball condition (that is, the EB condition of the complement, roughly speaking). The EB condition is also related to the strongly porous condition of [31] and the linear approximation property of [29]. Finally, if G is L–EB, then ∂G is satisfies the $1/L$–thinness condition of [13] which is equivalent to the $1/L$–porous condition in [36]. We will see below that we can work with the EB condition using the tools from [36].

We next prove that the EB condition is preserved under bilipschitz mappings. Note that it is trivial that a K–bilipschitz mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ maps L–EB domains onto (K^2L)–EB domains. We use a technique of Jussi Väisälä's to show that the EB condition is also preserved under K–bilipschitz mappings defined only in G. We need the following result, in which we denote $B^n(A,s) := \cup_{x\in A} B^n(x,s)$ for nonempty $A \subsetneq \mathbb{R}^n$. (In the statement of the lemma we applied a Euclidean similarity transformation, a trivial generalization of the original result.)

Lemma 5.2 (Theorem 3.17, [36]). *Let $x \in A \subsetneq \mathbb{R}^n$, $s > 0$ and $a > 0$ be such that $A \subseteq B^n(x,s) \subseteq B^n(A,a)$. Let $f\colon A \to \mathbb{R}^n$ be K–bilipschitz with $16K^3(K+1)a < s$. Then $B^n(f(x), s/(2K)) \subseteq B^n(f(A), Ka)$.*

Using this lemma the proof of the next theorem is similar to Väisälä's proof that porosity is preserved under bilipschitz mappings, [36, Theorem 4.2]. The proof is somewhat technical – the way to understand it is to set $\epsilon = 0$; then the claims are nice and simple, but, unfortunately, not quite true.

Theorem 5.3. *Let $G \subsetneq \mathbb{R}^n$ be a domain satisfying an L–EB condition and let $f\colon G \to f(G) \subsetneq \mathbb{R}^n$ be a K–bilipschitz mapping. Then $f(G)$ satisfies a K'–EB condition with $K' = \max\{4K^2L-1, 33K^3(K+1)\}$.*

Proof. It is easy to see that f can be extended to a K–bilipschitz mapping of $\overline{G}$. We denote this extension also by f.

Suppose that $f(G)$ is not L'–EB for $L' > \max\{4K^2L-1, 33K^3(K+1)\}$. This means that there exists a point $z \in \partial f(G)$ such that for every $w \in B^n(z,r')$ the ball $B^n(w, r'/L')$ intersects $f(G)$ in $B^n(z, r'(1+1/L'))$. From this it easily follows that for every sufficiently small $\epsilon > 0$, every point $x \in f(G) \cap B^n(z,\epsilon)$ and every $w \in B^n(x, r'')$ the ball $B^n(w, r''/L'')$ intersects $A := f(G) \cap B^n(x,s)$, where $r'' := r' - \epsilon$, $L'' := L'r''/r'$ and $s := r''(1+1/L'')$. This means that $B^n(x,r'') \subseteq B^n(A, r''/L'')$ and hence $B^n(x,s) \subseteq B^n(A,a)$ with $a := r''/L'' + s - r'' = 2r''/L''$.

It is clear that $A \subseteq B^n(x,s)$. Let us show that $16K^3(K+1)a < s$ for sufficiently small ϵ. Multiplying both sides of this inequality by L''/r'' and using the definition of a and s gives the equivalent inequality $32K^3(K+1) \leq L''+1$. Since $L'' \to L' \geq 33K^3(K+1)$ it follows that the inequality holds for small ϵ, hence we can apply Lemma 5.2 to the mapping f^{-1}.

This implies that $B^n(f^{-1}(x), s/(2K)) \subseteq B^n(G \cap f^{-1}(B^n(x,s)), Ka) \subseteq B^n(G, Ka)$. Since $|f^{-1}(z) - f^{-1}(x)| \le K\epsilon$ we conclude that for every $w \in B^n(f^{-1}(z), s/(2K) - K\epsilon)$ the ball $B^n(w, Ka)$ intersects G. This means that G is not $((s/(2K^2) - \epsilon)/a)$–EB. Letting $\epsilon \to 0$ this implies that G is not $((L'+1)/(4K^2))$–EB, and since G is L–EB by assumption we see that $(L'+1)/(4K^2) < L$ which contradicts the assumption $L' > 4K^2L - 1$ and hence proves the theorem. □

Definition 5.4. We say that a domain $G \subsetneq \mathbb{R}^n$ is a *quasiball* if there exists a quasiconformal homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $G = f(B^n)$.

For more information on quasiconformal mappings see e.g. [9, 26]. The function $\eta_{K,n}$ in the next lemma, from [1, 34], is such that for every quasiconformal homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with maximal dilatation constant K the inequality

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le \eta_{K,n}\left(\frac{|x-y|}{|x-z|}\right)$$

holds for every $x, y, z \in \mathbb{R}^n$, $x \neq z$. It was shown in [39, Theorem 1.8] that we can choose

$$\eta_{K,n}(1) \le \exp\{6(K+1)^2\sqrt{K-1}\};$$

hence we have a quantitative bound which is asymptotically sharp as $K \to 1$.

Lemma 5.5 (Lemma 5.8, [29])**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a quasiconformal homeomorphism with maximal dilatation less than or equal to K, $z \in S^{n-1}$ and $0 < r < \operatorname{diam}(f(S^{n-1}))$. Then $B^n(f(z), r) \setminus f(B^n)$ and $B^n(f(z), r) \cap f(B^n)$ both contain a ball of radius $t \ge r/(2\eta_{K,n}(1)^2)$.*

We can state the previous result in terms of the EB property:

Corollary 5.6. *Every quasiball G, defined by a mapping with maximal dilatation K, has the $(2\eta_{K,n}(1)^2 - 1)$–EB property.*

Proof. Let $z \in \partial G$ and $r > 0$. If $r < \operatorname{diam}(G)$, then Lemma 5.5 directly gives us a suitable ball for the verification of the EB condition. Suppose then that $r \ge \operatorname{diam}(G)$. By the lemma G contains a ball of diameter $\operatorname{diam}(G)/\eta_{K,n}(1)^2$. Draw a line through z and the center of this ball. Thus we find a second boundary point w_2 with $|w - w_2| \ge \operatorname{diam}(G)/\eta_{K,n}(1)^2$. Since G is contained in the ball $B^n(w_2, \operatorname{diam}(G))$, we find a ball B with center z and diameter $r - \operatorname{diam}(G)(1 - \eta_{K,n}(1)^{-2})$ in the set $B^n(w, r) \setminus B^n(w_2, \operatorname{diam}(G))$. We need to estimate the ratio of the radius of this ball to its center's distance from w:

$$\begin{aligned}\frac{r - \operatorname{diam}(G)(1 - \eta_{K,n}(1)^{-2})}{2|w-z|} &\ge \frac{r - \operatorname{diam}(G)(1 - \eta_{K,n}(1)^{-2})}{r + \operatorname{diam}(G)(1 - \eta_{K,n}(1)^{-2})} \\ &\ge \frac{\eta_{K,n}(1)^{-2}}{2 - \eta_{K,n}(1)^{-2}} = \frac{1}{2\eta_{K,n}(1)^2 - 1},\end{aligned}$$

where we used that $r \ge \operatorname{diam}(G)$ for the second inequality. Therefore this ball works in the EB condition with large r. □

6. Implications of the exterior ball condition

In this section we show that the EB property implies the comparison property and regularity in the Apollonian metric of Euclidean bilipschitz mappings.

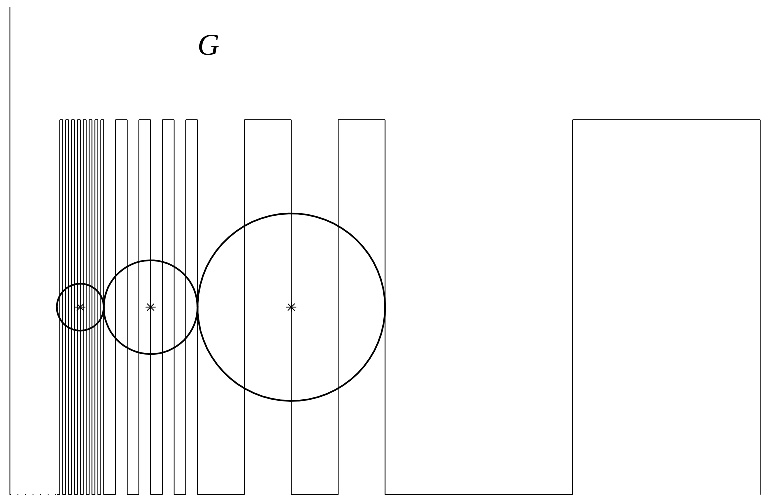

FIGURE 3. A domain with the comparison property which is not EB. Bars with $i = 0, 1, 2, 3$ shown.

Example 6.1. The domain $G := H^n \setminus [0, e_n]$ does not have the comparison property. For let $x_i = e_n + e_{n-1}/i$ and $y_i = e_n - e_{n-1}/i$. Then $j_G(x_i, y_i) = \log 3$ for every $i \geq 1$ but $\alpha_G(x_i, y_i) \to 0$ as $i \to \infty$.

An important consequence of the EB property for us is the following.

Theorem 6.2 (Theorem 1.3, [15]). *If $G \subsetneq \mathbb{R}^n$ has the L–EB property, then G has the comparison property with constant $L + \sqrt{L^2 - 1}$. This constant is the best possible one depending only on L.*

Combining Theorem 6.2 and Corollary 5.6 gives:

Corollary 6.3. *If $G \subsetneq \mathbb{R}^n$ is a K–quasiball, then there exists a constant L depending only on K and n such that $\frac{1}{L} j_G \leq \alpha_G$.*

The previous theorem shows that the EB condition implies the comparison property. The following example shows that the converse implication does not hold, i.e that not every comparison domain is an EB domain.

Example 6.4. Let $I_{i,j}$ be the closed rectangle with vertices

$$(1 + j2^{-i-1})2^{-i}e_1 + e_2, \quad (1 + (j+1)2^{-i-1})2^{-i}e_1 + e_2,$$
$$(1 + j2^{-i-1})2^{-i}e_1 \quad \text{and} \quad (1 + (j+1)2^{-i-1})2^{-i}e_1$$

for $i = 0, 1, 2, \ldots$ and $j = 0, 1, \ldots, 2^{i+1} - 1$. Let $G \subset \mathbb{R}^2$ be the upper right quadrant with the rectangles $I_{i,2j+1}$ removed for $i = 0, 1, 2, \ldots$ and $j = 0, 1, \ldots, 2^i - 1$. This domain, which is shown in Figure 3, does not have the EB property but has the comparison property.

Consider the points $3 \cdot 2^{-i-1}e_1 + e_2/2$, indicated by stars in the figure and the balls of radius 2^{-i-1} centered at these points. The EB property would imply that within every such ball we could choose a ball not contained in G the radius of which is greater than $2^{-i-2}/L$. However, the largest such ball has radius 2^{-i^2}, and so G does not have the EB property.

Suppose that there exist points $x, y \in G$ such that $\alpha_G(x,y) \le j_G(x,y)/3^4$. By Lemma 3.1 there would exist balls B_1 and B_2 in G with radii $4r/9$ and $d(B_1,B_2) = 2r/9$ such that the segment connecting their centers contains a boundary point z of G. It is clear this is not possible in our domain.

This example can be extended straightforwardly to $\mathbb{R}^n$ for $n \ge 3$.

The next theorem is a partial generalization of [10, Theorem 3.11] from the plane to higher dimensional space.

Theorem 6.5. *Let $G \subsetneq \mathbb{R}^n$ be a quasiball and let $f\colon G \to \mathbb{R}^n$ be bilipschitz with respect to the Apollonian metric. If $f(G)$ is a quasiball, then $f = g|_G$, where $g\colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is quasiconformal.*

Proof. Since by assumption G and $G' := f(G)$ are quasiballs, Corollary 6.3 implies that $j_G(x,y)/L \le \alpha_G(x,y) \le 2j_G(x,y)$ for $x, y \in G$ and similarly for G'. Thus, if $f\colon G \to G'$ is M–bilipschitz with respect to the Apollonian metric it follows that

$$j_G(x,y) \le L\alpha_G(x,y) \le LM\alpha_{G'}(f(x),f(y)) \le 2LMj_{G'}(f(x),f(y)),$$

where $x, y \in G$; similarly we may derive $j_{G'}(f(x),f(y)) \le 2LMj_G(x,y)$. Let $x \in G$ be fixed and let $y = y_r$ and $z = z_r$ be points in G such that $|x-y| = |x-z| = r$. We see that

$$\limsup_{r\to 0} \frac{|f(x)-f(y)|}{|f(x)-f(z)|} = \limsup_{r\to 0} \frac{j_{G'}(f(x),f(y))}{j_{G'}(f(x),f(z))} \le 4L^2M^2 \lim_{r\to 0} \frac{j_G(x,y)}{j_G(x,z)} = 4L^2M^2,$$

which is to say that f is quasiconformal in G with linear dilatation constant less than or equal to $4L^2M^2$. Since f maps the quasiball G onto the quasiball G', it can be extended to a quasiconformal mapping of $\overline{\mathbb{R}^n}$ by the reflection principle. □

Recall that the Apollonian bilipschitz modulus $\alpha_L(G)$ was defined in the introduction.

Proposition 6.6. *Let G be an L–EB domain. Then*

$$\alpha_K(G) \le 4K^2 \max\{4K^2L - 1, 33K^3(K+1)\}.$$

Proof. Let $f\colon G \to \mathbb{R}^n$ be a Euclidean K–bilipschitz mapping. It follows immediately from the Bernoulli inequality that f is K^2–bilipschitz with respect to the j_G metric. By Theorem 5.3, $f(G)$ satisfies the L'–EB condition with $L' = \max\{4K^2L-1, 33K^3(K+1)\}$. Let us denote $j := j_G(x,y)$, $j' := j_{f(G)}(f(x),f(y))$, $\alpha := \alpha_G(x,y)$ and $\alpha' := \alpha_{f(G)}(f(x),f(y))$ for short. Then, by Theorem 6.2 (since $L + \sqrt{L^2-1} \le 2L$), we find that

$$\frac{1}{4LK^2} \le \frac{j/(2L)}{2j'} \le \frac{\alpha}{\alpha'} \le \frac{2j}{j'/(2L')} \le 4K^2L',$$

which was to be shown. □

The next corollary follows by combining the previous proposition with Corollary 5.6.

Corollary 6.7. *There exists a function $\phi\colon [1,\infty) \to [1,\infty)$ depending only on K and n such that if $G \subsetneq \mathbb{R}^n$ is a K–quasiball, then $\alpha_L(G) \le \phi(L)$ for every $1 \le L < \infty$.*

7. The relationship between the bilipschitz and comparison properties

In this section we prove the relationship between the comparison property and $\alpha_L(G)$, which completes the proof of Theorem 1.3.

We first construct an auxiliary Euclidean bilipschitz mapping.

Lemma 7.1. *Let $0 < r < 1$ and define a mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by*

$$f(x_1, \dots, x_n) = \big(x_1 + (1 - |x|)r, x_2, \dots, x_n\big)$$

for $x \in B^n$ and $f(x) = x$ otherwise. Then f is $1/(1-r)$-bilipschitz.

Proof. It is clear that $f(B^n) = B^n$, $f(S^{n-1}) = S^{n-1}$ and $f(\mathbb{R}^n \setminus \overline{B^n}) = \mathbb{R}^n \setminus \overline{B^n}$, so we divide the proof into three parts, based on how many of the points are in B^n. First, let $x, y \in B^n$. We have $f(x) - f(y) = x - y + r(|y| - |x|)e_1$. Since $||y| - |x|| \le |x - y|$ it follows that

$$(1-r)|x-y| \le |x-y| - r||y|-|x|| \le |f(x)-f(y)| \le |x-y| + r||y|-|x|| \le (1+r)|x-y|.$$

Thus the bilipschitz constant in B^n is $1/(1-r) \ge 1 + r$. It is clear that f is bilipschitz also on the closure of B^n with the same constant.

Suppose next that $x \in B^n$ and $y \notin B^n$. Let $\{w\} := S^{n-1} \cap [x, y]$ and $\{z\} := S^{n-1} \cap [f(x), f(y)]$. Then

$$|f(x)-f(y)| \le |f(x)-f(w)| + |f(w)-f(y)| \le (1+r)|x-w| + |w-y| \le (1+r)|x-y|$$

and

$$|f(x)-f(y)| = |f(x)-z| + |z-f(y)| \ge (1-r)|x-z| + |z-y| \ge (1-r)|x-y|.$$

Finally, if $x, y \notin B^n$, then the bilipschitz condition is clear, since f is the identity in $\mathbb{R}^n \setminus B^n$. Therefore f is bilipschitz in all of $\mathbb{R}^n$, with constant $1/(1-r)$. □

The following is a quantitative version of (3)⇒(1) in Theorem 1.3.

Proposition 7.2. *Let $G \subsetneq \mathbb{R}^n$ be a domain such that $\alpha_L(G) < \infty$ for some $L > 1$. Then G has the comparison property with constant*

$$K \le \max\{25, (1+c^2)^2\},$$

where $c := 4/(L^{1/(2\alpha_L(G))} - 1)$.

Proof. Assume that there exist $z, w \in G$ such that $\alpha_G(z, w) \le j_G(z, w)/K$ for $K > 25$ (the claim is trivial otherwise).

Lemma 3.1 tells us that we can choose balls $B \subseteq \mathbb{R}^n$ and $B_1, B_2 \subseteq G \cap B$ such that B has radius r and B_1 and B_2 both have radii $r' := (1 - K^{-1/2})r/2$, $d(B_1, B_2) = 2(r - 2r')$ and that the segment connecting the centers of B_1 and B_2 intersects ∂G.

Let $x \in B_1$ and $y \in B_2$ be symmetric points in the spheres ∂B_1 and ∂B_2, that is, as in observation (A2) in Section 2.2. Denote $s := d(x, \partial B_1) = d(y, \partial B_2)$. It follows from the symmetry in ∂B_1 that

$$(r' - s)(r' + d + s) = |x_0 - x||x_0 - y| = r'^2,$$

where $d := d(B_1, B_2) = 2(r - 2r')$ and x_0 is the center of B_1. From this we solve $s = \sqrt{d^2/4 + dr'} - 1/(2d)$. Using the monotony of the Apollonian metric in the domain of definition for the first inequality, we find that

$$\begin{aligned}\alpha_G(x,y) &\le \sup_{a,b\in\partial(B_1\cup B_2)} \log\frac{|a-x|}{|a-y|}\frac{|b-y|}{|b-x|}\\ &= 2\log\left(1+\frac{d}{s}\right) = 2\log\left(1+\frac{d}{\sqrt{d^2/4+dr'}-d/2}\right)\\ &= 2\log\left(1+\frac{d}{r'}\big(\sqrt{1/4+r'/d}+1/2\big)\right)\\ &\le 2\log\big(1+2\sqrt{d/r'}\big),\end{aligned}$$

where the last inequality follows since $d/r' = 2(r/r'-2) = 4(1/(1-K^{-1/2})-1) = 4/(\sqrt{K}-1) \le 1$. Therefore we have shown that $\alpha_G(x,y) \le 2\log\big(1+4(\sqrt{K}-1)^{-1/2}\big)$.

Assume next without loss of generality that $\delta(x)\le\delta(y)$ and let $a\in\partial G$ be such that $|x-a| = \delta(x)$. Let f be as in Lemma 7.1 with the coordinate system chosen so that x is at the origin and $a = e_1$ with $r = 1-1/L$. Since $y\notin B^n$ we have $f(y) = y$ and we calculate

$$\alpha_{f(G)}(f(x),f(y)) \ge \log\frac{|f(y)-f(a)|}{|f(x)-f(a)|} = \log\frac{|y-a|}{|ra-a|} \ge \log(1/(1-r)) = \log L.$$

Since f is L–bilipschitz we have, by the definition of $\alpha_L(G)$,

$$\log L \le \alpha_{f(G)}(f(x),f(y)) \le \alpha_L(G)\alpha_G(x,y) \le 2\alpha_L(G)\log\big(1+4(\sqrt{K}-1)^{-1/2}\big),$$

from which the claim follows by solving for K. □

Proof of Theorem 1.3, (1)⇒(3). Assume first that G has the comparison property with constant K. Let L be a constant satisfying the bound $1\le L<L_0$, where L_0 is as in Corollary 3.8, and let $f\colon G\to\mathbb{R}^n$ be L–bilipschitz. Then, by the corollary, $f(G)$ has the comparison property with some constant K' depending on K and L. Hence

$$\frac{\alpha_G(x,y)}{\alpha_{f(G)}(f(x),f(y))} \le \frac{2j_G(x,y)}{j_{f(G)}(f(x),f(y))/K'} \le 2K'L^4$$

and

$$\frac{\alpha_{f(G)}(f(x),f(y))}{\alpha_G(x,y)} \le \frac{2j_{f(G)}(f(x),f(y))}{j_G(x,y)/K} \le 2KL^4.$$

It follows that $\alpha_L(G) \le 2L^4\max\{K,K'\}$ for these values of L. □

We conclude by considering a bilipschitz automorphism modulus. We define

$$\alpha'_L(G) := \sup_g \sup_{x,y\in G}\{\alpha_G(g(x),g(y))/\alpha_G(x,y)\},$$

where the first supremum is taken over all Euclidean L–bilipschitz mappings g mapping G onto G, again ignoring zero-denominator terms. The following result shows that the weaker bilipschitz condition $\alpha'_L(G)<\infty$ is equivalent to the comparison property.

Corollary 7.3. *Let $G\subsetneq\mathbb{R}^n$ be a domain. The following two conditions are equivalent:*

(1) *There exists a constant K such that $\frac{1}{K}j_G\le\alpha_G$.*
(2) *For every $L\ge 1$ we have $\alpha'_L(G)<\infty$.*

Proof. Assume first that G has the comparison property. Since $f(G) = G$ by assumption, $f(G)$ also has the comparison property. Thus we may argue as in the previous proof to show that G has the bilipschitz property. The converse implication follows from Theorem 7.2. □

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