Isometries of the quasihyperbolic metric

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Abstract

In this article we study curvature and isometries of the quasihyperbolic metric. We prove that, except for the trivial case of a half-plane, the isometries are exactly the similarity mappings. For the result we need to assume that the boundary of the domain is C^3 smooth.

1 Introduction

Let $D \subset \mathbb{R}^2$ be an open set and denote $\delta(x) = d(x, \partial D)$, the distance to the boundary. The quasihyperbolic metric in D is the conformal metric with the density $\delta(x)^{-1}$, in other words, the metric is given by

$$k_D(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds(z)}{\delta(z)},$$

where the infimum is taken over paths γ connecting x and y in D and ds represents integration with respect to arc-length.

The quasihyperbolic metric was first introduced in the seventies, and since then it has found innumerable applications, especially in the theory of quasiconformal mappings, see, e.g. [7, 8, 15, 21, 22]; new connections are still being made, for instance P. Jones and S. Smirnov [17] recently gave a criterion for removability of a set in the domain of definition of a Sobolev space in terms of the integrability of the quasihyperbolic metric, see also [18], and Z. Balogh and S. Buckley [1] used the metric in a geometric characterization of Gromov hyperbolic spaces.

Despite the prominence of the quasihyperbolic metric, there have been almost no investigations of its geometry. Three exceptions are the papers by G. Martin [21] and Martin and B. Osgood [22], the second of which was the main motivation for the approach presented in this paper, and the thesis by H. Lindén [20]. Part of the reason for this lack of geometrical investigations is probably that the density of the quasihyperbolic metric is not differentiable in the entire domain, which places the metric outside the standard framework of Riemanian metrics.

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At least two modifications of the quasihyperbolic metric have been proposed which do not suffer from this problem. J. Ferrand [6] suggested replacing the density δ^{-1} by

$$\sigma_D(x) = \sup_{a,b\in\partial D} \frac{|a-b|}{|a-x|\,|b-x|}.$$

Note that $\delta(x)^{-1} \le \sigma_D(x) \le 2\delta(x)^{-1}$, so the Ferrand metric and the quasihyperbolic metric are bilipschitz equivalent. Moreover, the Ferrand metric is Möbius invariant, whereas the quasihyperbolic metric is only Möbius quasi-invariant. A second variant was proposed more recently by R. Kulkarni and U. Pinkall [19], see also [16]. The K–P metric is defined by the density

$$\mu_D(x) = \inf\Big\{ \frac{2r}{(r-|x-z|)^2} : x\in B(z,r)\subset D \Big\}.$$

Equivalently, the infimum is taken over the hyperbolic densities of x in balls contained in D. This density satisfies the same estimate as Ferrand's density, i.e. $\delta(x)^{-1} \le \mu_D(x) \le 2\delta(x)^{-1}$, and the K–P metric is also Möbius invariant. Although the Ferrand and K–P metrics are in some sense better behaved than the quasihyperbolic metric, they suffer from the short-coming that it is very difficult to get a grip even of the density, even in simple domains.

Despite this, D. Herron, Z. Ibragimov and D. Minda [14] recently managed to solve the isometry problem of the K–P metric in most cases. By the isometry problem of the metric d we mean characterizing mappings $f\colon D\to\mathbb{R}^2$ with

$$d_D(x,y) = d_{f(D)}(f(x), f(y))$$

for all $x, y\in D$. Notice that in some sense we are here dealing with two different metrics, due to the dependence on the domain. Hence the usual way of approaching the isometry problem is by looking at some intrinsic features of the metric which are then preserved under the isometry. Since irregularities (e.g. cusps) in the domain often lead to more distinctive features, this implies that the problem is often easier for more complicated domains.

The work by Herron, Ibragimov and Minda [14] bears out this heuristic – they were able to show that all isometries of the K–P metric are Möbius mappings except in simply and doubly connected domains. Their proof is based on studying the curvature of the metric. For the quasihyperbolic metric, formulae for the curvature were worked out already in [22] (see Section 3, below), and were used in that paper to prove that all the isometries of the disc are similarity mappings. These will be our main tool in this paper. The other source of the ideas used below are the papers [9, 10, 11, 12] on isometries of some other similarity and Möbius invariant metrics.

There are three steps in characterizing quasihyperbolic isometries:

1. show that they are conformal;
2. show that they are Möbius; and
3. show that they are similarities.

The first step has been carried out by Martin and Osgood [22, Theorem 2.6] for completely arbitrary domains, so there is no more work to do there. In Section 4 we will use the results from [22] on the curvature of the quasihyperbolic metric, and some new ideas to prove that the conformal isometries are Möbius (second step). For this we need to assume that the boundary of the domain is at least C^3-smooth. In Section 2 we will work on the third step – we show that Möbius isometries are similarities provided the boundary is C^1. In Section 3 we study the Gaussian curvature of the quasihyperbolic metric, and the gradient of the curvature.

Notation

If $D \subset \mathbb{R}^2$, we denote by ∂D and $\overline{D}$ its boundary and closure, respectively. For $x \in D \subsetneq \mathbb{R}^2$ we denote $\delta(x) = d(x, \partial D) = \min\{|x - z| \colon z \in \partial D\}$. We tacitly identify $\mathbb{R}^2$ with $\mathbb{C}$, and speak about real and imaginary axes, etc. We will often work with a mapping $f \colon D \to \mathbb{R}^2$. In such cases we will use a prime to denote quantities on the image side, e.g. $x' = f(x)$, $D' = f(D)$ and $\delta'(x) = d(x, \partial D')$, and so on. By $B(x, r)$ we denote a disc with center x and radius r, and by $[x, y]$, $(x, y]$ the closed and half-open segment between x and y, respectively.

We denote by $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$ the one point compactification of $\mathbb{R}^2$. The cross-ratio $|a, b, c, d|$ is defined by

$$|a, b, c, d| = \frac{|a - c||b - d|}{|a - b||c - d|}$$

for distinct points $a, b, c, d \in \overline{\mathbb{R}^2}$, with the understanding that $|\infty - x|/|\infty - y| = 1$ for all $x, y \in \mathbb{R}^2$. A homeomorphism $f \colon \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|$$

for every quadruple of distinct points $a, b, c, d \in \overline{\mathbb{R}^2}$. A mapping of a subdomain of $\overline{\mathbb{R}^2}$ is Möbius, if it is a restriction of a Möbius mapping defined on $\overline{\mathbb{R}^2}$. For more information on Möbius mappings see e.g. [2, Section 3]. Note that a Möbius mapping can always be decomposed as $i \circ s$, where i is an inversion or the identity and s is a similarity. If i is an inversion in a circle with center x, then we say that i is centered at x, for short.

2 Isometries which are Möbius

Let D be a domain and $\zeta \in \partial D$. We say that ζ is *circularly accessible*, if there exists a disc $B \subset D$ such that $\zeta \in \partial B$.

Lemma 2.1. *Let $D \subsetneq \mathbb{R}^2$ be a Jordan domain with circularly accessible boundary, and let $f \colon D \to \mathbb{R}^2$ by a quasihyperbolic isometry which is also Möbius. Then, up to composition by similarity mappings, f is the identity or the inversion in a circle centered at a boundary point.*

Proof. Assume that f is not a similarity. Since f is a Möbius map, it is an inversion, up to similarities, which are always isometries of the quasihyperbolic metric. Thus it suffices to consider the case when f is an inversion in a unit sphere. Let us denote the center of this sphere by w.

Suppose first that $w \notin \overline{D}$ and let $\zeta \in \partial D$ be the closest boundary point to w. For simplicity we normalize the situation so that ζ lies on the positive real axis and $w = 0$. Since ζ is circularly accessible, we find a disc $B(z, r) \subset D$ which contains ζ in its closure. Since ζ is the closest boundary point to w, we see that z has to lie on the positive real axis, as well. Let x and y be points satisfying $\zeta < x < y \le \frac{\zeta(\zeta + 2r)}{\zeta + r}$. The right-hand inequality ensures that ζ is the closest boundary point to $[x, y]$, and that ζ' is the closest boundary point to $[x', y']$. Thus we find that $k_D(x, y) = \log \frac{|x - \zeta|}{|y - \zeta|}$ and $k_{D'}(x', y') = \log \frac{|x' - \zeta'|}{|y' - \zeta'|}$. Since f is the inversion in the unit sphere, we have

$$|x' - \zeta'| = \frac{|x - \zeta|}{|x|\,|\zeta|},$$

and similarly for y. Then the equation $\exp k_D(x, y) = \exp k_{D'}(x', y')$ gives us

$$\frac{|x - \zeta|}{|y - \zeta|} = \frac{|x - \zeta|}{|x|\,|\zeta|} \frac{|y|\,|\zeta|}{|y - \zeta|},$$

i.e. $|x| = |y|$. This contradiction shows that $w \in \overline{D}$. Since f maps D into $\mathbb{R}^2$, it is clear that $w \notin D$, so it follows that w is a boundary point. $\square$

We call D a C^k *domain*, if ∂D is locally the graph of a C^k function. Note that if D is a C^1 domain, then certainly every boundary point is circularly accessible.

Proposition 2.2. *Let $D \subsetneq \mathbb{R}^2$ be a C^1 domain, and let $f\colon D \to \mathbb{R}^2$ by a quasihyperbolic isometry which is also Möbius. If D is not a half-plane, then f is a similarity.*

Proof. We assume that f is not a similarity map. By the previous lemma we see that there is no loss of generality in considering only the case when f is the inversion centered at a boundary point. For simplicity of exposition, we normalize so that the origin is this center.

Let ζ be a boundary point of D distinct from 0 and let u be the inward pointing unit normal at ζ. For all sufficiently small $t > 0$, the point $x_t = \zeta + tu$ lies in D and its closest boundary point is ζ. For such $s < t$, we have

$$k_D(x_t, x_s) = \log \frac{t}{s}.$$

To estimate the distance of the image points, we use the inequality

$$j_{D'}(x', y') = \log\left(1 + \frac{|x' - y'|}{\min\{\delta'(x'), \delta'(y')\}}\right) \le k_{D'}(x', y'),$$

which is always valid (since $k_{D'}$ is the inner metric of $j_{D'}$, e.g. [8, Lemma 2.1]). We also need the formula

$$|x' - y'| = \frac{|x - y|}{|x|\,|y|}$$

for the length distortion of an inversion. Using these facts and the estimate $\delta'(x') \le |x' - \zeta'|$, we derive the inequality

$$\begin{aligned} k_{D'}(x', y') &\ge \log\Big(1 + \frac{|x' - y'|}{\min\{\delta'(x'), \delta'(y')\}}\Big) \\ &\ge \log\Big(1 + \frac{|x - y|/(|x|\,|y|)}{\min\{|x' - \zeta'|, |y' - \zeta'|\}}\Big) \\ &= \log\Big(1 + \frac{|x - y|\,|\zeta|}{|x|\,|y|\,\min\{|x - \zeta|/|x|, |y - \zeta|/|y|\}}\Big) \\ &= \log\Big(1 + \frac{|x - y|\,|\zeta|}{\min\{|y|\,|x - \zeta|, |x|\,|y - \zeta|\}}\Big). \end{aligned}$$

Applying this inequality to the points x_t and x_s as defined before, we have

$$k_{D'}(x'_t, x'_s) \ge \log\Big(1 + \frac{(t - s)\,|\zeta|}{\min\{t\,|x_t|, s\,|x_s|\}}\Big).$$

Let us choose $t = 2s$. Since $|x_{2s}|$ and $|x_s|$ both tend to $|\zeta|$ as $s \to 0$, we see that the second term in the minimum is smaller. Since the inversion is supposed to be an isometry, we can use the formula for $k_D(x_t, x_s)$ from before with the previous inequality to conclude that

$$\log \frac{2s}{s} \ge \log\Big(1 + \frac{(2s - s)\,|\zeta|}{s\,|x_s|}\Big).$$

Taking the exponential function gives $|x_s| \ge |\zeta|$. Since $x_s = \zeta + su$, this implies that $\langle \zeta - 0, u\rangle \le 0$ as $s \to 0$, where $\langle , \rangle$ denotes the scalar product.

Applying the same argument, but starting with points on the image side, we conclude that the opposite inequality is also valid. (There is actually a slight asymmetry here: the domain D' need not have circularly accessible boundary at the origin. However, it is clear that this does not affect the argument so far.) Thus it follows that $\langle \zeta - 0, u \rangle = 0$ for all boundary points. But since the boundary is assumed to be C^1, this implies that the domain is a half-plane. □

From [22, Theorem 2.8] we know that if $f\colon D \to \mathbb{R}^2$ is a quasihyperbolic isometry, then f is conformal in D. In dimensions three and higher every conformal mapping is Möbius. It is easy to see that the proofs in this section work also in the higher dimensional case. Therefore, we have proved the following result:

Corollary 2.3. *Let D be a C^1 domains in $\mathbb{R}^n$, $n \geq 3$, which is not a half-space. Then every quasihyperbolic isometry is a similarity mapping.*

Example 2.4. Note that if we do not assume C^1 boundary, then there are some further domains with non-trivial isometries: the punctured planes $\mathbb{R}^2 \setminus \{a\}$ and sector domains (i.e. domains whose boundary consists of two rays). In both cases inversions centered at the distinguished boundary point (a or the vertex of the sector). The previous proposition strongly suggests that these are all the examples.

3 Curvature of the quasihyperbolic metric

Let D be a domain in $\mathbb{R}^2$. We call a disc $B \subset D$ maximal, if it is not contained in any other disc contained in D. The set consisting of the centers of all maximal discs in D is called the *medial axis* of D and denoted by $\mathrm{MA}(D)$. The medial axis and differentiablility properties of the distance-to-the-boundary function have been studied e.g. in [3, 4, 5].

In a general domain the Gaussian curvature of the quasihyperbolic metric is not defined, since the distance-to-the-boundary function is not C^2. M. Heins [13] considered this situation for a quite general class of metric, and defined the notions of upper and lower curvature. Martin and Osgood worked with these curvatures in the context of the quasihyperbolic metric, see [22, Section 3] for details. However, if our domain is sufficiently regular (say C^2), and we are considering points not on the medial axis, then the upper and lower curvature agree, and define the curvature. In this case the curvature of k_D is given by

$$\mathcal{K}_D(z) = -\delta(z)^2 \triangle \log \delta(z),$$

[13, (1.3)] or [22, (3.1)]. On the medial axis this formula does not make sense, but the upper and lower curvatures still agree, and both equal $-\infty$, by [22, Corollary 3.12].

The next lemma is a specialization of Lemma 3.5, [22] to the case there the upper and lower curvatures agree.

Lemma 3.1 (Lemma 3.5, [22])**.** *Let G and $\tilde{G}$ be C^2 domains such that $B(z,r) \subset G \cap \tilde{G}$ and $\zeta \in (\partial G) \cap (\partial \tilde{G}) \cap (\partial B(z,r))$. If there is a neighborhood U of ζ such that $G \cap U \subset \tilde{G} \cap U$ and $d(z, \partial \tilde{G} \setminus U) > d(z, \partial \tilde{G})$, then $\mathcal{K}_G(z) \leq \mathcal{K}_{\tilde{G}}(z)$.*

Using this lemma we can derive the following very plausible statement, which says that the Gaussian curvature of the quasihyperbolic metric depends only on the curvature of the boundary at the closest boundary point. We sill need some more notation.

Let B be a disc with $\zeta \in (\partial B) \cap (\partial D)$. Then we call B the *osculating* disc at ζ if ∂B and ∂D have second order contact at ζ. Let D be at least a C^2 domain. Then there exists an osculating disc at every boundary point ζ. If this disc has radius r, then we define R_ζ to be r if the disc lies in the direction of the interior of D, and $-r$ otherwise. Note that the function $\zeta \mapsto 1/R_\zeta$ is C^{k-2} in a C^k domain, $k \geq 2$.

Proposition 3.2. *Let $D \subsetneq \mathbb{R}^2$ be a C^2 domain and $z \in D \setminus \mathrm{MA}(D)$ have closest boundary point $\zeta \in \partial D$. Then*

$$\mathcal{K}_D(z) = -\frac{R_\zeta}{R_\zeta - \delta(z)} = -\frac{1}{1 - \delta(z)/R_\zeta}.$$

If z lies on the medial axis, then $\mathcal{K}_D(z) = -\infty$.

Proof. The medial axis consists of points equidistant to two or more nearest boundary points, and of centers of osculating circles. For the former, the claim that $\mathcal{K}_D(z) = -\infty$ follows from [22, Corollary 3.12]. So we assume that z has a unique nearest boundary point, ζ.

We suppose further that $R_\zeta > 0$, the other case begin similar. Let $B(w, R_\zeta)$ be the osculating disc at ζ. We define $B_t = B(w + \frac{w-\zeta}{R_\zeta} t, R_\zeta + t)$, and note that ∂B_t contains ζ for all $t > -R_\zeta$. We have the formula

$$\mathcal{K}_{B(0,r)}(x) = -\frac{r}{|x|} = -\frac{r}{r - d(x, \partial B(0,r))}$$

for the curvature of the quasihyperbolic metric in a ball [22, Lemma 3.7], so we can calculate $\mathcal{K}_{B_t}(z)$ explicitly.

Using the previous lemma with $G = D$ and $\tilde{G} = B_t$ for $t > 0$ gives $\mathcal{K}_D(z) \le \mathcal{K}_{B_t}(z)$. If z is the center of B_0, then right-hand-side of the this inequality tends to $-\infty$ as $t \to 0$, which completes the proof of the claim regarding the medial axis. So we assume that z is not the center of B_0, and then we can apply the Lemma 3.1 with $G = B_t$ for $t < 0$ (sufficiently close to 0) and $\tilde{G} = D$ to get $\mathcal{K}_{B_t}(z) \le \mathcal{K}_D(z)$. Thus we have

$$\mathcal{K}_{B_{-t}}(z) \le \mathcal{K}_D(z) \le \mathcal{K}_{B_t}(z)$$

for small $t > 0$. Since $\mathcal{K}_{B_t}$ is continuous in t, we get $\mathcal{K}_D(z) = \mathcal{K}_{B_0}(z)$ as we let $t \to 0$. The proof is completed by applying the aforementioned formula for the curvature to the ball $B_0 = B(w, R_\zeta)$. □

Let $f \colon D \to \mathbb{R}^2$ be a C^1 mapping. By ∇f we denote the gradient of f, i.e. the vector $(\partial_1 f, \partial_2 f)$, and by $\tilde{\nabla} f(z)$ we denote $\delta(z)\nabla f(z)$. The reason for multiplying by $\delta(x)$ is that

$$\delta(y) = \lim_{x \to y} \frac{|x-y|}{k_D(x,y)},$$

so that the $\tilde{\nabla}$ operator is more natural in the setting where the quasihyperbolic but not the Euclidean distance is preserved (see (3.5), below).

We next present an explicit formula for $\tilde{\nabla}\mathcal{K}_D$. For this need a mapping which associates to every point in $D \setminus \mathrm{MA}(D)$ its closest boundary point. We call this mapping $\zeta = \zeta(z)$.

Lemma 3.3. *Let $D \subsetneq \mathbb{R}^2$ be a C^3 domain. Then*

$$\tilde{\nabla}\mathcal{K}_D(z) = (\mathcal{K}_D(z) + 1)\big[\mathcal{K}_D(z)\nabla\delta(z) - (\mathcal{K}_D(z) + 1)\nabla R_{\zeta(z)}\big]$$

for every z off the medial axis, where all differentiation is with respect to the variable z.

Proof. We use the formula from Proposition 3.2. Thus

$$\nabla\mathcal{K}_D(z) = -\nabla\frac{1}{1 - \delta(z)/R_\zeta} = \mathcal{K}_D(z)^2 \nabla\frac{\delta(z)}{R_\zeta} = \frac{\mathcal{K}_D(z)^2}{R_\zeta^2}(R_\zeta\nabla\delta(z) - \delta(z)\nabla R_\zeta),$$

where we understand ζ as a function of z. Note that R_ζ and δ are C^1, since D is C^3 and we are not on the medial axis. From Proposition 3.2 we also get $\frac{\delta(z)}{R_\zeta} = \frac{\mathcal{K}_D(z)+1}{\mathcal{K}_D(z)}$. Thus we continue the equation by

$$\begin{aligned}\tilde{\nabla}\mathcal{K}_D(z) &= \mathcal{K}_D(z)^2 \frac{\delta(z)}{R_\zeta}\Big(\nabla\delta(z) - \frac{\delta(z)}{R_\zeta}\nabla R_\zeta\Big) \\ &= (\mathcal{K}_D(z)+1)\big(\mathcal{K}_D(z)\nabla\delta(z) - (\mathcal{K}_D(z)+1)\nabla R_\zeta\big). \qquad \square\end{aligned}$$

We next show that $|\tilde{\nabla}\mathcal{K}|$ is an intrinsic quantity of the quasihyperbolic metric.

Lemma 3.4. *Let D be a C^3 domain. If $f\colon D \to \mathbb{R}^2$ is a quasihyperbolic isometry, then $|\tilde{\nabla}\mathcal{K}_D(z)| = |\tilde{\nabla}\mathcal{K}_{f(D)}(f(z))|$ for every $z \in D$.*

Proof. We know that f is conformal. For a unit vector u we find that

$$\begin{aligned}\big\langle\tilde{\nabla}\mathcal{K}_D(z), u\big\rangle &= \lim_{\varepsilon\to 0}\frac{\mathcal{K}_D(z+\varepsilon u) - \mathcal{K}_D(z)}{k_D(z+\varepsilon u, z)} \\ &= \lim_{\varepsilon\to 0}\frac{\mathcal{K}_{f(D)}(f(z+\varepsilon u)) - \mathcal{K}_{f(D)}(f(z))}{k_{f(D)}(f(z+\varepsilon u), f(z))}.\end{aligned} \tag{3.5}$$

Next we note that $f(z+\varepsilon u) = f(z) + \varepsilon f'(z)u + O(\varepsilon^2)$. Here $f'(z)u$ is understood as complex multiplication. Let us define another unit vector $\tilde{u} = \frac{f'(z)}{|f'(z)|}u$. Then we continue the previous equation by

$$\begin{aligned}\big\langle\tilde{\nabla}\mathcal{K}_D(z), u\big\rangle &= \lim_{\varepsilon\to 0}\frac{\mathcal{K}_{f(D)}(f(z)+\varepsilon f'(z)u) - \mathcal{K}_{f(D)}(f(z))}{k_{f(D)}(f(z)+\varepsilon f'(z)u, f(z))} \\ &= \lim_{\varepsilon\to 0}\frac{\varepsilon|f'(z)|\langle\nabla\mathcal{K}_{f(D)}(f(z)), \tilde{u}\rangle}{\varepsilon|f'(z)|\delta'(f(z))^{-1}} \\ &= \big\langle\tilde{\nabla}\mathcal{K}_{f(D)}(f(z)), \tilde{u}\big\rangle.\end{aligned}$$

Since u was an arbitrary unit vector, this implies that $|\tilde{\nabla}\mathcal{K}_D(z)| = |\tilde{\nabla}\mathcal{K}_{f(D)}(f(z))|$. $\square$

4 Isometries

We know that similarities are always quasihyperbolic isometries, and we want to show that in most cases these are the only ones. In view of the results in Section 2, it suffices for us to show that a quasihyperbolic isometry is a Möbius mapping, so this will be what we aim at in the proofs of this section.

A curve γ in D is a (quasihyperbolic) geodesic if

$$k_D(x,y) = k_D(x,z) + k_D(z,y)$$

for all $x, z, y \in \gamma$ in this order. It is clear from this definition that geodesics are preserved by isometries. A geodesic ray is a geodesic which is isometric to $\mathbb{R}^+$. For every $z \in D$ we easily find one geodesic ray, namely $[z, \zeta(z))$, which also happens to be a Euclidean line segment. The idea is to show that this geodesic is somehow special (from a quasihyperbolic point-of-view), so that it would map to a geodesic ray of the same kind.

Lemma 4.1. *Let $D \subsetneq \mathbb{R}^2$ be a C^2 domain with a boundary point ξ such that $1/R_\xi = 0$. Then every isometry $f\colon D \to \mathbb{R}^2$ of the quasihyperbolic metric is Möbius.*

Proof. Let $B \subset D$ be a non-maximal disc whose boundary contains ξ and let z denote the center of B. By Proposition 3.2 we find that $\mathcal{K}_D \equiv -1$ on the segment $\gamma = [z, \xi)$ Thus $\mathcal{K}_{f(D)} \equiv -1$ on γ', so $1/R'_{\zeta'(z')} = 0$ for every point z' on this curve. We consider two cases: either $\zeta'(z')$ is just a single point for all $z' \in \gamma'$, or it sweeps out a non-degenerate subcurve of the boundary $\partial D'$ as z' varies over γ'. (There is no third possibility, since ζ' is a continuous function on γ'.) In the single-point case we see that γ' has to be a line segment, since the boundary does not have corners. In this case we find that

$$k_D(x,y) = \left|\log \tfrac{|x-\xi|}{|y-\xi|}\right| \quad \text{and} \quad k_{D'}(x',y') = \left|\log \tfrac{|x'-\xi'|}{|y'-\xi'|}\right|,$$

where ξ' is the closest boundary point to the every point on γ'. But this easily implies that f is Möbius on γ. Since f is conformal it follows by uniqueness of analytic extension that f is a Möbius mapping on all of D.

So we consider the second case, that $\zeta'(z')$ sweeps out a non-degenerate subcurve of the boundary $\partial D'$. Since the curvature of the boundary at all these points is zero, it follows that the piece of the boundary is a line segment, L'.

Let $U' \subset D'$ be an open set such that $(\partial U') \cap (\partial D') = L'$ and the nearest boundary point of every point in U' lies in L'. Then the geometry of the quasihyperbolic metric in U is the same as in a half-plane, in particular $\mathcal{K}_{D'} \equiv -1$ on U'. Then $\mathcal{K}_D \equiv -1$ on $U = f^{-1}(U')$, so it follows that $(\partial U) \cap (\partial D) = L$, for some line segment L. So it follows that $f|_U$ is the restriction of a quasihyperbolic isometry of the half-plane. But these are only the Möbius mappings. Then we again conclude from the uniquness of analytic extension that f is a Möbius mapping on all of D. □

Let us call a domain strictly concave, if its complement is strictly convex.

Corollary 4.2. *Let $D \subsetneq \mathbb{R}^2$ be a C^2 domain which is not a half-plane, strictly convex or strictly concave. Then every quasihyperbolic isometry is a similarity mapping.*

Proof. Suppose that $1/R_\zeta \neq 0$ for all boundary points. Since $1/R_\zeta$ is continuous by assumption, this implies that it is either everywhere positive, or everywhere negative. In these cases we have a strictly convex and strictly concave domain, respectively, which was ruled out by assumption. So we find some point at which $1/R_\zeta = 0$. Then it follows from Lemma 4.1 that the isometry is Möbius and from Lemma 2.2 that it is a similarity. □

So we are left with only two types of domains that we cannot handle: strictly convex and strictly concave ones. As usual when working with isometries, the nicest domains turn out to be the most difficult. Unfortunately, we need to assume more regularity of the boundary in order to take care of these cases.

Theorem 4.3. *Let $D \subsetneq \mathbb{R}^2$ be a C^3 domain, which is not a half-plane. Then every isometry $f\colon D \to \mathbb{R}^2$ of the quasihyperbolic metric is a similarity mapping.*

Proof. In view of Corollary 4.2, we may restrict ourselves to the case when $\mathcal{K}_D(z) \neq -1$ for all $z \in D$. Let $z \in D \setminus \mathrm{MA}(D)$ and ζ be its nearest boundary point. We note that $\nabla\delta(z)$ and ∇R_ζ are perpendicular – first of all, $\nabla\delta(z)$ is parallel to $z - \zeta$; second, R_ζ is a constant in the direction of $z - \zeta$, since ζ is the closest boundary point to all points on this line (near z).

If D is bounded, then it is clear that R_ζ has a critical point. If D is unbounded, then we note that $1/R_\zeta$ cannot have any other limit than 0 at ∞ (although a limit need not exist, of course). Thus we see that R_ζ has a critical point in the unbounded case as well. Let ζ be a critical point of $\xi \mapsto R_\xi$ and fix a point $z \in D$ with $\mathcal{K}_D(z) \neq -\infty$ whose nearest boundary point is ζ. Of course, $\nabla R_\zeta = 0$ at the critical point ζ. Then it follows from Lemma 3.3 that

$$\tilde{\nabla}\mathcal{K}_D(z) = (\mathcal{K}_D(z) + 1)\mathcal{K}_D(z)\nabla\delta(z).$$

Since the curvature is intrinsic to the metric, we have $\mathcal{K}_{D'}(z') = \mathcal{K}_D(z)$. Also, $|\tilde{\nabla}\mathcal{K}_{D'}(z')| = |\tilde{\nabla}\mathcal{K}_D(z)|$ by Lemma 3.4, so we have

$$\big|(\mathcal{K}_D(z)+1)\mathcal{K}_D(z)\nabla\delta(z)\big| = \big|(\mathcal{K}_D(z)+1)\big[\mathcal{K}_D(z)\nabla\delta'(z') - (\mathcal{K}_D(z)+1)\nabla R'_{\zeta'(z')}\big]\big|$$

We know that $\mathcal{K}_D(z) \neq -1$ and that $\nabla\delta'(z')$ and $\nabla R'_{\zeta'(z')}$ are orthogonal. Thus the previous equation simplifies to

$$\big(\mathcal{K}_D(z)|\nabla\delta(z)|\big)^2 = \big(\mathcal{K}_D(z)|\nabla\delta'(z')|\big)^2 + \big((\mathcal{K}_D(z)+1)\big|\nabla R'_{\zeta'(z')}\big|\big)^2.$$

Since $|\nabla\delta| = 1$ off the medial axis for every domain, this equation implies that $\nabla R_{\zeta'} = 0$.

So for our point z, $\nabla\mathcal{K}_D(z)$ and $\nabla\mathcal{K}_{D'}(z')$ point to the nearest boundary point of z and z', respectively. Let $\gamma = [z,\zeta)$. Note that γ is a geodesic of the quasihyperbolic metric. Also, $\nabla\mathcal{K}_D(z)$ and γ are parallel at z. Now γ maps to some geodesic ray γ', and since f is a conformal mapping, γ' is parallel to $\nabla\mathcal{K}_{D'}(z')$ at z'. But $[z',\zeta')$ is a geodesic parallel to $\nabla\mathcal{K}_{D'}(z')$ at z', and since geodesics are unique (when the density is C^2, i.e. except possibly on the medial axis) we see that $\gamma' = [z',\zeta')$.

So we have shown that $f([z,\zeta)) = [z',\zeta')$. Moreover, we have

$$k_D(x,y) = \left|\log\tfrac{|x-\zeta|}{|y-\zeta|}\right| \quad \text{and} \quad k_{D'}(x',y') = \left|\log\tfrac{|x'-\zeta'|}{|y'-\zeta'|}\right|$$

for $x, y \in [z,\zeta)$. Thus we see that f is just a similarity on $[z,\zeta)$. But f is a conformal map, so this implies that f is a similarity in all of D. □

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