

# STARLIKENESS OF THE GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. In this paper, we derive conditions on the parameters  $a, b, c$  so that the function  $zF(a, b; c; z)$  is starlike in  $\mathbb{D}$ , where  $F(a, b; c; z)$  denotes the classical hypergeometric function. We give some consequences of our results including some mapping properties of convolution operator.

## 1. INTRODUCTION AND MAIN RESULTS

The Gaussian hypergeometric function  ${}_2F_1(a, b; c; z)$  given by the series

$$(1.1) \quad F(a, b; c; z) := {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} z^n \quad (|z| < 1),$$

where  $(a, 0) = 1$ ,  $(a, n+1) = (a+n)(a, n)$ ,  $n = 0, 1, 2, \dots$ , has appeared in the literature in many situations and contributed to various fields including conformal mappings, quasiconformal theory, and continued fractions [2, 13]. Here  $a, b, c$  are complex numbers and  $c \neq 0, -1, -2, -3, \dots$ . In the case of  $c = -m$ ,  $m = 0, 1, 2, \dots$ ,  $F(a, b; c; z)$  is defined if  $a = -k$  or  $b = -k$ , where  $j = 0, 1, 2, \dots$  and  $k \leq m$ . In this case,  $F(a, b; c; z)$  becomes a polynomial of degree  $k$  which we refer to as a hypergeometric polynomial. The hypergeometric function satisfies numerous identities (for example, see [25]) and we remark that the behaviour of the hypergeometric function  $F(a, b; c; z)$  near  $z = 1$  is classified into three cases according as  $\operatorname{Re}(c - a - b)$  is positive, zero, or negative. The case  $c = a + b$  is called the zero-balanced case and hypergeometric functions are unbounded for the case  $\operatorname{Re} c \leq \operatorname{Re}(a + b)$  as the asymptotic behavior at the singularity  $z = 1$  is well-known [25]. For  $z = x$ ,  $x \in (0, 1)$ ,  $a, b, c > 0$  the asymptotic behavior in the two cases  $a + b = c$  and  $a + b > c$  has been refined in [1] and [17], respectively. If  $\operatorname{Re}(c - a - b) > 0$  (see [25]), then

$$(1.2) \quad F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

For more details we refer to [2, 14, 25] and references therein. We recall some preliminaries.

Let  $\mathbb{D} := \{z : |z| < 1\}$  denote the unit disk of the complex plane  $\mathbb{C}$  and let  $\mathcal{H}$  denote the class of functions which are analytic in  $\mathbb{D}$ . As usual, let  $\mathcal{A}$  denote the class of all normalized functions  $f$  (i.e.  $f(0) = 0 = f'(0) - 1$ ) in  $\mathcal{H}$  and

$$\mathcal{S} = \{f \in \mathcal{A} \mid f \text{ is one-to-one in } \mathbb{D}\}.$$

A function  $f \in \mathcal{A}$  is called *starlike* (with respect to the origin 0), denoted by  $f \in \mathcal{S}^*$ , if  $tw \in f(\mathbb{D})$  whenever  $w \in f(\mathbb{D})$  and  $t \in [0, 1]$ . A function  $f \in \mathcal{A}$  that maps the unit disk  $\mathbb{D}$  onto a convex domain is called a *convex function*. Let  $\mathcal{K}$  denote the class of all

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functions  $f \in \mathcal{A}$  that are convex. For a given  $\delta < 1$ , a function  $f \in \mathcal{A}$  is called *starlike of order  $\delta$* , denoted  $f \in \mathcal{S}^*(\delta)$ , if

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \delta, \quad z \in \mathbb{D}.$$

For a given  $\delta < 1$ , a function  $f \in \mathcal{A}$  is called *convex of order  $\delta$* , denoted  $f \in \mathcal{K}(\delta)$ , if  $zf' \in \mathcal{S}^*(\delta)$ . It is well known that  $\mathcal{S}^*(0) = \mathcal{S}^*$  and  $\mathcal{K}(0) = \mathcal{K}$ . Clearly,  $\mathcal{S}^*(\delta) \subsetneq \mathcal{S}^*$  and  $\mathcal{K}(\delta) \subsetneq \mathcal{K}$  whenever  $0 < \delta < 1$ . A function  $f \in \mathcal{A}$  is said to be *close-to-convex* if  $\operatorname{Re}(f'(z)/g'(z)) > 0$  for all  $z \in \mathbb{D}$  and for some convex function  $g$  (need not be normalized). Let  $\mathcal{C}$  denote the union of all close-to-convex functions. A well-known fact is that

$$\mathcal{K} \subsetneq \mathcal{S}^*(1/2) \subsetneq \mathcal{S}^* \subsetneq \mathcal{C} \subsetneq \mathcal{S}.$$

For a detailed discussion on these classes, we refer to [5].

We are mainly interested in finding conditions on the triplet  $(a, b, c)$  so that  $zF(a, b, c; z)$  is starlike in  $\mathbb{D}$ . The geometric problem of starlikeness and close-to-convexity of  $zF(a, b, c; z)$  has been discussed by a number of authors (e.g. [7, 12, 18, 22]). But the exact range on the triplet  $(a, b, c)$  leading to solutions to the problem remains open for each of these classes, as well as to a number of other classical subclasses of  $\mathcal{S}$ .

Although the convexity of  $F(a, b, c; z)$  is addressed in [12] and later extended in [18], results concerning the convexity of the normalized function  $zF(a, b, c; z)$  do not seem to have discussed in detail. A condition for  $zF(a, b, c; z)$  to be convex in  $\mathbb{D}$  is given, for example, in [24] which we state as

**Lemma A.** *If  $a, b > 0$  and  $c > a + b + 2$ , then a sufficient condition for  $zF(a, b, c; z)$  to be convex in  $\mathbb{D}$  is that*

$$(1.3) \quad \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ 1 + \frac{3ab}{c-a-b-1} + \frac{(a,2)(b,2)}{(c-a-b-2,2)} \right] \leq 2.$$

We also observe that,  $1+z = F(-1, -1; 1; z)$  is convex in  $\mathbb{D}$  whereas the normalized analytic function  $z(1+z) = zF(-1, -1; 1; z)$  is not even univalent. Further, for  $f \in \mathcal{A}$  to be in  $\mathcal{K}$ , it is necessary that  $|f''(0)| \leq 2$ . According to a well-known result of Bieberbach, for  $f$  to belong  $\mathcal{S}$  it is necessary that  $|f''(0)| \leq 4$ . In view of these important facts we easily have the following preliminary result.

**Proposition 1.4.** *The function  $zF(a, b, c; z)$  is not convex in  $\mathbb{D}$  for  $|ab| > |c|$ ,  $c \neq 0, -1, -2, \dots$ . The function  $zF(a, b, c; z)$  is not univalent for  $|ab| > 2|c|$ .*

The non-sharp conditions of the above type, respectively, for nonunivalence, nonstarlikeness, nonconvexity may be obtained in a number of ways. The following lemma due to Miller and Mocanu [11] is the main tool used for our results concerning the starlikeness property of the hypergeometric function  $zF(a, b, c; z)$ .

**Lemma B.** *Let  $\Omega \subset \mathbb{C}$ . Suppose that  $\psi : \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C}$  satisfies the condition  $\psi(ir, s; z) \notin \Omega$  when  $r$  is real and  $s \leq -(1+r^2)/2$ . If  $p$  is analytic in  $\mathbb{D}$ , with  $p(0) = 1$  and  $\psi(p(z), zp'(z); z) \in \Omega$  for  $z \in \mathbb{D}$ , then  $\operatorname{Re}(p(z)) > 0$  in  $\mathbb{D}$ .*

We now state our main results and some of their consequences as corollaries.

**Theorem 1.5.** *Let  $a, b$  and  $c$  be nonzero real numbers such that  $F(a, b, c; z)$  has no zeros in  $\mathbb{D}$ . Then  $zF(a, b, c; z)$  is a starlike of order  $\beta \in [0, 1)$  if*

$$(1) \quad c \geq 1 + a + b - ab/\tilde{\beta} \quad (\text{equivalently, } C \geq 0);$$

- (2)  $C + \tilde{\beta} \geq 2A$ ; and  
 (3)  $(\tilde{\beta} + 2\tilde{\beta}^2)C + 2BD + D^2 \geq 0$ ,

where  $\tilde{\beta} = 1 - \beta$ ,  $A = \tilde{\beta}^2 - \tilde{\beta}(a + b) + ab$ ,  $B = \tilde{\beta}(a + b) - 2\tilde{\beta}^2$ ,  $C = \tilde{\beta}\tilde{c} + ab$ ,  $D = \tilde{\beta}\tilde{c}$ , and  $\tilde{c} = c - 1 - (a + b)$ .

*Remark 1.6.* From the proof of Theorem 1.5, it is clear that the theorem is applicable also when  $a$  is a nonzero complex number and  $b = \bar{a}$ . In particular, when  $a$  is negative integer with  $b = a$ , then the conclusion of Theorem 1.5 will be for the hypergeometric polynomials.  $\square$

If we set  $\beta = 1/2$ , then a computation shows that Theorem 1.5 takes the following simple form.

**Corollary 1.7.** *Let  $a, b$  and  $c$  be nonzero real numbers such that  $F(a, b; c; z)$  has no zeros in  $\mathbb{D}$ . Then  $zF(a, b; c; z)$  is a starlike of order  $1/2$  if*

$$c \geq \max\{1 + a + b - 2ab, 1 + 2ab, 1 + |a - b|\}.$$

Finding sufficient conditions for the function  $F(a, b; c; z)$  to be nonvanishing in  $\mathbb{D}$  is a different problem and has to be dealt with separately. For example, in view of the following derivative formula

$$(1.8) \quad F'(a, b; c; z) = \frac{d}{dz}(F(a, b; c; z)) = \frac{ab}{c}F(a + 1, b + 1; c + 1; z),$$

Lemma 2 in [11] gives that if  $a, b, c$  are nonzero real and satisfy

$$c \geq b \geq 0 \quad \text{and} \quad a \in [-1, 1) \cup [c - 1, c + 1],$$

then  $F(a, b; c; z) \neq 0$  in  $\mathbb{D}$ . In [18] the second and third author obtained various sufficient coefficient conditions to show that  $zF(a, b; c; z)$  is close-to-convex, and which in turn would imply that  $F(a, b; c; z) \neq 0$  in  $\mathbb{D}$ .

Our next result determines conditions on  $a, b, c$  for the odd Gaussian hypergeometric function  $zF(a, b; c; z^2)$  to be starlike (although one could still state a general result directly from Theorem 1.5).

**Corollary 1.9.** *Suppose that  $F(a, b; c; z) \neq 0$  in  $\mathbb{D}$ , and in addition,  $a, b, c$  satisfy the condition*

$$c \geq \max\{1 + a + b - 2ab, 1 + 2ab, 1 + |a - b|\}.$$

*Then  $zF(a, b; c; z^2)$  is in  $\mathcal{S}^*$ .*

*Proof.* If we let  $f(z) = zF(a, b; c; z)$  and  $h(z) = f(z^2)/z$ , then we have

$$\frac{zh'(z)}{h(z)} = 2\frac{z^2f'(z^2)}{f(z^2)} - 1.$$

By Corollary 1.7,  $f$  is starlike of order  $1/2$  and therefore, we conclude that  $h$  is starlike in  $\mathbb{D}$ .  $\square$

Corollaries 1.10 and 1.12 below follow immediately from Theorem 1.5 by some simple algebraic manipulation of the sufficient conditions. Hence there is no need to get into the details here, but we do provide some hints for one case.

**Corollary 1.10.** *Let  $a, b$  and  $c$  be nonzero real numbers such that  $F(a, b; c; z)$  has no zeros in  $\mathbb{D}$ . Then  $zF(a, b; c; z)$  is starlike of order  $1 - \frac{a+b}{2} \in [0, 1)$  if*

$$(1.11) \quad c \geq 1 + \frac{(a - b)^2}{a + b}.$$

*Proof.* Set  $\beta = 1 - \frac{a+b}{2}$  in Theorem 1.5. Then  $B = 0$ , and the condition  $C \geq 0$  in Theorem 1.5 is equivalent to (1.11). Condition (2) is equivalent to

$$C \geq -\left(\frac{a+b}{2}\right) - \frac{(a-b)^2}{2}$$

which is clearly true by (1.11). Because  $C \geq 0$  and  $B = 0$ , the condition (3) obviously holds. The conclusion follows.  $\square$

**Corollary 1.12.** *Let  $a, b$  and  $c$  be nonzero real numbers such that  $F(a, b; c; z)$  has no zeros in  $\mathbb{D}$ . Then  $zF(a, b; c; z)$  is a starlike if*

- (1)  $c \geq \max\{1 + a + b - ab, 2 + 2ab - (a + b)\}$ ; and
- (2)  $(c - 1)(c - 2) \geq a^2 + b^2 - ab - a - b$ .

At the end of the paper we discuss a few open problems.

*Remark 1.13.* Concerning the zero-freeness of  $F(a, b; c; z)$  in the unit disk  $\mathbb{D}$ , there is nothing to prove if either  $a = -1$  or  $b = -1$ . For example, if  $a = -1$  and  $0 \leq |b| \leq |c| \neq 0$  then we have  $F(-1, b; c; z) = 1 - (b/c)z \neq 0$ . Moreover, using the Euler integral representation for the hypergeometric function, one has (see [19])

$$(1.14) \quad F(a, b; c; z) = \int_0^1 \lambda(t) \frac{1}{(1-tz)^a} dt, \quad z \in \mathbb{D}, \quad \operatorname{Re} c >, \operatorname{Re} b > 0,$$

where

$$(1.15) \quad \lambda(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} t^{b-1} (1-t)^{c-b-1}.$$

Since  $\operatorname{Re}((1-tz)^{-a}) > 0$  in  $\mathbb{D}$  for  $-1 \leq a \leq 1$ , we observe from (1.14) that  $F(a, b; c; z) \neq 0$  whenever  $-1 \leq a \leq 1$  and  $c > b > 0$ . Also, one can use the well-known identity

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

and apply the above process for  $F(c-a, c-b; c; z)$  to conclude that  $F(a, b; c; z) \neq 0$  whenever  $c-a \in [-1, 1]$  and  $c > b > 0$ .

In general, there are ways to obtain sufficient conditions on the parameters  $a, b, c$  so that  $F(a, b; c; z) \neq 0$  in  $\mathbb{D}$ . For instance, there are a number of techniques which provide conditions so that the normalized analytic function  $zF(a, b; c; z)$  is univalent or close-to-convex in  $\mathbb{D}$  although the explicit range on the parameters  $a, b, c$  remains an open problem. These conditions in particular give sufficient conditions for the zero-freeness of  $F(a, b; c; z)$  in  $\mathbb{D}$ . For a detailed discussion, we refer to [9, 10, 14, 18, 22] as each has a different approach and explicit conditions are not known for either of these classes. To mention a simpler form, we recall the following result which derives from the theory of prestarlike functions (compare [20, p.61]) (see also [10]): If  $c \geq b \geq a > 0$ , then  $zF(a, b; c; z)$  is starlike of order  $1 - a/2$  and therefore  $F(a, b; c; z) \neq 0$  for all  $z \in \mathbb{D}$ .  $\square$

As an immediate reference concerning zero-freeness result for  $F(a, b; c; z)$  in the unit disk, we include below a summarized version of another result from [8, Theorem 1].

**Lemma C.** *Suppose that  $a, b, c$  satisfy any one of the following:*

- (1)  $0 < a \leq b \leq c$
- (2)  $-1 \leq a < 0 < b \leq c$
- (3)  $a < 0 < b$  and  $c \geq -a + b + 1$
- (4)  $0 < a \leq c \leq b \leq c + 1$
- (5)  $1 < a \leq c \leq b \leq c + a - 1$ .

Then  $F(a, b; c; z) \neq 0$  for all  $z \in \mathbb{D}$ .

At the end of the paper we discuss a few open problems.

## 2. APPLICATIONS TO CONVOLUTIONS

Equation (1.8) gives

$$\frac{c-1}{(a-1)(b-1)} zF'(a-1, b-1; c-1; z) = zF(a, b; c; z).$$

Using this one can easily transform Theorem 1.5 so as to obtain conditions for  $F(a, b; c; z)$  being convex of order  $\beta$  (note that  $F(a, b; c; z)$  is not the usual normalized function) as the definition for convexity of  $f \in \mathcal{H}$  continues to hold as long as  $f'(z) \neq 0$  in  $\mathbb{D}$ .

In addition to this remark, we now recall two identities that are indeed easy to verify by comparing the coefficients of  $z^n$  on both sides of them:

$$(2.1) \quad z(zF(1, b; c; z))' = zF(2, b; c; z) \quad \text{and} \quad z(zF(a, b; 2; z))' = zF(a, b; 1; z).$$

Using the first equation in (2.1), one can obtain the following results.

**Theorem 2.2.** *Let  $b$  and  $c$  be nonzero real numbers such that  $F(2, b; c; z)$  has no zeros in  $\mathbb{D}$ . Then  $zF(1, b; c; z)$  is convex of order  $\beta \in [0, 1)$  if*

- (1)  $c \geq 3 + b - 2b/\tilde{\beta}$  (equivalently,  $C \geq 0$ );
- (2)  $C + \tilde{\beta} \geq 2A$ ; and
- (3)  $(\tilde{\beta} + 2\tilde{\beta}^2)C + 2BD + D^2 \geq 0$ ,

where  $\tilde{\beta} = 1 - \beta$ ,  $A = \tilde{\beta}^2 - \tilde{\beta}(2+b) + 2b$ ,  $B = \tilde{\beta}(2+b) - 2\tilde{\beta}^2$ ,  $C = \tilde{\beta}\tilde{c} + 2b$ ,  $D = \tilde{\beta}\tilde{c}$ , and  $\tilde{c} = c - 3 - b$ .

As in the corollaries in Section 1, we can obtain corollaries concerning the convexity of  $zF(1, b; c; z)$ . For instance, a counterpart of Corollary 1.12 follows.

**Corollary 2.3.** *Let  $b$  and  $c$  be nonzero real numbers such that  $F(2, b; c; z)$  has no zeros in  $\mathbb{D}$ . Then  $zF(1, b; c; z)$  is a convex if  $c \geq \max\{3 - b, 3b\}$ .*

Moreover, using the second equation in (2.1), one can also derive results concerning the convexity of  $zF(a, b; 2; z)$ .

For  $f, g \in \mathcal{H}$  with  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  and  $g(z) = \sum_{k=0}^{\infty} b_k z^k$ , the (Hadamard) convolution of  $f$  and  $g$  is defined by  $(f * g)(z) := f(z) * g(z) = \sum_{k=0}^{\infty} a_k b_k z^k$  which is analytic in  $\mathbb{D}$ , i.e.  $f * g \in \mathcal{H}$ . For subsets  $\mathcal{M}$  and  $\mathcal{N}$  of  $\mathcal{H}$ , we define

$$\mathcal{M} * \mathcal{N} := \{f * g : f \in \mathcal{M}, g \in \mathcal{N}\}.$$

In [21], among other things Ruscheweyh and Sheil-Small proved that the class  $\mathcal{K}$  is closed under convolution, i.e.,

$$(2.4) \quad \mathcal{K} * \mathcal{K} \subset \mathcal{K}$$

settling the Pólya-Schoenberg conjecture. Moreover,

$$(2.5) \quad \mathcal{K} * \mathcal{S}(\beta) \subset \mathcal{S}(\beta) \quad \text{and} \quad \mathcal{K} * \mathcal{K}(\beta) \subset \mathcal{K}(\beta)$$

and a similar containment relation holds for the class of close-to-convex functions. In fact a general result has been proved later. To state our application, we need some preparation.

Using integral representation of the hypergeometric function  $F(a, b; c; z)$  we have the operator [3]

$$V_{a,b;c}(f)(z) := zF(a, b; c; z) * f(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt$$

where  $\operatorname{Re} a > 0$ ,  $\operatorname{Re} b > 0$ ,  $\operatorname{Re}(c + 1) > \operatorname{Re}(a + b)$  and

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1} (1-t)^{c-a-b} F\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right).$$

In particular, the Carlson–Schaffer operator  $G_{b,c}(f)(z)$  defined as  $G_{b,c}(f)(z) := V_{1,b;c}(f)(z)$  is given by

$$G_{b,c}(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \quad \lambda(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} t^{b-1} (1-t)^{c-b-1},$$

for  $\operatorname{Re} c > \operatorname{Re} b > 0$ ,  $f \in \mathcal{A}$ . Several basic geometric properties of this operator are known in the literature, for example see [3, 15, 16]. Thus, the convolution  $zF(a, b; c; z) * f(z)$  can be regarded as an extension of the study of integral operators of functions  $f$  in suitable subclasses  $\mathcal{A}$ , a classical topic in geometric function theory. Indeed, Bernardi [4] proved that for  $\gamma \in \mathbb{N}$

$$V_{1,\gamma+1;\gamma+2}(\mathcal{F}) \subset \mathcal{F} \text{ whenever } \mathcal{F} = \mathcal{S}^*, \mathcal{K} \text{ or } \mathcal{C}.$$

An interesting observation is that the Bernardi transform is related to zero-balanced hypergeometric function with functions in  $\mathcal{A}$ . We now recall a challenging problem in itself from [15, Problem 1.2].

*Problem 2.6.* Find all possible classes of functions  $\mathcal{F}_1, \mathcal{F}_2$  and conditions on the parameters  $(a, b, c)$  such that  $V_{a,b;c}(\mathcal{F}_1) \subset \mathcal{F}_2 \subset \mathcal{S}$ . In particular, can we extend the known properties of the Bernardi operator  $V_{1,\gamma+1;\gamma+2}(f)$  to the hypergeometric operator  $V_{a,b;c}(f)$ ?

From our present investigation, we can state conditions on  $a, b, c$  such that

$$V_{a,b;c}(\mathcal{K}) \subset \mathcal{F} \text{ or } V_{a,b;c}(\mathcal{S}^*) \subset \mathcal{F}$$

for some choices of  $\mathcal{F}$ , namely,  $\mathcal{K}$  or  $\mathcal{S}^*$  or  $\mathcal{C}$ .

In view of Theorems 1.5 and 2.2, one can obtain a number of convolution results. The details of the proofs of the following results are left as exercises.

**Theorem 2.7.** *Let  $b$  and  $c$  be nonzero real numbers such that  $F(2, b; c; z)$  has no zeros in  $\mathbb{D}$ . In addition, suppose that  $c \geq \max\{3 - b, 3b\}$ . Then we have*

- (1)  $V_{1,b;c}(\mathcal{S}^*) \subset \mathcal{S}^*$ ,
- (2)  $V_{1,b;c}(\mathcal{C}) \subset \mathcal{C}$ .

*Proof.* By Corollary 2.3,  $zF(1, b; c; z)$  is convex. Thus, by (2.5),  $f(z) * zF(1, b; c; z)$  is also starlike whenever  $f$  is starlike. Similar inclusion holds for close-to-convex case.  $\square$

Denote by  $\mathcal{S}_1^*$  the class of functions  $f \in \mathcal{A}$  for which

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \text{ for } z \in \mathbb{D}.$$

A well-known fact which may be found in [23] is that  $\mathcal{S}_1^* \equiv \mathcal{S}^* \equiv \mathcal{S}$  in case each function  $f$  in these classes has the property that  $f^{(n)}(0) < 0$  for all  $n \geq 2$ . Using this result, Silverman [24, Theorem 2] proved the following result regarding the class  $\mathcal{S}_1^*$ , in the case when all except the first of the Taylor coefficients of  $zF(a, b; c; z)$  are negative.

**Theorem D.** *If  $a, b > -1$ ,  $c > 0$ , and  $ab < 0$ , then a necessary and sufficient condition for  $zF(a, b; c; z)$  to be in  $\mathcal{S}^*$  ( $\mathcal{S}_1^*$ ) is that  $c \geq a + b + 1 - ab$ . The condition  $c \geq a + b + 1 - ab$  is necessary and sufficient for  $zF(a, b; c; z)$  to be in  $\mathcal{S}$ .*

Now we state the final result.

**Theorem 2.8.** *Let  $a, b, c$  be real numbers and define  $\alpha := (a - 1)(b - 1)$ ,  $\beta := ab - 1$  and  $\gamma := (a + 1)(b + 1)$ . Suppose that*

- (1)  $c + 1 \geq |1 + \gamma|$ ;
- (2)  $c - 1 \geq |1 - \alpha|$ ; and
- (3)  $2c^2 + 2 - 4\beta^2 - 2(1 - \alpha)(1 + \gamma) > -\sqrt{((c - 1)^2 - (1 - \alpha)^2)((c + 1)^2 - (1 + \gamma)^2)}$ .

*If  $F(a, b; c; z) \neq 0$  in  $\mathbb{D}$ , then  $zF(a, b; c; z) \in \mathcal{S}_1^*$ .*

Clearly, one can use Theorem 2.8 to generate a number of results as above for the convolution operator  $V_{a,b;c}(f)$ .

### 3. PROOFS OF MAIN THEOREMS

We start with the proof of Theorem 1.5.

**Proof of Theorem 1.5.** Let  $\phi(z) = zF(a, b; c; z)$  and  $p(z)$  be defined by

$$\frac{z\phi'(z)}{\phi(z)} = \beta + (1 - \beta)p(z).$$

Then,  $p(z)$  is analytic in  $\mathbb{D}$  and  $p(0) = 1$ . It is well-known that the hypergeometric function  $z \mapsto F(z) := F(a, b; c; z)$  satisfies the second order (hypergeometric) differential equation

$$(3.1) \quad z(1 - z)F''(z) + [c - (1 + a + b)z]F'(z) - abF(z) = 0.$$

A simple computation yields that

$$(3.2) \quad \begin{aligned} & (1 - z)(1 - \beta)zp'(z) + (1 - z)(1 - \beta)^2p^2(z) + p(z) \left( (1 - \beta)[c - 1 - (a + b)z] - 2(1 - \beta)^2(1 - z) \right) \\ & + (1 - \beta)^2(1 - z) - (1 - \beta)[c - 1 - (a + b)z] - abz = 0. \end{aligned}$$

Let us now define

$$\begin{aligned} \psi(r, s; z) = & (1 - z)\tilde{\beta}s + (1 - z)\tilde{\beta}^2r^2 + \left( \tilde{\beta}[c - 1 - (a + b)z] - 2\tilde{\beta}^2(1 - z) \right)r \\ & + \tilde{\beta}^2(1 - z) - \tilde{\beta}[c - 1 - (a + b)z] - abz. \end{aligned}$$

Then (3.2) is equivalent to

$$\psi(p(z), zp'(z); z) = 0.$$

We want to apply Lemma B to conclude the proof of the theorem. Therefore, to prove that  $\operatorname{Re}(p(z)) > 0$  in  $\mathbb{D}$ , we must show that the assumptions of our theorem implies that  $\psi(ir, s; z)$  does not have a zero for  $z \in \mathbb{D}$  and  $s \leq -(1 + r^2)/2$  with all  $r \in \mathbb{R}$ . We group the terms of  $\psi$  as follows:

$$\begin{aligned} \psi(ir, s; z) = & \left[ \tilde{\beta}s - \tilde{\beta}^2r^2 + \tilde{\beta}^2 - \tilde{\beta}(a + b) + ab + i(\tilde{\beta}(a + b) - 2\tilde{\beta}^2)r \right] (1 - z) \\ & - \tilde{\beta}\tilde{c} - ab + i\tilde{\beta}\tilde{c}r \\ =: & \left[ \tilde{\beta}s - \tilde{\beta}^2r^2 + A + iBr \right] (1 - z) - C + iDr, \end{aligned}$$

where

$$A = \tilde{\beta}^2 - \tilde{\beta}(a + b) + ab, \quad B = \tilde{\beta}(a + b) - 2\tilde{\beta}^2, \quad C = \tilde{\beta}\tilde{c} + ab, \quad D = \tilde{\beta}\tilde{c}.$$

Thus we find that the zero of  $\psi$  is at

$$z_0 = 1 + \frac{-C + iDr}{\tilde{\beta}s - \tilde{\beta}^2r^2 + A + iBr}.$$

Further it is a simple exercise to see that

$$|z_0|^2 = 1 + \frac{-2(\tilde{\beta}s - \tilde{\beta}^2r^2 + A)C + C^2 + (2BD + D^2)r^2}{(\tilde{\beta}s - \tilde{\beta}^2r^2 + A)^2 + B^2r^2}.$$

Since we want the zero not to occur in the unit disk (in order to satisfy the conditions of the lemma), we have to show that  $|z_0| \geq 1$  for all the relevant parameter values. Since the denominator in the second term of our expression for  $|z_0|$  is always non-negative, we see that this is equivalent to showing that

$$-2(\tilde{\beta}s - \tilde{\beta}^2r^2 + A)C + C^2 + (2BD + D^2)r^2 \geq 0.$$

This condition has to be satisfied for all  $s \leq -(1 + r^2)/2$  and all real  $r$ . Since  $C \geq 0$  (by assumption) we see that the inequality need only be checked for the largest value of  $s$ , i.e.  $s = -(1 + r^2)/2$ , in which case we need that

$$\begin{aligned} & \left( \tilde{\beta}(1 + r^2) + 2\tilde{\beta}^2r^2 - 2A \right) C + C^2 + (2BD + D^2)r^2 \\ &= \left[ (\tilde{\beta} + 2\tilde{\beta}^2)C + 2BD + D^2 \right] r^2 - 2AC + C^2 + \tilde{\beta}C \end{aligned}$$

is non-negative. By assumption, the square bracket and the constant term are both non-negative, so this is clear. Hence, by Lemma B, we conclude that  $\operatorname{Re}(p(z)) > 0$  in  $\mathbb{D}$  which shows that  $\phi(z) \in \mathcal{S}^*(\beta)$ .  $\square$

*Remark 3.3.* The conditions given in the previous theorem are optimal in the sense that  $\psi(ir, s; z)$  has a zero in the unit disk if and only if the conditions hold. Thus, in particular, using differential subordination with this equation cannot give better results. The verification of this statement is left as an exercise.

**Proof of Theorem 2.8.** By the definition of the class  $\mathcal{S}_1^*$ , it suffices to show that

$$\operatorname{Re} \left( \frac{\phi(z)}{z\phi'(z)} \right) > \frac{1}{2}, \quad z \in \mathbb{D},$$

where  $\phi(z) = zF(a, b; c; z) =: zF(z)$ . Define

$$p(z) = 2 \left( \frac{\phi(z)}{z\phi'(z)} - \frac{1}{2} \right).$$

Then  $p(z)$  is analytic in  $\mathbb{D}$ ,  $p(0) = 1$  and

$$(1 + p(z))zF'(z) = (1 - p(z))F(z).$$

Since  $z \mapsto F(z)$  satisfies the differential equation (3.1), the above substitution in (3.1) easily results in

$$\psi(p(z), zp'(z); z) = 0$$

where

$$\begin{aligned} \psi(r, s; z) &= 2s(1 - z) + r^2[c - 2 + (a - 1)(b - 1)z] \\ &\quad + 2r[1 + (ab - 1)z] - [c - (a + 1)(b + 1)z]. \end{aligned}$$

Again, we wish to apply Lemma B to conclude the proof of the theorem. Therefore, to prove that  $\operatorname{Re}(p(z)) > 0$  in  $\mathbb{D}$ , we must show that the assumptions of our theorem implies that  $\psi(ir, s; z) \neq 0$  for  $z \in \mathbb{D}$  and  $s \leq -(1 + r^2)/2$  with all  $r \in \mathbb{R}$ .

Let us denote  $\alpha := (a - 1)(b - 1)$ ,  $\beta := ab - 1$  and  $\gamma := (a + 1)(b + 1)$ . Then

$$\psi(ir, s; z) = 2s(1 - z) - r^2(c - 2 + \alpha z) + 2ir(1 + \beta z) - c + \gamma z.$$

Solving  $\psi(ir, s; z) = 0$  for  $z$  gives

$$|z|^2 = \frac{(2s - r^2(c - 2) - c)^2 + 4r^2}{(-2s - \alpha r^2 + \gamma)^2 + 4\beta^2 r^2}.$$

Thus  $\psi(ir, s; z) \neq 0$  for  $z \in \mathbb{D}$  if and only if the

$$\frac{(2s - r^2(c - 2) - c)^2 + 4r^2}{(-2s - \alpha r^2 + \gamma)^2 + 4\beta^2 r^2} \geq 1$$

which, after simplification, gives

$$(2s - r^2(c - 2) - c)^2 + 4(1 - \beta^2)r^2 - (2s + \alpha r^2 - \gamma)^2 \geq 0.$$

In this expression we see that the coefficient of  $s^2$  is 0, and the coefficient of  $s$  is

$$4((-\alpha - c + 2)r^2 + \gamma - c).$$

By assumption,  $-\alpha - c + 2 < 0$  and  $\gamma - c < 0$ , so the coefficient of  $s$  is negative, and it suffices to check the inequality with  $s = -(1 + r^2)/2$ .

Setting  $u = r^2$  and  $2s = -(1 + r^2)$  gives the inequality

$$(u(c - 1) + c + 1)^2 + 4(1 - \beta^2)u - ((1 - \alpha)u + \gamma + 1)^2 \geq 0.$$

This is a second degree polynomial inequality in  $u$ , and we want this inequality to hold for all  $u \geq 0$ . One easily checks that  $Au^2 + 2Bu + C \geq 0$  for all  $u \geq 0$  if and only if  $A \geq 0$ ,  $C \geq 0$  and  $B \geq -\sqrt{AC}$ , where

$$\begin{aligned} A &= (c - 1)^2 - (1 - \alpha)^2, \\ B &= c^2 - 1 + 2(1 - \beta^2) - (1 - \alpha)(1 + \gamma), \\ C &= (c + 1)^2 - (1 + \gamma)^2. \end{aligned}$$

These are exactly the conditions of the theorem. □

#### 4. CONCLUSION

We conclude the paper with a number of problems.

*Problem 4.1.* (1) For  $a, b, c > 0$ ,  $a + b < c$ , the function  $g(z) = F(a, b; c; z)$  is bounded, see (1.2). Therefore there are numbers  $m, M$  such that

$$\mathbb{D}(m) = \{z \in \mathbb{C} : |z| < m\} \subset g(\mathbb{D}) \subset \mathbb{D}(M).$$

We do not know the least value of  $M$  in (1.2), nor the greatest value of  $m$ .

(2) We next recall a couple of open problems from [18, 6.4], which also are still open, as far as we know. There exist positive numbers  $\delta_1, \delta_2$  such that for  $a \in (0, \delta_1)$  and  $b \in (0, \delta_2)$  the normalized functions  $z {}_2F_1(a, b; a + b; z)$  and  $z {}_2F_1(a, b; a + b; z^2)$  map the unit disk  $\mathbb{D}$  into a strip domain. For example, each of the functions

$$-\log(1 - z) = z {}_2F_1(1, 1; 2; z) \quad \text{and} \quad \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right) = z {}_2F_1(1, 1/2; 3/2; z^2)$$

have this property. The problem is to find the exact range of the constants  $\delta_1, \delta_2$  and conditions on  $a$  and  $b$  satisfying the stated property.

We recall that the Koebe function  $k(z) := z/(1 - z)^2 = zF(1, 2; 1; z)$  maps  $\mathbb{D}$  onto the complement of the ray  $\{w = u + iv \in \mathbb{C} : u = 0, v \leq -1/4\}$ . Note that  $k(\mathbb{D})$  is a domain which is starlike with respect to the origin. This function raises the following question: suppose  $a, b, c > 0$  with  $c < a + b$ . Do there exist  $\delta_3, \delta_4 > 0$  such that for  $a \in (0, \delta_3)$  and  $b \in (0, \delta_4)$  the functions  $z {}_2F_1(a, b; c; z)$  and  $z {}_2F_1(a, b; c; z^2)$  have the property that the

image domain is completely contained in a sector type domain where the “angle” depends on  $a + b - c$ ?

Also, we see that the function  $c(z) := z/(1 - z^2) = zF(1, 1; 1; z^2)$  maps  $\mathbb{D}$  onto a slit domain, namely, the complement of the ray  $\{w = u + iv \in \mathbb{C} : u = 0, |v| \geq 1/2\}$ . Again, we note that  $c(\mathbb{D})$  is a domain which is starlike with respect to the origin. Thus, it is natural to look at the geometric link while passing from the zero-balanced case  $c = a + b$  to the nonzero-balanced case  $c < a + b$  in  ${}_2F_1(a, b; c; z^2)$ .

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