HARNACK'S INEQUALITY FOR $p(\cdot)$-HARMONIC FUNCTIONS WITH UNBOUNDED EXPONENT p

PETTERI HARJULEHTO, PETER HÄSTÖ*, AND VISA LATVALA

Abstract. We study properties of the function $u = \lim_{\lambda\to\infty} u_\lambda$, where u_λ is the solution of the $\min\{p(\cdot), \lambda\}$-Laplacian Dirichlet problem with bounded Sobolev boundary function. Here $p : \Omega \to (n, \infty]$ is a variable exponent such that $1/p$ is Lipschitz continuous. We derive Bloch-type estimates and using them we prove Harnack's inequality in cases of unbounded but finite exponent.

1. Introduction

During the last ten years, function spaces with variable exponent have attracted a lot of interest, see the surveys by Diening, Hästö and Nekvinda [9] and Samko [32]. Apart from interesting theoretical considerations, these investigations were motivated by a proposed application of variable exponent spaces to modeling electrorheological fluids, see Růžička [30, 31] and Acerbi & Mingione [2]. More recently, another application emerged, as Chen, Levine & Rao [6] proposed a variable exponent formulation for the problem of image restoration.

Partial differential equations related to Sobolev spaces with variable exponent have also been investigated by several researchers. The paradigmatic Dirichlet minimization problem,

$$\inf_u \int_\Omega |\nabla u|^{p(x)}\, dx$$

for $u - w \in W_0^{1,p(\cdot)}(\Omega)$, where $w \in W^{1,p(\cdot)}(\Omega)$ is the boundary function, has been investigated e.g. in [1, 13, 20, 24] and the corresponding Euler-Lagrange equation

$$\operatorname{div}(p(x)|\nabla u|^{p(x)-2}\nabla u) = 0$$

e.g. in [3, 4, 5, 12, 15, 22, 29, 33]. However, all of these investigations have been limited to the case when p is bounded away from 1 and ∞. Investigations of the $p(\cdot)$-harmonic functions usually also assume that p is log-Hölder continuous, but also stronger continuity conditions have been used, for instance in [1, 3, 15]. Recently, the authors [21] considered minimizers in the case $p \to 1$. The main points of that work are described in Section 3.

In this paper we investigate the opposite limit, as $p \to \infty$. Previous investigations of limit cases of the variable exponent (e.g. [11, 16, 18] on Sobolev inequalities when $p \nearrow n$ or $p \searrow n$, and [8, 17] on maximal inequalities when $p \searrow 1$) suggest that going to the limit will be quite difficult. Our approach is based on solving the Dirichlet problem for $p_\lambda = \min\{p, \lambda\}$, a truncated version of the exponent p we are really interested in, and then letting $\lambda \to \infty$. The challenge is to obtain estimates which are independent of the upper

Date: August 15, 2019.

2000 *Mathematics Subject Classification.* 35J60 (35B20, 35J25, 46E35).

Key words and phrases. Non-standard growth, variable exponent, Laplace equation, Dirichlet energy, solution, Caccioppoli estimate.

* Corresponding author.

This research was supported by the Academy of Finland, INTAS and the Emil Aaltonen Foundation.

bound of the intermediate exponents. This is only partially achieved in the method presented in this article: for our results we need a growth condition on the exponent. Although this condition allows us to approach the set where the exponent is infinite, it does not allow us to cross into this set.

The structure of this article is as follows. We start by reviewing the basics of variable exponent spaces. In Section 3 we present the method and results from the case $p \to 1$. Then we consider minimizers in the one-dimensional case in Section 4. In Section 5 we prove Bloch type estimates which are used in Section 6 for the proof of Harnack's inequality for minimizers in the unbounded case $p : \Omega \to (n, \infty)$.

2. Preliminaries

By $\Omega \subset \mathbb{R}^n$ we denote a bounded open set. A measurable function $p\colon \Omega \to [1, \infty]$ is called a *variable exponent*, and we denote for $A \subset \Omega$

$$p_A^+ := \operatorname*{ess\,sup}_{x\in A} p(x), \quad p_A^- := \operatorname*{ess\,inf}_{x\in A} p(x), \quad p^+ := p_\Omega^+ \quad \text{and} \quad p^- := p_\Omega^-.$$

For $\lambda > 1$ we denote $p_\lambda(x) = \min\{\lambda, p(x)\}$. By Ω_∞ we always denote the set where p equals infinity, $\Omega_\infty := \{x \in \Omega : \ p(x) = \infty\}$. By C we denote a generic constant, whose value may change between appearances even within a single line.

The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For a constant function p, it coincides with the standard Lebesgue space. Often it is assumed that $p^+ < \infty$, since this condition is known to imply many desirable features for $L^{p(\cdot)}(\Omega)$. Spaces with $p^+ = \infty$ have been investigated in [8, 10, 25].

We define a *modular* on the set of measurable functions by setting

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)}\, dx + \operatorname*{ess\,sup}_{x\in\Omega_\infty} |u(x)|.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular $\varrho_{L^{p(\cdot)}(\Omega)}(u/\mu)$ is finite for some $\mu > 0$. The Luxemburg norm on this space is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\Big\{\mu > 0 : \ \varrho_{L^{p(\cdot)}(\Omega)}(\tfrac{u}{\mu}) \leqslant 1\Big\}.$$

In the case of norms and modulars taken over the whole set Ω we also use an abbreviated notation where $L^{p(\cdot)}(\Omega)$ in the subscript is replaced simply by $p(\cdot)$. Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space.

If E is a measurable set with a finite measure, and p and q are variable exponents satisfying $q \leqslant p$, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$ and the norm of the embedding does not exceed $1 + |E|$. In particular this implies that every function $u \in L^{p(\cdot)}(\Omega)$ also belongs to $L^{p_\Omega^-}(\Omega)$. The variable exponent Hölder inequality takes the form

$$\int_\Omega fg\, dx \leqslant 3\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

where p' is the pointwise conjugate exponent, $1/p(x) + 1/p'(x) \equiv 1$. Recall also the following fundamental relationship between norm and modular:

Lemma 2.1. *If* $u \in L^{p(\cdot)}(\Omega)$*, then* $\|u\|_{p(\cdot)} \leqslant 1$ *if and only if* $\varrho_{p(\cdot)}(u) \leqslant 1$.

For all the preceding results we refer to Kováčik & Rákosník [25].

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space

$W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

We define *the Sobolev space with zero boundary values*, $W_0^{1,p(\cdot)}(\Omega)$, as the closure of the set of compactly supported $W^{1,p(\cdot)}(\Omega)$-functions with respect to the norm $\|\cdot\|_{1,p(\cdot)}$. Note that we do not use smooth functions in this definition, since this class of functions is not always dense in the variable exponent Sobolev spaces [34].

Assume that p is bounded. We say that a function $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is a (*weak*) $p(\cdot)$-*supersolution* in Ω if

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx \geqslant 0 \tag{2.2}$$

for every non-negative test function $\varphi \in W_0^{1,p(\cdot)}(\Omega)$. A function u is a $p(\cdot)$-*subsolution* in Ω if $-u$ is a $p(\cdot)$-supersolution in Ω, and a $p(\cdot)$-*solution* in Ω if it is both a $p(\cdot)$-super- and a $p(\cdot)$-subsolution in Ω. A function $u \in W^{1,p(\cdot)}(\Omega)$ is a $p(\cdot)$-*minimizer* if

$$\int_\Omega |\nabla u|^{p(x)}\, dx \leqslant \int_\Omega |\nabla v|^{p(x)}\, dx$$

for every $v \in W^{1,p(\cdot)}(\Omega)$ with $u - v \in W_0^{1,p(\cdot)}(\Omega)$. If we assume that $u \in W^{1,p(\cdot)}(\Omega)$, then u is a $p(\cdot)$-minimizer if and only if it is a $p(\cdot)$-solution. For every $f \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ there exists a $p(\cdot)$-minimizer with $u - f \in W_0^{1,p(\cdot)}(\Omega)$ [24].

The bounded variable exponent p is said to be $\log$-*Hölder continuous* if there is a constant $L > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{L}{\log(e + 1/|x-y|)}$$

for all $x, y \in \Omega$. A bounded exponent p is $\log$-Hölder continuous in Ω if and only if there exists a constant $C > 0$ such that

$$|B|^{p_B^- - p_B^+} \leqslant C$$

for every ball $B \subset \Omega$ [7, Lemma 3.2].

Under the log-Hölder condition smooth functions are dense in variable exponent Sobolev spaces [7].

3. MINIMIZERS FOR $p \to 1$

In this section we give an overview of our recent study [21] of the $p(\cdot)$-harmonic functions in the case $p^- = 1$. In this case we have problems already with the existence of a solution with given boundary values. Even when a minimizer exists, it may be discontinuous, and so there can be no question of it satisfying Harnack's inequality. A critical feature which the limits $p \to 1$ and $p \to \infty$ have in common is that both lead us out of the realm of reflexive Sobolev spaces. In the case $p \to 1$, we need a new space with some crucial properties of the space $\mathrm{BV}(\Omega)$ of function of bounded variation.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Recall that the function $u \in L^1(\Omega)$ is of *bounded variation*, $u \in \mathrm{BV}(\Omega)$, if

$$\sup\{ \int_\Omega u \operatorname{div}\varphi\, dx : \varphi \in C_0^1(\Omega;\mathbb{R}^n),\ |\varphi| \leqslant 1 \} < \infty.$$

It is well-known that for $u \in \mathrm{BV}_{loc}(\Omega)$ there is a Radon measure μ on Ω and a μ-measurable function $\sigma\colon \Omega \to \mathbb{R}^n$ such that $|\sigma| = 1$ μ-almost everywhere and

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega \varphi \cdot \sigma \, d\mu$$

for every $\varphi \in C_0^1(\Omega; \mathbb{R}^n)$. The measure μ is called the *total variation measure* and it is denoted by $\|\nabla u\|$.

Assume that p is a bounded variable exponent and define $Y = \{ x \in \Omega : p(x) = 1 \}$. For $u \in \mathrm{BV}(\Omega) \cap W^{1,p(\cdot)}(\Omega \setminus Y)$ and a Borel set $E \subset \Omega$, we define a new modular by

$$\varrho_{\mathrm{BV}^{p(\cdot)}(E)}(u) := \|\nabla u\|(Y \cap E) + \varrho_{L^{p(\cdot)}(E \setminus Y)}(\nabla u).$$

The norm for functions in $L^1_{\mathrm{loc}}(\Omega)$ is given by a Luxemburg-type definition:

$$\|u\|_{\mathrm{BV}^{p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \inf\{\lambda > 0 : \varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u/\lambda) \leqslant 1\}.$$

With this norm we define the space $\mathrm{BV}^{p(\cdot)}(\Omega)$ to consist of those measurable functions $u\colon \Omega \to \mathbb{R}$ for which $\|u\|_{\mathrm{BV}^{p(\cdot)}(\Omega)} < \infty$. We denote $u \in \mathrm{BV}^{p(\cdot)}_{loc}(\Omega)$, if $u \in \mathrm{BV}^{p(\cdot)}(U)$ for every open set $U \subset\subset \Omega$.

We write $p^\lambda = \max(p, \lambda)$ for $\lambda > 1$. Assume that $f \in L^\infty(\Omega) \cap W^{1,p^\delta(\cdot)}(\Omega)$ for some $\delta > 1$. Then for any $\lambda \in (1, \delta)$ there exists a $p^\lambda(\cdot)$-solution with boundary value function f [24]. This unique solution is denoted by u^λ. By a $p(\cdot)$-solution with boundary value function f we mean a limit of such solutions.

The main results from [21] regarding this limit are contained in the following theorem. Here we say that p is *strongly log-Hölder continuous* in Ω if it is log-Hölder continuous in Ω and

$$\lim_{x \to y} \big|p(x) - 1\big| \log \frac{1}{|x - y|} = 0$$

for every $y \in Y$.

Theorem 3.1. *Let p be strongly* log*-Hölder continuous and bounded. Let $f \in W^{1,p^\delta(\cdot)}(\Omega) \cap L^\infty(\Omega)$ for some $\delta > 1$ and denote by u^λ a $p^\lambda(\cdot)$-solution in Ω with boundary value function f, for any $\lambda \in (1, \delta)$. Then there exists a function $u \in \mathrm{BV}^{p(\cdot)}_{loc}(\Omega) \cap L^\infty(\Omega)$ with the following properties.*

(i) *There exists a sequence $(\lambda_j) \to 1$ such that $u^{\lambda_j} \to u$ in $L^{p(\cdot)}(\Omega)$ and $u^{\lambda_j} \rightharpoonup u$ in $W^{1,p(\cdot)}_{loc}(\Omega \setminus Y)$.*
(ii) *u is a $p(\cdot)$-solution in $\Omega \setminus Y$.*
(iii) *for every $D \subset\subset \Omega$ the function u belongs to $W^{1,p(\cdot)}(D \setminus Y)$.*
(iv) *u is a local minimizer of the energy $\varrho_{\mathrm{BV}^{p(\cdot)}}$, i.e. for every compact $F \subset \Omega$ and every $v \in \mathrm{BV}^{p(\cdot)}_{loc}(\Omega)$ with $v = u$ in $\Omega \setminus F$ we have*

$$\varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u) \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(v).$$

The proofs of these results are based on moving back and forth between the solutions u^λ and regularized versions of the same. The crucial question is to keep control of the modular $\varrho_{\mathrm{BV}^{p(\cdot)}}$ in this process and arrive at the minimization property, (iv) in the previous theorem.

Estimates in one direction follow immediately from the weak lower semicontinuity of the modular. The much more difficult part relates to the upper semicontinuity. We restricted our consideration to mollified versions $v_\delta = v * \varphi_\delta$ of the function $v \in \mathrm{BV}^{p(\cdot)}(\Omega)$. The main tool in the proof of the previous theorem is the following approximation result:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded and let p be a bounded, strongly $\log$-Hölder continuous exponent in Ω. If $v \in \mathrm{BV}^{p(\cdot)}(\Omega)$ and $F \subset \Omega$ is closed, then*

$$\limsup_{\delta\to 0} \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(v_\delta) \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(v).$$

Fortunately, it turns out that we can skip these complicated approximation procedures in the limit $p \to \infty$ considered in the rest of this article. The reason for this difference is, essentially, that the space $W^{1,p^\lambda(\cdot)}(\Omega) \cap L^\infty(\Omega)$ decreases as $\lambda \to 1$, whereas in the case considered below we have an increasing sequence of spaces, and so we can consider the solution in any intermediate space of our choice. The details of existence of the limit function in the case $p \to \infty$ are given in the beginning of Section 6. On the other hand, the use of Caccioppoli type estimates is more problematic in the case $p \to \infty$ than it is in the case $p \to 1$.

Next, we turn to the one-dimensional case for some intuition on what is going on at the limit of the minimization problem.

4. The one-dimensional case

Consider minimizing

$$\varrho_{p(\cdot)}(u') = \int_{-1}^{1} |u'(x)|^{p(x)}\, dx \tag{4.1}$$

where u is absolutely continuous with boundary values a and b, $a < b$. If we assume that the minimizer u exists and is absolutely continuous, then one easily finds (by the Euler–Lagrange equation) that its derivative is of the form

$$u'(x) = \Big(\frac{c}{p(x)}\Big)^{\frac{1}{p(x)-1}}, \tag{4.2}$$

where the constant $c > 0$ is chosen such that $u(-1) = a$ and $u(1) = b$. The next theorem answers the question of existence.

Theorem 4.3 (Theorem 3.2, [19]). *Let $p\colon (-1,1) \to (1,\infty)$ be bounded. The minimization problem (4.1) with boundary values a and b, $a < b$, has a unique absolutely continuous minimizer if and only if there exists a constant $\tilde{c} \geqslant 1$ such that*

$$b - a \leqslant \int_{-1}^{1} \Big(\frac{\tilde{c}}{p(x)}\Big)^{\frac{1}{p(x)-1}} dx < \infty.$$

In this case the derivative of the minimizer is given by (4.2) for appropriate $c \in (0, \tilde{c}]$.

Using the previous expression for the derivative, we can easily prove the Harnack inequality. Example 3.10 from [22] shows that the constant in the Harnack inequality can not be independent of the solution u.

Theorem 4.4 (The Harnack inequality). *Let $1 < p^- \leqslant p^+ < \infty$ and $I \subset \mathbb{R}$ be an open interval. If $u \in W^{1,p(\cdot)}(I)$ is a minimizer with boundary values 0 and $a > 0$, then*

$$\sup_{y\in B(x,r)} u(y) \leqslant C(p^-, p^+, a) \inf_{y\in B(x,r)} u(y)$$

for every $x \in I$ and every r with $B(x, 2r) \subset I$.

Proof. We assume without loss of generality that $I = (-1, 1)$. We start by noting that $t^{-1/(t-1)}$ lies between e^{-1} and 1 for all $t > 1$. Using this and integrating (4.2) we obtain

$$\sup_{B(x,r)} u = \int_{-1}^{x+r} \Big(\frac{c}{p(y)}\Big)^{\frac{1}{p(y)-1}} dy \leqslant \int_{-1}^{x+r} c^{\frac{1}{p(y)-1}}\, dy \leqslant (x + r + 1) \max\Big\{c^{\frac{1}{p^+-1}}, c^{\frac{1}{p^--1}}\Big\}$$

and

$$\inf_{B(x,r)} u = \int_{-1}^{x-r} \Big(\frac{c}{p(y)}\Big)^{\frac{1}{p(y)-1}}\, dy \geqslant \frac{1}{e}\int_{-1}^{x-r} c^{\frac{1}{p(y)-1}}\, dy \geqslant \frac{1}{e}(x-r+1)\min\Big\{c^{\frac{1}{p^+-1}}, c^{\frac{1}{p^--1}}\Big\}$$

Taking into account that $\frac{x+r+1}{x-r+1} \leqslant 3$, we obtain

$$\frac{\sup_{B(x,r)} u}{\inf_{B(x,r)} u} \leqslant 3e\max\Big\{c^{\frac{1}{p^+-1}-\frac{1}{p^--1}}, c^{\frac{1}{p^--1}-\frac{1}{p^+-1}}\Big\}.$$

Here the constant c is from (4.2) and it depends on the boundary values of u and p. Hence c can be estimated in terms of p^-, p^+ and a. □

For the rest of this section we focus on one-dimensional examples which include an unbounded exponent. With these examples we are able to introduce some crucial ideas of this paper in an elementary context. Throughout the examples and the rest of the paper we denote $p_\lambda = \min(p,\lambda)$ for $\lambda > 1$.

Example 4.5. Let $p : (-1,1) \to \mathbb{R}$ be an exponent which is increasing and finite on $(-1,0)$, even on $(-1,1)$, with $p^- > 1$, and $\lim_{x\to 0} p(x) = \infty$. For every $\lambda < \infty$, we get a p_λ-minimizer of (4.1) with boundary values $-a$ and $a > 0$ by formula (4.2). We denote this minimizer by u_λ and the corresponding constant by c_λ. It is easy to see that c_λ remains bounded as $\lambda \to \infty$, since the energy of u_λ has to be smaller than or equal to the energy of $x \mapsto (4ax+3a)\chi_{(-1,-1/2]}(x) + a\chi_{(-1/2,1)}(x)$, which is given by

$$\int_{-1}^{-\frac{1}{2}} (4a)^{p_\lambda(x)}\, dx \leqslant (4a)^{p(-1/2)} + 1 < \infty. \tag{4.6}$$

Thus we can find a sequence $\lambda_i \to \infty$ such that $c_{\lambda_i} \to c \in [0,\infty)$.

Let i_0 be such that $c_{\lambda_i} \leqslant c+1$ for every $i \geqslant i_0$. Then for any $i \geqslant i_0$ we obtain

$$\Big(\frac{c_{\lambda_i}}{p_{\lambda_i}(t)}\Big)^{\frac{1}{p_{\lambda_i}-1}} \leqslant (c+1)^{\frac{1}{p^--1}}.$$

Since

$$\lim_{i\to\infty}\Big(\frac{c_{\lambda_i}}{p_{\lambda_i}(t)}\Big)^{\frac{1}{p_{\lambda_i}-1}} = \Big(\frac{c}{p(t)}\Big)^{\frac{1}{p(t)-1}}$$

a.e. in $[-1,1]$ we find by dominated convergence that

$$u_{\lambda_i}(x) = \int_{-1}^{x}\Big(\frac{c_{\lambda_i}}{p_{\lambda_i}(t)}\Big)^{\frac{1}{p_{\lambda_i}(t)-1}} dt - a \to \int_{-1}^{x}\Big(\frac{c}{p(t)}\Big)^{\frac{1}{p(t)-1}} dt - a =: u_\infty(x)$$

as $i \to \infty$. It is clear that u_∞ is increasing and it attains the boundary values $\pm a$ at ± 1, respectively. It is also clear that $c > 0$ since otherwise u_∞ would be a constant function $-a$. See Figure 1 for examples.

We conclude this example by proving that u_∞ *minimizes the energy* (4.1) *among functions which are absolutely continuous on* $[-1,1]$ *and attain the boundary values* $\pm a$ *at* ± 1, *respectively.* To show this, assume to the contrary that v is an admissible function which has smaller $p(\cdot)$-energy than u_∞. By the dominated convergence theorem with $|v'|^{p(x)}+1$ as a majorant, we have

$$\int_{-1}^{1} |v'(x)|^{p_{\lambda_i}(x)} dx \to \int_{-1}^{1} |v'(x)|^{p(x)} dx.$$

Since the u'_{λ_i} are uniformly bounded, dominated convergence implies that

$$\varrho_{p(\cdot)}(u'_\infty) = \lim_{i\to\infty} \varrho_{p_{\lambda_i}(\cdot)}(u'_{\lambda_i}).$$

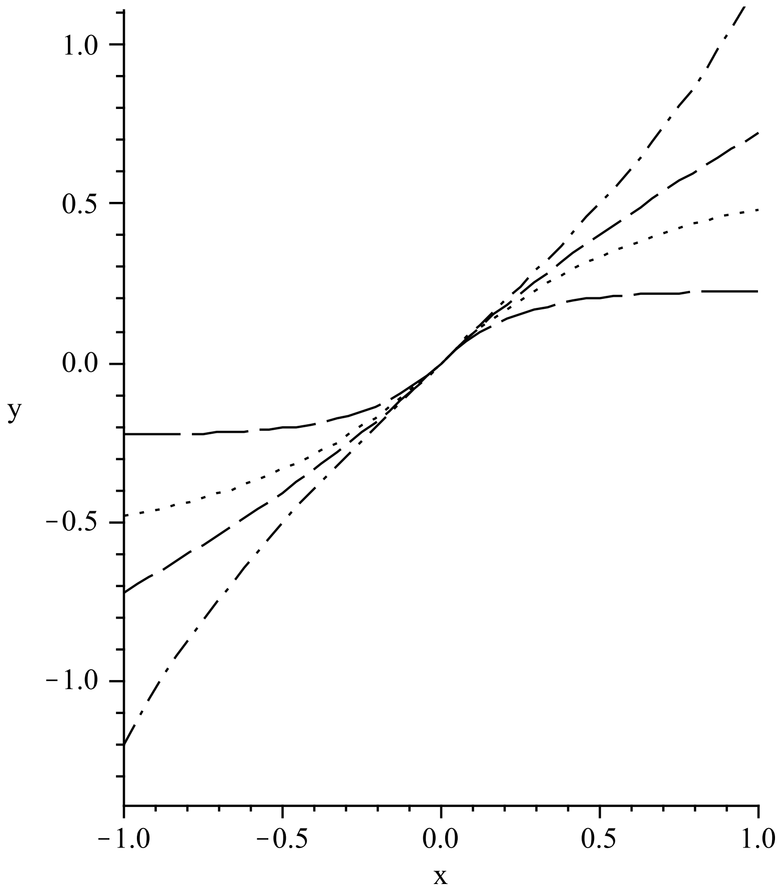

FIGURE 1. Four minimizers for $p(x) = \frac{3}{|x|}$ with different boundary values.

Since $\varrho_{p(\cdot)}(v') < \varrho_{p(\cdot)}(u'_\infty)$ we conclude that $\varrho_{p_{\lambda_i}(\cdot)}(v') < \varrho_{p_{\lambda_i}(\cdot)}(u'_{\lambda_i})$ for some i. This is a contradiction since each u_{λ_i} was assumed to be a $p_{\lambda_i}(\cdot)$-minimizer.

Example 4.7. Assume next that p is increasing and finite on $(-1, -\frac{1}{4})$, $p = \infty$ on $[-\frac{1}{4}, 0]$, $\lim_{x\to(-\frac{1}{4})^-} p(x) = \infty$ and p is even. As in Example 4.5 we see that there exists a limit constant $c \in [0, \infty)$ of the minimizer constants c_{λ_i}. If $c > 0$, then

$$\lim_{i\to\infty} \Big(\frac{c_{\lambda_i}}{p_{\lambda_i}(t)}\Big)^{\frac{1}{p_{\lambda_i}(t)-1}} = 1 \quad \text{for } t \in [-\tfrac{1}{4}, \tfrac{1}{4}],$$

and if $c = 0$, then the same limit belongs to $[0, 1]$. In both cases we have

$$\lim_{i\to\infty} \Big(\frac{c_{\lambda_i}}{p_{\lambda_i}(t)}\Big)^{\frac{1}{p_{\lambda_i}-1}} = \Big(\frac{c}{p(t)}\Big)^{\frac{1}{p(t)-1}}$$

a.e. in $[-1, 1] \setminus [-\frac{1}{4}, \frac{1}{4}]$. As above, dominated convergence implies that

$$\begin{aligned} u_{\lambda_i}(x) &= \int_{-1}^{x} \Big(\frac{c_{\lambda_i}}{p_{\lambda_i}(t)}\Big)^{\frac{1}{p_{\lambda_i}(t)-1}}\,dt - a \\ &\to \int_{-1}^{x} \lim_{i\to\infty} \Big(\frac{c_{\lambda_i}}{p_{\lambda_i}(t)}\Big)^{\frac{1}{p_{\lambda_i}(t)-1}}\,dt - a =: u_\infty(x). \end{aligned} \tag{4.8}$$

In particular,

$$u'_\infty(t) = \lim_{i\to\infty} \Big(\frac{c_{\lambda_i}}{p_{\lambda_i}(t)}\Big)^{\frac{1}{p_{\lambda_i}(t)-1}}$$

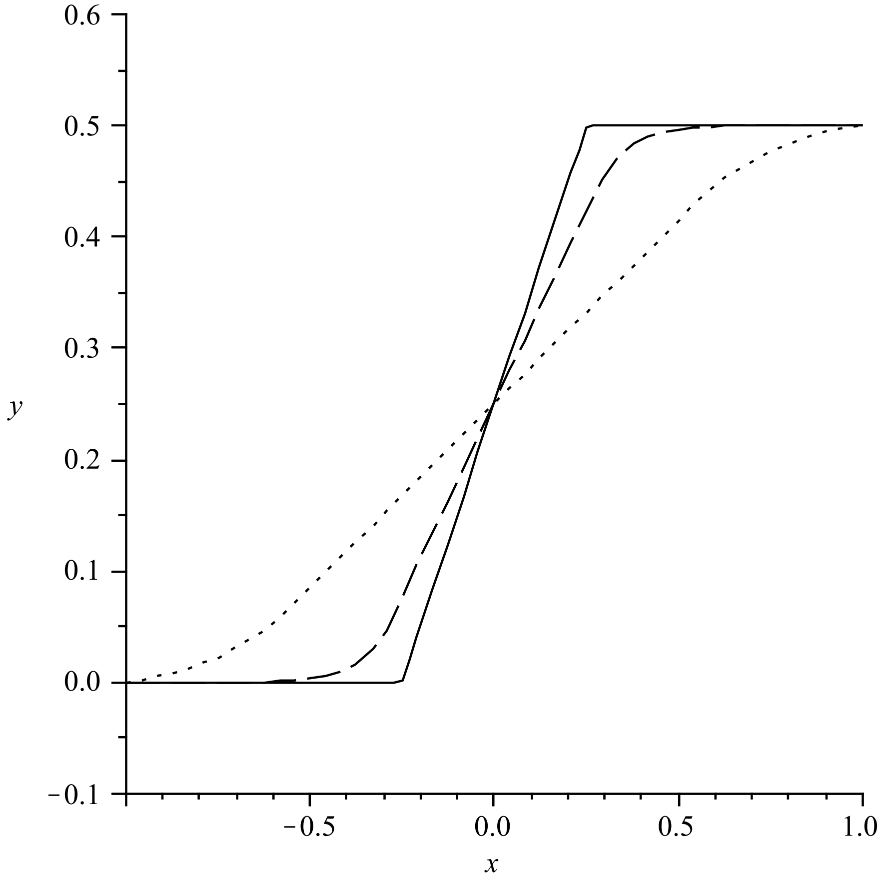

Figure 2. A limit function and two solutions with $\lambda = 10, 100$.

a.e. in $[-1, 1]$. By the arguments of Example 4.5 we infer that u_∞ minimizes the energy (4.1) among absolutely continuous functions v with boundary values $\pm a$ at ± 1, respectively. Notice that any such function v with finite energy necessarily satisfies $0 \leqslant v' \leqslant 1$ a.e. in $[-\frac{1}{4}, \frac{1}{4}]$.

Example 4.9. As our third example we consider the explicit exponent

$$p(x) = \begin{cases} \frac{3}{|x|-\frac{1}{4}}, & |x| > \frac{1}{4} \\ \infty, & |x| \leqslant \frac{1}{4} \end{cases}$$

and choose boundary values 0 and $\frac{1}{2}$. This exponent satisfies the requirements of Example 4.7. Suppose that $c > 0$. Then $u'_\infty > \chi_{[-\frac{1}{4}, \frac{1}{4}]}$, which is impossible, since the total mass of u'_∞ is $\frac{1}{2}$. Therefore $c = 0$ and $u'_\infty = 0$ a.e. in $[-1, 1] \setminus (-\frac{1}{4}, \frac{1}{4})$ from which it follows that $u'_\infty = 1$ a.e. in $(\frac{1}{4}, \frac{1}{4})$. Figure 2 presents the limit function u_∞ (line) and p_λ-solutions with λ equal to 10 (dot) and 100 (dash). The constants c_λ are approximately equal to 4.6×10^{-4} and 1.6×10^{-14}.

Note that the limit function u_∞ equals 0 on $(-1, -\frac{1}{4})$. Therefore, *Harnack's inequality does not hold for u_∞ in the form of Theorem 4.4.*

5. Bloch type estimates for bounded supersolutions

Recall that $\Omega \subset \mathbb{R}^n$ is a bounded open set. We want to prove estimates independent of p^+ for bounded supersolutions. For this purpose we assume throughout this section that

$1 < p^- \leqslant p^+ < \infty$ and that $1/p$ is Lipschitz continuous. In particular, $1/p$ is log-Hölder continuous, and we may estimate the norm of 1 over a ball by

$$\|1\|_{L^{p(\cdot)}(B)} \approx |B|^{1/p_B} \tag{5.1}$$

[8, Lemma 6.2]. Here p_B denotes the harmonic average of p,

$$p_B := \Big(\fint_B \frac{1}{p(x)}\, dx \Big)^{-1}.$$

The following Caccioppoli estimate is a modification of [22, Lemma 3.2]. The new feature in the estimate is the choice of a test function which includes the variable exponent. This has both advantages and disadvantages: we need to assume that p is differentiable almost everywhere, but, on the other hand, we avoid terms involving p^+, which would be impossible to control later.

Lemma 5.2 (Caccioppoli estimate)**.** *Let $p\colon \Omega \to (1,\infty)$ be an exponent with $1 < p^- \leqslant p^+ < \infty$ and such that $1/p$ is Lipschitz continuous. Let $u > \delta > 0$ be a supersolution in Ω and let $\eta\colon \Omega \to [0,1]$ be a Lipschitz function with compact support in Ω satisfying $\eta \log \frac{1}{\eta} \leqslant a\,|\nabla \eta|$ a.e. in $\{\eta > 0\}$ for some constant $a > 0$. Then for any $\gamma < 0$ we have*

$$\int_\Omega |\nabla u|^{p(x)} \eta^{p(x)} u^{\gamma-1}\, dx \leqslant \int_\Omega \Big(\tfrac{1}{|\gamma|}(p(x) + a|\nabla p(x)|)\Big)^{p(x)} u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)}\, dx.$$

Proof. We want to test u with the function $\psi := u^\gamma \eta^{p(\cdot)}$. Since η has compact support, it is enough to show that $\psi \in W^{1,p(\cdot)}(\Omega)$ in order for it to be a valid test function. Since $u^\gamma \eta^{p(\cdot)} \leqslant \delta^\gamma$, we obtain $\psi \in L^{p(\cdot)}(\Omega)$. For the derivative we have

$$\begin{aligned} |\nabla \psi| &\leqslant \big|\gamma u^{\gamma-1}\eta^p \nabla u + u^\gamma p \eta^{p-1}\nabla\eta + u^\gamma \eta^p \log \eta \nabla p\big| \\ &\leqslant |\gamma|\,\delta^{\gamma-1}|\nabla u|\,\eta^{p(x)} + p^+\delta^\gamma|\nabla\eta| + \delta^\gamma a|\nabla\eta|\,|\nabla p|. \end{aligned}$$

a.e. in $\{\eta > 0\}$. Since η has a compact support in Ω and $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ by the definition of being a supersolution, we see that $|\nabla\psi| \in L^{p(\cdot)}(\Omega)$.

Since u is a supersolution and ψ is a non-negative test function we find that

$$\begin{aligned} 0 &\leqslant \int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\psi\, dx \\ &= \int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \Big(\gamma u^{\gamma-1}\eta^{p(x)}\nabla u + u^\gamma p(x)\eta^{p(x)-1}\nabla\eta + u^\gamma\eta^{p(x)}\log\eta\,\nabla p(x)\Big)\, dx. \end{aligned}$$

Since $|\gamma| = -\gamma$ and $|\log\eta| = \log\frac{1}{\eta}$ we get

$$\begin{aligned} &\int_\Omega p(x)\,|\gamma|\,|\nabla u|^{p(x)}\eta^{p(x)}u^{\gamma-1}\, dx \\ &\quad\leqslant \int_\Omega p(x)^2|\nabla u|^{p(x)-1}u^\gamma\eta^{p(x)-1}|\nabla\eta| + p(x)|\nabla u|^{p(x)-1}u^\gamma\eta^{p(x)}\log\tfrac{1}{\eta}\,|\nabla p(x)|\, dx \\ &\quad\leqslant \int_\Omega p(x)(p(x) + a|\nabla p(x)|)|\nabla u|^{p(x)-1}u^\gamma\eta^{p(x)-1}|\nabla\eta|\, dx, \end{aligned} \tag{5.3}$$

where, in the second step, we used the assumption $\eta\log\frac{1}{\eta} \leqslant a\,|\nabla\eta|$.

We next apply the ε-version of Young's inequality,

$$fg \leqslant \Big(\frac{1}{\varepsilon}\Big)^{p-1}\frac{f^p}{p} + \varepsilon\frac{g^{p'}}{p'}$$

with $f = u^{\frac{\gamma+p(x)-1}{p(x)}}|\nabla\eta|$, $g = |\nabla u|^{p(x)-1}\eta^{\frac{p(x)}{p'(x)}}u^{\gamma-\frac{\gamma+p(x)-1}{p(x)}}$, and $\varepsilon(x) = \dfrac{|\gamma|}{p(x)+a|\nabla p(x)|}$. This gives

$$|\nabla u|^{p(x)-1}u^\gamma\eta^{p(x)-1}|\nabla\eta| \leqslant \tfrac{1}{p(x)}\varepsilon(x)^{1-p(x)}u^{\gamma+p(x)-1}|\nabla\eta|^{p(x)} + \tfrac{1}{p'(x)}\varepsilon(x)|\nabla u|^{p(x)}\eta^{p(x)}u^{\gamma-1}.$$

Using this in (5.3) implies that

$$\begin{aligned}&\int_\Omega p(x)\,|\gamma|\,|\nabla u|^{p(x)}\eta^{p(x)}u^{\gamma-1}\,dx\\ &\qquad\leqslant \int_\Omega |\gamma|\,\varepsilon(x)^{-p(x)}u^{\gamma+p(x)-1}|\nabla\eta|^{p(x)}\,dx + \int_\Omega |\gamma|\,(p(x)-1)|\nabla u|^{p(x)}\eta^{p(x)}u^{\gamma-1}\,dx.\end{aligned}$$

Dividing both sides by $|\gamma|$ and moving the second integral on the right hand side to the left hand side completes the proof. □

Lemma 5.4. *Let $p\colon \Omega \to (1,\infty)$ be an exponent with $1 < p^- \leqslant p^+ < \infty$ such that $1/p$ is Lipschitz continuous. Let u be a nonnegative supersolution in Ω and let $R \in (0,1]$. Assume further that $\eta\colon \Omega \to [0,R]$ is a Lipschitz function with support in a closed ball $\overline{B} \subset\subset \Omega$ of radius R and such that $\eta\log\frac{R}{\eta} \leqslant a\,|\nabla\eta|$ a.e. in $\{\eta > 0\}$ for some constant $a > 0$. Then*

$$\int_B \Big|\nabla\log(u+R)\Big|^{p(x)}\eta^{p(x)}\,dx \leqslant \int_B \Big(\tfrac{p(x)+a|\nabla p(x)|}{s-1}\Big)^{p(x)}(u+R)^{p(x)-s}R^{s-p(x)}|\nabla\eta|^{p(x)}\,dx,$$

where $s = p^-_B$.

Proof. Since $R \leqslant 1$ and $s = p^-_{B_R}$, we have

$$\begin{aligned}\int_B \Big|\nabla\log(u+R)\Big|^{p(x)}\eta^{p(x)}\,dx &= \int_B |\nabla u|^{p(x)}(u+R)^{-p(x)}\eta^{p(x)}\,dx\\ &\leqslant \int_B |\nabla u|^{p(x)}(u+R)^{-s}R^{s-p(x)}\eta^{p(x)}\,dx\\ &= R^s\int_B |\nabla u|^{p(x)}(u+R)^{-s}(\eta/R)^{p(x)}\,dx.\end{aligned}$$

Next we use Lemma 5.2 for the functions $u+R$ and η/R with $\gamma = 1-s$. This gives

$$\begin{aligned}\int_B \Big|\nabla\log(u+R)\Big|^{p(x)}\eta^{p(x)}\,dx &\leqslant R^s\int_B \Big(\tfrac{p(x)+a|\nabla p(x)|}{s-1}\Big)^{p(x)}(u+R)^{p(x)-s}|\tfrac{1}{R}\nabla\eta|^{p(x)}\,dx\\ &= \int_B \Big(\tfrac{p(x)+a|\nabla p(x)|}{s-1}\Big)^{p(x)}(u+R)^{p(x)-s}R^{s-p(x)}|\nabla\eta|^{p(x)}\,dx.\end{aligned}$$ □

Corollary 5.5. *Let $p\colon \Omega \to (1,\infty)$ be an exponent with $1 < p^- \leqslant p^+ < \infty$ such that $1/p$ is L-Lipschitz continuous. Let u be a nonnegative supersolution in Ω and let $B := B(x_0,R) \subset\subset \Omega$ with $R \leqslant 1$. Then*

$$\int_{\frac12 B} \Big|\nabla\log(u+R)\Big|^{p(x)}(\tfrac{\alpha}{2}R)^{p(x)}\,dx \leqslant \int_B \Big(\alpha\,\tfrac{p(x)-R\log\alpha|\nabla p(x)|}{s-1}\Big)^{p(x)}(u+R)^{p(x)-s}R^{s-p(x)}\,dx$$

for $0 < \alpha \leqslant \frac{1}{e}$, where $s = p^-_B$.

Proof. We choose $\eta(x) = \alpha(R-|x-x_0|)_+$, so that its support is $\overline{B}$. Since the function $t \mapsto t\log\frac{1}{t}$ is increasing on $(0,\frac{1}{e})$ and $\eta/R \leqslant \alpha$, we find that

$$\frac{\eta}{R}\log\frac{R}{\eta} \leqslant \alpha\log\frac{1}{\alpha} = |\nabla\eta|\log\frac{1}{\alpha}$$

in $\{\eta > 0\}$. Since $\eta \geqslant \frac{\alpha}{2}R$ in $B_{\frac12 R}$, the claim follows from Lemma 5.4 with $a = R\log\frac{1}{\alpha}$. □

We are now ready to present the Bloch-type estimate which we will see in the next section to imply the Harnack inequality.

Theorem 5.6. *Let $p\colon \Omega \to (n,\infty)$ be an exponent with $n < p^- \leqslant p^+ < \infty$ such that $1/p$ is L-Lipschitz continuous, and let u be a bounded non-negative supersolution in Ω. Then there exists a constant C which depends only on $\|u\|_\infty$, n and L such that if $B := B(y,R)$ for $y \in \Omega$ satisfies*

$$R \leqslant \min\Big\{\frac{1}{s\log s}, \frac{1}{4Ls}, \operatorname{dist}(y,\partial\Omega)\Big\}, \tag{5.7}$$

where $s = p_B^-$, then

$$\int_{\frac12 B} |\nabla \log(u+R)|^n \, dx \leqslant C.$$

Proof. In the proof, we denote by $C_{\log}$ any constant which depends only on $\log$-Hölder constant and dimension n. Note that (5.7) is surely satisfies whenever

$$R \leqslant \min\Big\{\frac{1}{p^+\log p^+}, \frac{1}{4Lp^+}, \operatorname{dist}(y,\partial\Omega)\Big\},$$

so for every point there exists some R which satisfies the condition. Set $B' := \frac12 B$ and $v := \log(u+R)$. By Hölder's inequality and (5.1) we obtain

$$\begin{aligned}\int_{B'} |\nabla v|^n \, dx &\leqslant 3\|1\|_{L^{(p(\cdot)/n)'}(B')}\| \, |\nabla v|^n\|_{L^{p(\cdot)/n}(B')} \\ &\approx |B|^{\frac{p_{B'}-n}{p_{B'}}}\|\nabla v\|^n_{L^{p(\cdot)}(B')} = \|R^{1-\frac{n}{p_{B'}}}\nabla v\|^n_{L^{p(\cdot)}(B')}.\end{aligned}$$

Since $1/p$ is $\log$-Hölder continuous, we have $R^{1/p(x)} \leqslant C_{\log} R^{1/p(y)}$ for all $x,y \in B'$ and therefore $R^{-n/p_{B'}} \leqslant C_{\log} R^{-n/p(x)}$ for all $x \in B'$.

We denote $\Lambda := \max\{\sup_B u + R, 1\}$ and obtain

$$\|R^{1-\frac{n}{p_{B'}}}\nabla v\|_{L^{p(\cdot)}(B')} \leqslant C_{\log}\tfrac{\Lambda}{\beta}\Big\|\tfrac{\beta}{2\Lambda}R^{\frac{p(\cdot)-n}{p(\cdot)}}\nabla v\Big\|_{L^{p(\cdot)}(B')} \tag{5.8}$$

for any $\beta > 0$. We will next show that $\beta > 0$ can be chosen depending on the parameters indicated in the statement of the theorem such that

$$\int_{B'} R^{p(x)-n}\Big(\tfrac{\beta}{2\Lambda}\Big)^{p(x)}|\nabla v(x)|^{p(x)} \, dx \leqslant 1. \tag{5.9}$$

Then it follows from Lemma 2.1 that the norm is also less than one, so (5.8) implies that

$$\|R^{1-n/p_{B'}}\nabla v\|_{L^{p(\cdot)}(B')} \leqslant C_{\log}\tfrac{\Lambda}{\beta}.$$

To find a suitable $\beta > 0$, we first apply Corollary 5.5 with $\alpha = \frac{\beta}{\Lambda}$ and obtain, for $\beta \leqslant \frac{\Lambda}{e}$, the inequality:

$$\begin{aligned}&\fint_{B'} (\tfrac{\beta R}{2\Lambda})^{p(x)}|\nabla v(x)|^{p(x)} \, dx \\ &\leqslant 2^n \fint_B \Big(\beta^{\frac{p(x)+R\log\left(\frac{\Lambda}{\beta}\right)|\nabla p(x)|}{s-1}}\Big)^{p(x)} (u+R)^{p(x)-s}R^{s-p(x)}\Lambda^{-p(x)} \, dx.\end{aligned} \tag{5.10}$$

By the definition of Λ we have $(u+R)^{p(x)-s}\Lambda^{-p(x)} \leqslant \Lambda^{-s}$. Since $1/p$ is L-Lipschitz continuous, we conclude that $p(x) \leqslant s + 2LRsp(x)$. Hence $R^{s-p(x)} \leqslant R^{-2LRsp(x)}$. Since the function $r \mapsto r\log\frac{1}{r}$ is increasing in $(0,e^{-1})$ and $R \leqslant (s\log s)^{-1}$, we conclude that

$$sR\log\tfrac{1}{R} \leqslant s(s\log s)^{-1}\log(s\log s) \leqslant 2.$$

Hence $R^{-2LRs} \leqslant e^{4L}$, and we conclude that

$$(u+R)^{p(x)-s}R^{s-p(x)}\Lambda^{-p(x)} \leqslant e^{4Lp(x)}\Lambda^{-s} \leqslant e^{4Lp(x)}. \tag{5.11}$$

The estimate $p(x) \leqslant s + 2LRsp(x)$, with the assumption on R, also implies that

$$p(x) \leqslant \frac{s}{1-2LRs} \leqslant 2s.$$

Taking into account that $|\nabla p(x)| \leqslant Lp(x)^2 \leqslant 4Ls^2$ and recalling that $\frac{\Lambda}{\beta} \geqslant e$ we obtain

$$p(x) + R\log(\tfrac{\Lambda}{\beta})|\nabla p(x)| \leqslant \log(\tfrac{\Lambda}{\beta})(2s+4LRs^2) \leqslant 3\log(\tfrac{\Lambda}{\beta})s \leqslant 6\log(\tfrac{\Lambda}{\beta})(s-1). \tag{5.12}$$

Here we have used the assumptions $R < (4Ls)^{-1}$ and $s \geqslant n \geqslant 2$.

Using estimates (5.11) and (5.12) in (5.10) gives

$$\fint_{B'} (\tfrac{\beta R}{2\Lambda})^{p(x)}|\nabla v(x)|^{p(x)}\,dx \leqslant 2^n \fint_B (6e^{4L}\beta\log(\tfrac{\Lambda}{\beta}))^{p(x)}\,dx \leqslant \fint_B (12e^{4L}\beta\log(\tfrac{\Lambda}{\beta}))^{p(x)}\,dx.$$

Finally, choosing β such that

$$12e^{4L}\beta\log(\tfrac{\Lambda}{\beta}) \leqslant 1,$$

we conclude that the right hand side of the previous inequality is at most one, which concludes the proof of (5.9). □

6. The limit function and the Harnack inequality

In this section we assume that $p\colon \Omega \to (n,\infty]$. For $\lambda > 1$ we write $p_\lambda = \min\{p,\lambda\}$. Assume that $f \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ with $\int_\Omega |\nabla f|^{p(x)}dx < \infty$. For $\lambda > 1$ let u_λ be the unique minimizer of

$$\int_\Omega |\nabla u|^{p_\lambda(x)}dx$$

in the set $\{u \in W^{1,p_\lambda(\cdot)}(\Omega) : u - f \in W_0^{1,p_\lambda(\cdot)}(\Omega)\}$, see [20, 24].

We will use the following notation:

$$t^{p(x)} = \begin{cases} t^{p(x)} & \text{for } 1 \leqslant p(x) < \infty;\\ 0 & \text{for } p(x) = \infty \text{ and } 0 < t < 1;\\ 1 & \text{for } p(x) = \infty \text{ and } t = 1;\\ \infty & \text{for } p(x) = \infty \text{ and } t > 1.\end{cases}$$

For every $\lambda > \lambda' > 1$ we obtain

$$\begin{aligned}\int_\Omega |\nabla u_\lambda|^{p_{\lambda'}(x)}dx &\leqslant \int_\Omega 1 + |\nabla u_\lambda|^{p_\lambda(x)}dx\\ &\leqslant |\Omega| + \int_\Omega |\nabla f|^{p_\lambda(x)}dx \leqslant 2|\Omega| + \int_\Omega |\nabla f|^{p(x)}dx.\end{aligned} \tag{6.1}$$

Thus $\varrho_{p_{\lambda'}}(\nabla u_\lambda)$ is uniformly bounded in λ, and it follows that $\|\nabla u_\lambda\|_{L^{p_{\lambda'}(\cdot)}(\Omega)}$ is also uniformly bounded.

Since $|u_\lambda| \leqslant \sup|f|$, we obtain that $\|u_\lambda\|_{1,p_{\lambda'}(\cdot)}$ is uniformly bounded. Fix a monotone sequence $\lambda'_k \to \infty$. Since $W^{1,p_{\lambda'_1}(\cdot)}(\Omega)$ is a reflexive Banach space, we find a sequence $\lambda_i \to \infty$ and a function u_∞ such that $u_{\lambda_i} \rightharpoonup u_\infty$ in $W^{1,p_{\lambda'_1}(\cdot)}(\Omega)$. For $k \geqslant 2$, $W^{1,p_{\lambda'_k}(\cdot)}(\Omega)$ is also a reflexive Banach space, so we can find a subsequence, denoted again by (u_{λ_i}) such that $u_{\lambda_{i_j}} \rightharpoonup u_\infty$ in $W^{1,p_{\lambda'_k}(\cdot)}(\Omega)$. Note that we have the same limit function, since we moved to a subsequence. By a diagonal process, we may assume that as $\lambda_i \to \infty$ we have $u_{\lambda_i} \rightharpoonup u_\infty$ in $W^{1,p_{\lambda'_k}(\cdot)}(\Omega)$ for every λ'_k and therefore

$$u_{\lambda_i} \rightharpoonup u_\infty \text{ in } W^{1,p_{\lambda'}(\cdot)}(\Omega) \text{ for all } \lambda'.$$

By the Rellich-Kondrachov compactness theorem, there exists a subsequence, denoted again by (u_{λ_i}), that converges to u_∞ locally uniformly in Ω.

Since the modular is weakly lower semicontinuous [21, Lemma 2.1] we obtain by (6.1) that

$$\int_\Omega |\nabla u_\infty|^{p_{\lambda'}(x)} dx \leqslant \liminf_{i\to\infty} \int_\Omega |\nabla u_{\lambda_i}|^{p_{\lambda'}(x)} dx \leqslant 2|\Omega| + \int_\Omega |\nabla f|^{p(x)} dx.$$

Since $|\nabla u_\infty(x)|^{p_{\lambda'}(x)} \to |\nabla u_\infty(x)|^{p(x)}$ for almost every x as $\lambda' \to \infty$, Fatou's Lemma implies that

$$\int_\Omega |\nabla u_\infty|^{p(x)}\, dx \leqslant 2|\Omega| + \int_\Omega |\nabla f|^{p(x)}\, dx.$$

It follows that $|\nabla u_\infty(x)| \leqslant 1$ for almost every $x \in \Omega_\infty$.

We collect these results in the following theorem.

Theorem 6.2. *Let $p\colon \Omega \to (n,\infty]$. Assume that $f \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ with $\int_\Omega |\nabla f|^{p(x)} dx < \infty$. Let u_λ be the Dirichlet p_λ-energy minimizer for the boundary value function f. Then there exist a sequence (λ_i) converging to infinity and a function $u_\infty \in W^{1,p(\cdot)}(\Omega)$ such that (u_{λ_i}) converges locally uniformly to u_∞ in Ω. Moreover, $\int_\Omega |\nabla u_\infty|^{p(x)}\, dx$ is finite and $|\nabla u_\infty| \leqslant 1$ almost everywhere in $\{p = \infty\}$.*

We call any function u_∞ from the previous theorem a $p(\cdot)$-harmonic function with boundary value function f.

We finish this paper with the Harnack inequality for the limit function u_∞ of a finite but potentially unbounded exponent. First we show how the Bloch estimate implies the Harnack inequality.

For the proof we need the concept of a weakly monotone function. Following [28] we say that $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is a $p(\cdot)$-*weakly monotone function* if for every pair (m, M) of real numbers, $m < M$, and every $D \subset\subset \Omega$ the conditions

$$(u - M)_+, (m - u)_+ \in W^{1,p(\cdot)}_0(D)$$

imply that $m \leqslant u \leqslant M$. As in [26, Lemma 2.7] one easily shows that every $p(\cdot)$-solution is $p(\cdot)$-weakly monotone. When $p^- > n$ every $p(\cdot)$-solution is continuous and hence monotone in the sense of Lebesgue, i.e. it attains its extrema on the boundary of any compact set in its domain of definition.

Lemma 6.3. *Assume that $u\colon \Omega \to [0,\infty)$ is a continuous function that is monotone in the sense of Lebesgue. If $B(y, 2R) \subset \Omega$ and*

$$\fint_{B(y,2R)} |\nabla \log(u + R)|^n\, dx \leqslant A,$$

then u satisfies the Harnack inequality

$$\sup_{B(y,R)} (u + R) \leqslant C \inf_{B(y,R)} (u + R)$$

with constant depending only on A and n.

Proof. Denote $v = \log(u + R)$ and observe that v is monotone, since u is monotone. Since $v \in W^{1,n}(B(y, 2R))$ it is also n-weakly monotone and thus by the proof of [28], Theorem 1, we have the oscillation estimate

$$(\operatorname{osc}_{B(y,R)} v)^n \leqslant C \int_{B(y,2R)} |\nabla v|^n \, dx \leqslant CA.$$

By exponentiating this we conclude the Harnack inequality

$$\sup_{B(y,R)} (u + R) \leqslant \exp((CA)^{\frac{1}{n}}) \inf_{B(y,R)} (u + R). \qquad \square$$

We next combine the previous lemma with the estimates from the previous section.

Theorem 6.4. *Let $p \colon \Omega \to (n, \infty)$ be such that $p^- > n$ and $1/p$ is L-Lipschitz continuous. Assume that $f \in W^{1,p(\cdot)}(\Omega)$ is a bounded non-negative function with $\int_\Omega |\nabla f|^{p(x)} dx < \infty$. Then any $p(\cdot)$-harmonic function u_∞ with boundary value function f satisfies the Harnack inequality*

$$\sup_{B(y,R)} (u_\infty + R) \leqslant C \inf_{B(y,R)} (u_\infty + R)$$

in balls centered at $y \in \Omega$ with radius

$$R < \frac{1}{16} \min \Big\{ \frac{1}{p(y) \log p(y)}, \frac{1}{L p(y)}, \operatorname{dist}(y, \partial\Omega)) \Big\}.$$

The constant C in the Harnack inequality depends only on $\|f\|_\infty$, n and L.

Proof. Each u_{λ_i} satisfies the Bloch-type estimate, Theorem 5.6, with the constant independent of λ_i. Thus by Lemma 6.3 we have

$$\sup_{B(y,R)} (u_{\lambda_i} + R) \leqslant C \inf_{B(y,R)} (u_{\lambda_i} + R),$$

where the constant is independent of λ_i. Since $u_{\lambda_i} \to u_\infty$ locally uniformly, the claim follows. $\square$

Note that the previous results apply only in the case when $p < \infty$ a.e. in Ω. However, we also get partial results for the general case by applying these results in the domain $\Omega \setminus \Omega_\infty$.

Remark 6.5. Moser's iteration and De Giorgi's method give the Harnack inequality

$$\sup_{B(y,R)} (u + R) \leqslant C \inf_{B(y,R)} (u + R)$$

for solutions and minimizers if p is $\log$-Hölder continuous with $1 < p^- \leqslant p^+ < \infty$. Here the constant depends on $\|u\|_{L^t(B(y,4R))}$ for some small $t > 0$. For the proofs see [4, 22, 23].

The method presented in this paper gives an alternative proof for the Harnack inequality for solutions if p is $\log$-Hölder continuous with $n < p^- \leqslant p^+ < \infty$. The constant depends on $\|u\|_{L^\infty(B(y,4R))}$, the $\log$-Hölder constant of p and p^+. We avoid the assumption that $1/p$ be Lipschitz by replacing the test function $u\eta^{p(\cdot)}$ by $u\eta^{p^+}$ in the Caccioppoli inequality, Lemma 5.2.

Remark 6.6. We conclude the paper by briefly commenting on some estimates that do not hold for solutions.

(a) Suppose that p is $\log$-Hölder continuous, $n < p^- \leqslant p^+ < \infty$, and every solution u satisfies an improved version of the Caccioppoli estimate:

$$\int_\Omega |\nabla u|^{p(x)} |u|^{-p(x)} \eta^{p(x)} \, dx \leqslant C \int_\Omega |\nabla \eta|^{p(x)} \, dx, \tag{6.7}$$

where the constant C does not depend on u. Then using the method given in Sections 5 and 6, we obtain

$$\sup_B u \leqslant C \inf_B u, \tag{6.8}$$

where C is independent of u. By [22, Example 3.10], this is not possible and thus (6.7) does NOT hold.

(b) Suppose next that $1/p$ is log-Hölder continuous, $p^- > n$ and each u_λ satisfies inequality (6.8), where the constant C is independent of p^+_λ (but may depend on $\|u\|_\infty$). Then the limit function u_∞ satisfies the inequality (6.8) as well. This is impossible by Example 4.9 and thus if inequality (6.8) holds, then the constant cannot be independent of p^+.

Acknowledgement

We would like to thank the referee for a meticulous reading and detailed corrections.

References

[1] E. Acerbi and G. Mingione: Regularity results for a class of functionals with nonstandard growth, *Arch. Ration. Mech. Anal.* **156** (2001), 121–140.

[2] E. Acerbi and G. Mingione: Regularity results for stationary electro-rheological fluids, *Arch. Ration. Mech. Anal.* **164** (2002), no. 3, 213–259.

[3] E. Acerbi and G. Mingione: Gradient estimates for the $p(x)$-Laplacean system, *J. Reine Angew. Math.* **584** (2005), 117–148.

[4] Yu. Alkhutov: The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition, *Differ. Uravn.* **33** (1997), no. 12, 1651–1660, 1726. (Russian) [*Differ. Equ.* **33** (1997), no. 12, 1653–1663 (1998).]

[5] Yu. Alkhutov and O. Krasheninnikova: Continuity at boundary points of solutions of quasilinear elliptic equations with a nonstandard growth condition, *Izv. Ross. Akad. Nauk Ser. Mat.* **68** (2004), no. 6, 3–60. (Russian) [Translation in *Izv. Math.* **68** (2004), no. 6, 1063–1117.]

[6] Y. Chen, S. Levine and M. Rao: Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* **66** (2006), no. 4, 1383–1406.

[7] L. Diening: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Math. Inequal. Appl.* **7** (2004), no. 2, 245–253.

[8] L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta and T. Shimomura: Maximal functions in variable exponent spaces: limiting cases of the exponent, Preprint (2007).

[9] L. Diening, P. Hästö and A. Nekvinda: Open problems in variable exponent Lebesgue and Sobolev spaces, *FSDONA04 Proceedings* (P. Drabek and J. Rakosnik (eds.); Milovy, Czech Republic, 2004), 38–58.

[10] D. Edmunds, J. Lang and A. Nekvinda: On $L^{p(x)}$ norms, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **455** (1999), no. 1981, 219–225.

[11] D. Edmunds and J. Rákosník: Sobolev embeddings with variable exponent, II, *Math. Nachr.* **246/247** (2002), 53–67.

[12] X.-L. Fan: Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form, *J. Differential Equations* **235** (2007), no. 2, 397–417.

[13] X.-L. Fan and D. Zhao: A class of De Giorgi type and Hölder continuity, *Nonlinear Anal.* **36** (1999), 295–318.

[14] X.-L. Fan and D. Zhao: On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424–446.

[15] X.-L. Fan, Q. Zhang and Y. Zhao: A strong maximum principle for $p(x)$-Laplace equations. (Chinese) *Chinese Ann. Math. Ser. A* **24** (2003), no. 4, 495–500. [Translation in *Chinese J. Contemp. Math.* **24** (2003), no. 3, 277–282.]

[16] T. Futamura and Y. Mizuta: Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent, *Math. Inequal. Appl.* **8** (2005), no. 4, 619–631.

[17] T. Futamura an Y. Mizuta: Maximal functions for Lebesgue spaces with variable exponent approaching 1, *Hiroshima Math. J.* **36** (2006), no. 1, 23–28.
[18] P. Harjulehto and P. Hästö: Sobolev inequalities for variable exponents attaining the values 1 and n, *Publ. Mat.*, to appear.
[19] P. Harjulehto, P. Hästö and M. Koskenoja: The Dirichlet energy integral on intervals in variable exponent Sobolev spaces, *Z. Anal. Anwendungen* **22** (2003), no. 4, 911–923.
[20] P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen: The Dirichlet energy integral and variable exponent Sobolev spaces, *Potential Anal.* **25** (2006), no. 3, 205–222.
[21] P. Harjulehto, P. Hästö and V. Latvala: Minimizers of the variable exponent, non-uniformly convex Dirichlet energy, *J. Math. Pures Appl. (9)* **89** (2008), no. 2, 174–197.
[22] P. Harjulehto, J. Kinnunen and T. Lukkari: Unbounded supersolutions of nonlinear equations with nonstandard growth, *Bound. Value Probl.* **2007**, Article ID 48348, 20 pages, doi:10.1155/2007/48348.
[23] P. Harjulehto, T. Kuusi, T. Lukkari, N. Marola and M. Parviainen: Harnack's inequality for quasiminimizers with non-standard growth conditions, *J. Math. Anal. Appl.* **344** (2008), no. 1, 504–520.
[24] P. Hästö: On the variable exponent Dirichlet energy integral, *Comm. Pure Appl. Anal.* **5** (2006), no. 3, 413–420.
[25] O. Kováčik and J. Rákosník: On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41(116)** (1991), 592–618.
[26] P. Koskela, J. J. Manfredi and E. Villamor: Regularity theory and traces of $\mathcal{A}$-harmonic functions, *Trans. Amer. Math. Soc.* **348**, no. 2, (1996), 755–766.
[27] V. Latvala: BMO-invariance of quasiminimizers, *Ann. Acad. Sci. Fenn. Math.* **29** (2004), 407–418.
[28] J. J. Manfredi: Weakly monotone functions, *J. Geom. Anal.* **4**, no. 2, (1994), 393–402.
[29] M. Mihăilescu and V. Rădulescu: On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.* **135** (2007), no. 9, 2929–2937.
[30] M. Růžička: *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin, 2000.
[31] M. Růžička: Modeling, mathematical and numerical analysis of electrorheological fluids, *Appl. Math.* **49** (2004), no. 6, 565–609.
[32] S. Samko: On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators, *Integral Transforms Spec. Funct.* **16** (2005), no. 5-6, 461–482.
[33] V. Zhikov: On some variational problems, *Russial J. Math. Physics* **5** (1997), 105–116.
[34] V. Zhikov: Density of smooth functions in Sobolev-Orlicz spaces, *J. Math. Sci. (N. Y.)* **132** (2006), no. 3, 285–294.

DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 68, FI-00014 UNIVERSITY OF HELSINKI, FINLAND.
E-mail address: `petteri.harjulehto@helsinki.fi`

DEPARTMENT OF MATHEMATICAL SCIENCES, P.O. BOX 3000, FI-90014 UNIVERSITY OF OULU, FINLAND
E-mail address: `peter.hasto@helsinki.fi`
URL: `http://cc.oulu.fi/~phasto/`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JOENSUU, P.O. BOX 111, FI-80101 JOENSUU, FINLAND.
E-mail address: `visa.latvala@joensuu.fi`