RESEARCH ARTICLE

Boundedness of solutions of the non-uniformly convex, non-standard growth Laplacian

P. Harjulehto[a] , P. Hästö[b,*] and V. Latvala[c]

[a] *Department of Mathematics and Statistics, P.O. Box 68, FI-00014 University of Helsinki, Finland*;
[b] *Department of Mathematical Sciences, P.O. Box 3000, FI-90014 University of Oulu, Finland*;
[a] *Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland*

We study the variable exponent $p(\cdot)$-Laplace equation

$$-\Delta_{p(\cdot)} u = 0$$

when p attains the value 1. We prove a removability result and local boundedness for its solutions and obtain a weak continuity result. Our results apply also to $(\mathcal{A}, \mathcal{B})$-solutions of $p(\cdot)$-growth.

AMS Subject Classification: 35J60 (35B20, 35J25)

Keywords: Non-standard growth, variable exponent, Laplace equation, Dirichlet energy, local boundedness, solution, Hausdorff measure, capacity

1. Introduction

In this paper we deal with variable exponent function spaces and PDE with non-standard growth. For background and references we refer to the surveys [12, 22, 35], the monograph [11] or the recent papers [8, 13, 25, 27, 32]. There have been hundreds of papers in this field in recent years, motivated in part by an application in modeling electrorheological fluids [3, 34] and a variable exponent formulation for the problem of image restoration [6]. The paradigmatic Dirichlet minimization problem,

$$\inf_u \int_\Omega |\nabla u|^{p(x)}\, dx$$

for $u - w \in W_0^{1,p(\cdot)}(\Omega)$, where $w \in W^{1,p(\cdot)}(\Omega)$ is the boundary value function, and the corresponding Euler-Lagrange equation

$$\operatorname{div}\big(p(x)|\nabla u|^{p(x)-2}\nabla u\big) = 0$$

This research was supported by the Academy of Finland, INTAS and the Emil Aaltonen Foundation.
*Corresponding author. Email: peter.hasto@helsinki.fi

have been studied e.g. in [2, 5, 7, 9, 16, 17, 36, 38, 40]. These investigations were limited to the case when p is bounded away from 1 and ∞.

Recently, we have studied the behavior of minimizers when p approaches 1 in [20] and when p approaches ∞ in [21]. The latter case has since been studied also by Lindqvist and Lukkari [28] and Manfredi, Rossi and Urbano [30, 31], but here we concern ourselves again with the limit $p \to 1$. Provided that the set where $p = 1$ is small, we prove that the BV-Sobolev solution obtained by a limiting argument in [20] is in fact in the Sobolev space, and is a weak solution in the normal sense. Then we prove some regularity results, including local boundedness in Section 4.

Let us now state our problem and results more precisely (for notation and further definitions, see Section 2). Let $1 \leqslant p^- \leqslant p^+ < \infty$ and let $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{B} : \Omega \times \mathbb{R}^n \to \mathbb{R}$, be Carathéodory functions, i.e. continuous in the first variable and measurable in the second one. We say that $\mathcal{A}$ and $\mathcal{B}$ have *$p(\cdot)$-type structure* if there exist constants $\Lambda \geqslant 1$ and $0 < \epsilon_b \leqslant 1$ such that

$$\begin{aligned} |\mathcal{A}(x,z)| &\leqslant \Lambda\,(1+|z|^{p(x)-1}),\\ \mathcal{A}(x,z)\cdot z &\geqslant \tfrac{1}{\Lambda}\,|z|^{p(x)},\\ |\mathcal{B}(x,z)| &\leqslant \Lambda\,\big(1+|z|^{p(x)-\epsilon_b}\big). \end{aligned}$$

We call a function $u \in W^{1,p(\cdot)}_{\rm loc}(\Omega)$ a *(weak)* $(\mathcal{A},\mathcal{B})$-*solution* in Ω, if

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi + \mathcal{B}(x,\nabla u)\varphi\,dx = 0 \qquad (\star)$$

for every test function $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ with a compact support on Ω. In the special case $\mathcal{A}(x,z) = p(x)\,|z|^{p(x)-2}z$ and $\mathcal{B} = 0$, we talk about *$p(\cdot)$-solutions.*

Our previous studies have been written for $p(\cdot)$-solutions and $p(\cdot)$-supersolutions, although it was clear that some extensions to the context of $(\mathcal{A},\mathcal{B})$-solutions are possible. In a recent paper [1] Adamovicz and Hästö argued that solutions to the equation with

$$\mathcal{A}(x,z) = |z|^{p(x)-2}z \quad\text{and}\quad \mathcal{B}(x,z) = |z|^{p(x)-2}\log|z|\,z\cdot\nabla p$$

are in many ways more similar to solutions of the classical p-Laplace equation than are $p(\cdot)$-solutions. The reason is that this choice of $\mathcal{B}$ is a generalization of the strong form of the equation $-\Delta_p u = 0$. See [1] for more details. In order to cover both the results from [20] and from [1], as well as other possible future generalizations, we state the next theorem for $(\mathcal{A},\mathcal{B})$-solutions. Thus it holds in particular in the cases considered in these two papers.

Theorem 1.1. *Assume that $p\colon \Omega \to [1,\infty)$ is bounded* log-*Hölder continuous such that $H^{n-1}(\{\,p=1\,\}) = 0$. If $u \in L^\infty(\Omega)$ is an $(\mathcal{A},\mathcal{B})$*-solution *in $\Omega \setminus \{\,p=1\,\}$, then u is an $(\mathcal{A},\mathcal{B})$*-solution *in Ω.*

In Section 4 we prove a boundedness result for solutions of our equation by modifying the Moser iteration scheme presented in [4]. For constant exponents $p > 1$ these ideas go back to [37]. Here we require the stronger, although natural, assumption

$$|\mathcal{B}(x,z)| \leqslant \Lambda\,\big(1+|z|^{p(x)-1}\big)$$

to make the proof work. Unfortunately, the equation from [1] is not covered in this case. We note that an attempt to use the De Giorgi method to prove boundedness of solutions of the equation from [1] also failed. Notice however that even the case $\mathcal{B} \equiv 0$ is interesting here since we allow the exponent p attain the value 1 in a possibly large set.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded, and let $p\colon \Omega \to [1, \infty)$ be a bounded* log-*Hölder continuous exponent. Let B be a ball with $4B \Subset \Omega$ and let u be an $(\mathcal{A}, \mathcal{B})$-solution in Ω with $|\mathcal{B}(x, z)| \leqslant \Lambda\,(1 + |z|^{p(x)-1})$. Assume that $s > p^+_{4B} - p^-_{4B}$. Then*

$$\operatorname*{ess\,sup}_{B} |u| \leqslant C\left(\left(\fint_{2B} |u|^t\,dx\right)^{\frac{1}{t}} + |B|^{\frac{1}{n}}\right)$$

for every $t > 0$, where the constant C depends only on n, Λ, p^+, $C_{\log}$, t, and the $L^{ns}(4B)$-norm of u.

Since the exponent p is uniformly continuous, we can take for example $s = p^-_\Omega/n$ by choosing B small enough. Thus the constants in the estimates are finite for all solutions u on a scale that depends only on p. From this result we obtain the following corollary.

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded, and let $p\colon \Omega \to [1, \infty)$ be a bounded* log-*Hölder continuous exponent. Then every $(\mathcal{A}, \mathcal{B})$-solution is continuous outside a set $E \subset \{\, p = 1\,\}$ with $\mathcal{H}^{n-1}(E) = 0$.*

Note here that even though we obtain only minimal regularity, this is the first result to give any regularity of solutions of $-\Delta_{p(\cdot)} u = 0$ in the case $p^- = 1$. This seems to be true even for the 1-Laplace equation.

2. Preliminaries

Conventions

The following notation will be used throughout the rest of this article, often without further mention. By $A \Subset B$ we mean that $\overline{A} \subset B$. By $\Omega \subset \mathbb{R}^n$ we denote a bounded open set. A bounded measurable function $p\colon \Omega \to [1, \infty)$ is called a *variable exponent*, and we denote for $A \subset \Omega$

$$p^+_A := \sup_{x\in A} p(x), \quad p^-_A := \inf_{x\in A} p(x), \quad p^+ := \sup_{x\in\Omega} p(x), \quad p^- := \inf_{x\in\Omega} p(x).$$

For $\lambda > 1$ we denote $p_\lambda(x) := \max\{\lambda, p(x)\}$. By Y (for "yksi", meaning one in Finnish) we always denote the set where p equals one, $Y := \{x \in \Omega : \ p(x) = 1\}$.

We write simply $A \lesssim B$ if there is a constant c such that $A \leqslant cB$. Here c depends only on Λ, p^+, n, and $C_{\log}$ if not otherwise stated. We also use the notation $A \approx B$ in a similar way.

Variable exponent spaces

The results of this section can be found in [11]; most were first proved in [26].

We define a *modular* by setting

$$\varrho_{L^{p(\cdot)}(\Omega)}(f) := \int_\Omega |f(x)|^{p(x)}\,dx.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions defined on Ω for which the modular is finite. The Luxemburg norm on this space is defined as

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\Big\{\lambda > 0 :\ \varrho_{L^{p(\cdot)}(\Omega)}\big(\tfrac{f}{\lambda}\big) \leqslant 1\Big\}.$$

In the case of norms and modulars taken over the whole set Ω we also use an abbreviated notation where $L^{p(\cdot)}(\Omega)$ in the subscript is replaced simply by $p(\cdot)$. Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For a constant function p it coincides with the standard Lebesgue space.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

If E is a measurable set with a finite measure, p and q are variable exponents satisfying $q \leqslant p$, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$. In particular this implies that every function $u \in W^{1,p(\cdot)}(\Omega)$ also belongs to $W^{1,p^-_\Omega}(\Omega)$. The variable exponent Hölder inequality takes the form

$$\int_\Omega fg\,dx \leqslant 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

$1/p(x) + 1/p'(x) \equiv 1$.

The variable exponent p is said to be log-*Hölder continuous* if there is a constant $C_{\log}$ such that

$$|p(x) - p(y)| \leqslant \frac{C_{\log}}{\log(e + 1/|x-y|)}$$

for all $x, y \in \Omega$. This condition was first used by Zhikov [39] and Fan [15] in the mid-nineties. A crucial fact is that this condition implies the local boundedness of the maximal operator. This was shown by Diening [10], who also noted that

$$|B|^{p^-_B} \approx |B|^{p^+_B}$$

if and only if p is log-Hölder continuous (note that we consider only the case of bounded domains). Under the log-Hölder condition smooth functions are dense in variable exponent Sobolev spaces (cf. [41]) and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$.

Capacity and Hausdorff measure

Recall that

$$C_0(\Omega) := \{u \in C(\Omega) :\ \operatorname{spt} u \Subset \Omega\},$$

Suppose that K is a compact subset of Ω. We denote

$$R_{p(\cdot)}(K,\Omega) := \{u \in W^{1,p(\cdot)}(\Omega) \cap C_0(\Omega) \colon u \geqslant 1 \text{ on } K\}$$

and define

$$\operatorname{cap}^*_{p(\cdot)}(K,\Omega) := \inf_{u \in R_{p(\cdot)}(K,\Omega)} \int_\Omega |\nabla u|^{p(x)}\,dx.$$

Further, if $U \subset \Omega$ is open, then

$$\operatorname{cap}_{p(\cdot)}(U,\Omega) := \sup_{\substack{K \subset U\\ \text{compact}}} \operatorname{cap}^*_{p(\cdot)}(K,\Omega),$$

and for an arbitrary set $E \subset \Omega$

$$\operatorname{cap}_{p(\cdot)}(E,\Omega) := \inf_{\substack{E \subset U \subset \Omega\\ U \text{ open}}} \operatorname{cap}_{p(\cdot)}(U,\Omega).$$

The number $\operatorname{cap}_{p(\cdot)}(E,\Omega)$ is called the *variational $p(\cdot)$-capacity of E (relative to Ω)*. If $p^+ < \infty$, then $\operatorname{cap}^*_{p(\cdot)}(K,\Omega) = \operatorname{cap}_{p(\cdot)}(K,\Omega)$ for every compact set $K \subset \Omega$ and relative $p(\cdot)$-capacity is an outer capacity, outer measure and Choquet capacity. For the proofs see [11, Section 9.2].

Our first result shows that the variational $p(\cdot)$-capacity and Hausdorff $(n-1)$-measure of the set $Y = \{x \in \Omega : \ p(x) = 1\}$ vanish simultaneously.

Proposition 2.1. *Let p be a bounded* log*-Hölder continuous exponent. Then*

$$\operatorname{cap}_{p(\cdot)}(Y,\Omega) \lesssim H^{n-1}(Y).$$

Moreover, $H^{n-1}(Y) = 0$ if and only if $\operatorname{cap}_{p(\cdot)}(Y,\Omega) = 0$.

Proof. We prove first the inequality of the statement under the assumption that $Y \Subset \Omega$. For any $\delta > 0$, we may cover the set Y by balls $B_i := B(x_i, r_i)$ such that $B_i \cap Y \neq \emptyset$ and $r_i < \min\{\delta, \operatorname{dist}(x_i, \partial\Omega)/2\}$ for every $i = 1, 2, \ldots$ Then

$$\operatorname{cap}_{p(\cdot)}(Y,\Omega) \leqslant \sum_{i=1}^{\infty} \operatorname{cap}_{p(\cdot)}(\overline{B_i},\Omega).$$

Let u_i be the Lipschitz function which equals 1 in $\overline{B}_i$, $2 - |x_i - y|/r_i$ for $y \in 2B_i \setminus \overline{B}_i$, and 0 elsewhere. Then u_i is a suitable test function for $\operatorname{cap}_{p(\cdot)}\big(\overline{B}_i,\Omega\big)$. We obtain by log-Hölder continuity that

$$\operatorname{cap}_{p(\cdot)}\big(\overline{B_i},\Omega\big) \leqslant \int_{2B_i} |\nabla u_i|^{p(x)}\,dx \approx r_i^{\,n-p^-_{2B_i}} \approx r_i^{n-1}.$$

Now the inequality of the proposition follows from the definition of Hausdorff-measure once we take the infimum over all possible covers and let $\delta \to 0$. The general case can be easily treated by considering closed subsets $Y_i = \{\, x \in Y : d(x,\partial\Omega) \geqslant \frac{1}{i} \,\}$ of Y and applying the first part of the proof. In particular, $\operatorname{cap}_{p(\cdot)}(Y,\Omega) = 0$ if $H^{n-1}(Y) = 0$.

Assume that $\operatorname{cap}_{p(\cdot)}(Y,\Omega)=0$. Since $L^{p(\cdot)}(\Omega)$ embeds continuously into $L^1(\Omega)$, we can easily infer that $\operatorname{cap}_1(Y,\Omega)=0$. Thus $H^{n-1}(Y)=0$ by [14, Theorem 3, p. 193]. □

3. Existence of solutions

Assume that p is a bounded variable exponent with $Y=\{x\in\Omega:\ p(x)=1\}\neq\emptyset$. Recall that $p_\lambda=\max\{p,\lambda\}$ for $\lambda>1$. Fix $\delta>1$ and a bounded boundary value function $f\in W^{1,p_\delta(\cdot)}(\Omega)$; then we can find a $p_\lambda(\cdot)$-solution u_λ for the boundary value function f by [24] for every $1<\lambda<\delta$. In [20, Proposition 6.1] we proved the following results concerning subsequences of (u_λ).

Lemma 3.1. *Let p be a bounded continuous exponent and let (λ_j) be a sequence decreasing to 1. Let (u_{λ_j}) be a sequence of $p_{\lambda_j}(\cdot)$-solutions in Ω with a bounded boundary value function $f\in W^{1,p_\delta(\cdot)}(\Omega)$ for some $\delta>1$. Then there exists a subsequence (λ_j) and $u\in L^\infty(\Omega)$ such that*

(1) $u_{\lambda_j}\to u$ in $L^{p_\delta(\cdot)}_{\rm loc}(\Omega)$ for $\delta\in[1,\frac{n}{n-1})$;
(2) $u_{\lambda_j}\rightharpoonup u$ in $W^{1,p(\cdot)}_{\rm loc}(\Omega\setminus Y)$;
(3) u is a $p(\cdot)$-solution in $\Omega\setminus Y$.

We show next that if the set Y is small enough then the function u from the previous lemma is a $p(\cdot)$-solution in the whole set Ω. Notice that Proposition 2.1 and the following theorem directly imply Theorem 1.1.

Theorem 3.2. *Assume that $p\colon\Omega\to[1,\infty)$ is bounded and uniformly continuous such that $\operatorname{cap}_{p(\cdot)}(Y,\Omega)=0$. If $u\in L^\infty(\Omega)$ is an $(\mathcal{A},\mathcal{B})$-solution in $\Omega\setminus Y$, then u is an $(\mathcal{A},\mathcal{B})$-solution in Ω.*

Proof. Let $U\Subset\Omega$ be open and write $Y_U:=Y\cap\overline{U}$. Since the relative capacity of Y is zero, so is that of Y_U. Because Y_U is compact we may choose a sequence (ψ_j) of functions in $W^{1,p(\cdot)}(\Omega)\cap C_0(\Omega)$, $0\leqslant\psi_j\leqslant 1$, such that $\psi_j=1$ in Y_U and $|\nabla\psi_j|\to 0$ in $L^{p(\cdot)}(\Omega)$. Since ψ_j has zero boundary values and p is continuous we obtain by the Poincaré inequality [19, Theorem 4.3] that

$$\|\psi_j\|_{L^{p(\cdot)}(\Omega)}\lesssim\|\nabla\psi_j\|_{L^{p(\cdot)}(\Omega)},$$

and thus $\psi_j\to 0$ in $L^{p(\cdot)}(\Omega)$. By taking a subsequence if necessary, we may assume that ψ_j and $|\nabla\psi_j|$ converge to zero almost everywhere. We write $\tilde\psi_j:=\min\{1,2\psi_j\}$. Now $\tilde\psi_j=1$ in an open set containing Y_U.

We show first that $u\in W^{1,p(\cdot)}(U\setminus Y)$. Here we follow the proof of [29, Lemma 2.115, p. 126]. Let $\eta\in C_0^\infty(\Omega)$ be a non-negative function with $0\leqslant\eta\leqslant 1$ that equals 1 on U and write $\eta_j:=(1-\tilde\psi_j)\eta$. Then $\eta_j^{p^+}u$ is a suitable test function of u, so that

$$\int_\Omega\mathcal{A}(x,\nabla u)\cdot\nabla(n_j^{p^+}u)+\mathcal{B}(x,\nabla u)n_j^{p^+}u\,dx=0.$$

We use the structural assumptions and estimate $|u|$ by $\|u\|_\infty$. This gives

$$\int_\Omega\eta_j^{p^+}|\nabla u|^{p(x)}\,dx\lesssim\int_\Omega(1+|\nabla u|^{p(x)-1})|\nabla\eta_j|\,\eta_j^{p^+-1}+(1+|\nabla u|^{p(x)-\epsilon_b})\eta_j^{p^+}\,dx,$$

where the constant in $\lesssim$ depends also on $\|u\|_\infty$. By Young's inequality and $\eta_j \leqslant 1$,

$$|\nabla u|^{p(x)-1}|\nabla \eta_j|\,\eta_j^{p^+-1} \leqslant \delta \eta_j^{p^+}|\nabla u|^{p(x)} + c_\delta |\nabla \eta_j|^{p(x)}$$

and

$$|\nabla u|^{p(x)-\epsilon_b}\eta_j^{p^+} \leqslant \delta |\nabla u|^{p(x)}\eta_j^{p^+} + c_\delta$$

for arbitrary $\delta > 0$. Hence we may absorb the terms with ∇u into the left-hand side and obtain

$$\int_\Omega \eta_j^{p^+}|\nabla u|^{p(x)}\,dx \lesssim \int_\Omega |\nabla \eta_j|\,\eta_j^{p^+-1} + |\nabla \eta_j|^{p(x)} + \eta_j^{p^+} + 1\,dx \lesssim \int_\Omega |\nabla \eta_j|^{p(x)} + 1\,dx.$$

The right-hand side is bounded with respect to j since η_j is a converging sequence in $W^{1,p(\cdot)}(\Omega)$. By Fatou's lemma, we conclude that $\eta^{p^+}|\nabla u|^{p(\cdot)} \in L^1(\Omega\setminus Y)$, so that $\nabla u \in L^{p(\cdot)}(U\setminus Y)$. Since u is bounded, we have $u \in W^{1,p(\cdot)}(U\setminus Y)$.

Then we follow [18, Theorem 6.2] and show that $u \in W^{1,p(\cdot)}(U)$. By the previous step of the proof, the functions $u_j := (1-\tilde\psi_j)u$ belong to $W^{1,p(\cdot)}(U\setminus Y_U)$. Moreover, $u_j = 0$ in an open neighborhood of Y_U, so it is clear that u_j can be extend by 0 to Y_U and we conclude that $u_j \in W^{1,p(\cdot)}(U)$. We easily calculate that

$$\begin{aligned}
&\varrho_{L^{p(\cdot)}(U)}(|\nabla(u_i-u_j)|)\\
&\quad\leqslant \int_U (|\tilde\psi_j| + |\tilde\psi_i|)^{p(x)}|\nabla u|^{p(x)}dx + \int_U (|\nabla\tilde\psi_j| + |\nabla\tilde\psi_i|)^{p(x)}|u|^{p(x)}dx\\
&\quad\lesssim \int_U (|\psi_j| + |\psi_i|)^{p(x)}|\nabla u|^{p(x)}dx + \left(\|u\|_\infty^{p^+}+1\right)\int_U |\nabla\psi_j|^{p(x)} + |\nabla\psi_i|^{p(x)}dx.
\end{aligned}$$

For the first integral on the right hand side we find by the theorem of dominated convergence, with $2^{p^+}|\nabla u|^{p(x)}$ as majorant, that it tends to zero as $i,j\to\infty$. The second integral on the right hand side tends to zero as $i,j\to\infty$ since $\nabla\psi_j \to 0$ in $L^{p(\cdot)}(\Omega)$. Since $|u_i - u_j| \leqslant (|\psi_j| + |\psi_i|)\operatorname{ess\,sup}_U |u|$ we conclude that $\|u_i - u_j\|_{L^{p(\cdot)}(U)} \to 0$ as $i,j\to\infty$. Thus (u_i) is a Cauchy sequence in $W^{1,p(\cdot)}(U)$. Since $W^{1,p(\cdot)}(U)$ is a Banach space, (u_i) converges to some function w in $W^{1,p(\cdot)}(U)$. By the point-wise convergence, $w = u$ a.e. in $U\setminus Y$, and so $w = u$ a.e. in U since Y has measure zero. Therefore $u \in W^{1,p(\cdot)}(U)$, and since $U \Subset \Omega$ was arbitrary, we obtain $u \in W^{1,p(\cdot)}_{\rm loc}(\Omega)$.

We conclude the proof by showing that u satisfies $(\star)$ for every $\varphi \in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ with compact support in Ω. Let $K := \operatorname{spt}\varphi \Subset \Omega$ and let U be an open set with $K \subset U \Subset \Omega$. Note that the previously defined functions ψ_j are also appropriate test functions for the capacity of Y_K.

Since $\tilde\psi_i = 1$ in an open set containing Y_K and thus we can test u with the function $\varphi(1-\tilde\psi_i)$, which has a compact support in $\Omega\setminus Y$. This yields

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi + \mathcal{B}(x,\nabla u)\varphi\,dx = \int_\Omega \mathcal{A}(x,\nabla u)\cdot(\tilde\psi_i\nabla\varphi + \varphi\nabla\tilde\psi_i) + \mathcal{B}(x,\nabla u)\varphi\tilde\psi_i\,dx.$$

Therefore we get by the structural assumptions on $\mathcal{A}$ and $\mathcal{B}$ that

$$\begin{aligned}
&\bigg| \int_\Omega \mathcal{A}(x, \nabla u)\cdot \nabla \varphi + \mathcal{B}(x,\nabla u)\varphi \, dx \bigg| \\
&\lesssim \int_U (1+|\nabla u|^{p(x)-1})\big(|\varphi|\,|\nabla \tilde\psi_i| + \tilde\psi_i\,|\nabla\varphi|\big) + (1+|\nabla u|^{p(x)})|\varphi|\,\tilde\psi_i \, dx \qquad (3.3)\\
&\lesssim \int_U |\nabla u|^{p(x)-1}(|\nabla \psi_i| + \psi_i |\nabla \varphi|) + \big(1 + |\nabla\varphi| + |\nabla u|^{p(x)}\big)\psi_i + |\nabla \psi_i| \, dx.
\end{aligned}$$

Here the constant in $\lesssim$ depends also on $\sup|\varphi|$. Since $\psi_i \to 0$ in $W^{1,1}(U)$, the last term tends to 0; since $1 + |\nabla\varphi| + |\nabla u|^{p(x)} \in L^1(U)$, the second term tends to 0 by dominated convergence. For the first term we use Hölder's inequality:

$$\begin{aligned}
&\int_U |\nabla u|^{p(x)-1}\big(|\nabla\tilde\psi_i| + \tilde\psi_i|\nabla\varphi|\big)\, dx \\
&\qquad \lesssim \big\| \, |\nabla u|^{p(\cdot)-1}\big\|_{L^{p'(\cdot)}(U)} \big(\|\nabla \psi_i\|_{L^{p(\cdot)}(U)} + \|\psi_i \nabla\varphi\|_{L^{p(\cdot)}(U)}\big).
\end{aligned}$$

To control the right hand side, we first note that

$$\varrho_{L^{p'(\cdot)}(U)}\big(|\nabla u|^{p(\cdot)-1}\big) = \varrho_{L^{p(\cdot)}(U)}(|\nabla u|) < \infty.$$

Thus the corresponding norm is bounded since $p^+ < \infty$. Since $\psi_i|\nabla\varphi| \lesssim |\nabla\varphi|$ and $\psi_i|\nabla\varphi| \to 0$ a.e., it follows by dominated convergence that $\|\psi_i\nabla\varphi\|_{L^{p(\cdot)}(U)} \to 0$ and $\|\psi_i\|_{W^{1,1}(U)} \to 0$. Thus the right hand side of (3.3) tends to zero, and since the left hand side is independent of i, we conclude that

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi + \mathcal{B}(x,\nabla u)\varphi\, dx = 0.$$

Hence u is a solution for $(\star)$ in Ω. □

4. Local boundedness

In this section we show that $(\mathcal{A},\mathcal{B})$-solutions are always locally bounded. Note no restriction on Y is needed in this section. We follow the proof of the case $p^- > 1$, which is by Yu. Alkhutov [4], and add some ideas from [23]. These proofs are based on Moser's iteration. Here we need to assume a stronger condition on $\mathcal{B}$ than in the previous section, namely, that

$$|\mathcal{B}(x,z)| \leqslant \Lambda\,(1 + |z|^{p(x)-1}), \qquad (4.1)$$

for some constant Λ.

Throughout the section, we fix a ball $B_R := B(x,R)$ with a radius $R \leqslant 1$ such that $B_{4R} := B(x,4R) \Subset \Omega$. For brevity, we write $v := \max\{u,0\} + R$, where u is a fixed solution of $(\star)$ in Ω with $\mathcal{B}$ satisfying (4.1). Additionally, we abbreviate $m := p^-_{B_{4R}}$ and $M := p^+_{B_{4R}}$ in the proofs.

Lemma 4.2. *Let* $p\colon \Omega \to [1,\infty)$ *be a bounded* log-*Hölder continuous exponent and* *let $R \leqslant \varrho < r \leqslant 3R \leqslant 3$. Then*

$$\left(\fint_{B_\varrho} v^{\beta n' p^-_{B_{4R}}} \, dx \right)^{\frac{1}{n'}} \leqslant C \beta^{p^-_{B_{4R}}} \left(\frac{r}{r-\varrho} \right)^{p^+_{B_{4R}}} \fint_{B_r} v^{(\beta-1)p^-_{B_{4R}} + p(x)} \, dx$$

for $\beta \geqslant 1$. The constant C depends only on n, p^+, Λ, and the log-*Hölder constant of p.*

Proof. Choose a non-negative $\eta \in C_0^\infty(B_r)$ with $\eta \leqslant 1$ to be specified below. Let G be a function on $[0, \infty)$ with $G'(t) := \beta t^{\beta-1}$. The function G_j is defined by the cut-off derivative, $G_j'(t) := \beta \min\{t, j\}^{\beta-1}$. Fixing the origin, we see that

$$G_j(t) = \begin{cases} t^\beta, & \text{for } 0 \leqslant t \leqslant j, \\ j^\beta + \beta j^{\beta-1}(t-j), & \text{for } t \geqslant j. \end{cases}$$

We further define

$$H_j(\xi) := \int_R^\xi G_j'(t)^m \, dt$$

for $\xi \geqslant R$.

First we show that $\psi := H_j(v)\eta^M$ is a suitable test function. Since η has compact support in Ω, it is enough to show that $\psi \in W^{1,p(\cdot)}(\Omega)$. Since

$$|H_j(v)| \leqslant \frac{\beta^m}{(\beta-1)m+1} j^{(\beta-1)m+1} + \beta^m j^{(\beta-1)m} v,$$

we find that $\psi \in L^{p(\cdot)}(\Omega)$. For the gradient we have

$$|\nabla \psi| \leqslant M\eta^{M-1} |\nabla \eta| H_j(v) + \eta^M G_j'(v)^m |\nabla v| \leqslant C(\eta) M H_j(v) + \eta^M (\beta j^{\beta-1})^m |\nabla v|,$$

and hence $|\nabla \psi| \in L^{p(\cdot)}(\Omega)$.

Since u is an $(\mathcal{A}, \mathcal{B})$-solution and ψ is an admissible test function, we have

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \psi + \mathcal{B}(x, \nabla u) \psi \, dx = 0.$$

Note that $\nabla v = 0$ and $H_j(v) = H_j(R) = 0$ whenever $u \leqslant 0$. If $u > 0$, then $\nabla v = \nabla u$. Hence $\mathcal{A}(x, \nabla u) \cdot \nabla \psi = \mathcal{A}(x, \nabla v) \cdot \nabla \psi$ and similarly for $\mathcal{B}$. Taking into account the structural conditions on $\mathcal{A}$ and $\mathcal{B}$, we obtain

$$\int_\Omega |\nabla v|^{p(x)} G_j'(v)^m \eta^M \, dx \lesssim \Lambda^2 \int_\Omega (1 + |\nabla v|^{p(x)-1}) H_j(v)(\eta + |\nabla \eta|) \eta^{M-1} \, dx,$$

where the constant depends only on p^+. Let us denote $\xi := \eta + |\nabla \eta|$.

We estimate the term involving ∇u on the right hand side with Young's inequality for the exponents $p(x)$ and $p'(x)$. For $p(x) > 1$ and $0 < \sigma \leqslant \frac{1}{2}$ this yields that

$$\begin{aligned} &H_j(v)|\nabla v|^{p(x)-1}\eta^{M-1}\xi \\ &\quad = G_j'(v)^{-\frac{m}{p'(x)}} H_j(v)\xi\eta^{M-\frac{M}{p'(x)}-1} \cdot G_j'(v)^{\frac{m}{p'(x)}} |\nabla v|^{p(x)-1}\eta^{\frac{M}{p'(x)}} \\ &\quad \leqslant c(\sigma) G_j'(v)^{-m(p(x)-1)} H_j(v)^{p(x)}\xi^{p(x)}\eta^{M-p(x)} + \sigma G_j'(v)^m |\nabla v|^{p(x)}\eta^M. \end{aligned}$$

If $p(x) = 1$, then the above inequality is trivial. By combining the previous inequalities with an appropriate choice of σ (depending only on Λ and p^+) and recalling $\eta \leqslant \chi_{B_r}$, we obtain that

$$\begin{aligned} \int_\Omega |\nabla v|^{p(x)} G_j'(v)^m \eta^M \, dx \leqslant c \int_{B_r} & G_j'(v)^{-m(p(x)-1)} H_j(v)^{p(x)} \xi^{p(x)} + H_j(v)\xi \, dx \\ & + \frac{1}{2} \int_\Omega G_j'(v)^m |\nabla v|^{p(x)} \eta^M \, dx. \end{aligned}$$

Then we absorb the second integral on the right hand side into the left hand side and use the trivial estimate $|\nabla v|^m \leqslant 1 + |\nabla v|^{p(x)}$ to derive

$$\begin{aligned} \int_\Omega |\nabla v|^m G_j'(v)^m \eta^M \, dx \lesssim \int_{B_r} & G_j'(v)^m + H_j(v)\xi \\ & + G_j'(v)^{-m(p(x)-1)} H_j(v)^{p(x)} \xi^{p(x)} \, dx. \end{aligned}$$

Since η vanishes outside B_r, we get

$$\begin{aligned} \fint_{B_r} |\nabla (G_j(v)\eta^{\frac{M}{m}})|^m \, dx &= \fint_{B_r} \Big| G_j(v) \frac{M}{m} \eta^{\frac{M}{m}-1} \nabla \eta + \eta^{\frac{M}{m}} G_j'(v) \nabla v \Big|^m \, dx \\ &\lesssim \fint_{B_r} G_j(v)^m |\nabla \eta|^m + G_j'(v)^m + H_j(v)\xi \\ &\qquad + G_j'(v)^{-m(p(x)-1)} H_j(v)^{p(x)} \xi^{p(x)} \, dx. \end{aligned}$$

Next we use the Sobolev–Poincaré inequality

$$\left(\fint_{B_r} \left(\frac{|w|}{R} \right)^{n'm} dx \right)^{\frac{1}{n'm}} \lesssim \left(\fint_{B_r} |\nabla w|^m \, dx \right)^{\frac{1}{m}}$$

for the function $w = G_j(v)\eta^{\frac{M}{m}} \in W_0^{1,m}(B_r)$. We obtain that

$$\begin{aligned} \left(\fint_{B_r} \left(\frac{G_j(v)\eta^{\frac{M}{m}}}{R} \right)^{n'm} dx \right)^{\frac{1}{n'}} &\lesssim \fint_{B_r} |\nabla (G_j(v)\eta^{\frac{M}{m}})|^m \, dx \\ &\lesssim \fint_{B_r} G_j(v)^m |\nabla \eta|^m + G_j'(v)^m + H_j(v)\xi \\ &\qquad + G_j'(v)^{-m(p(x)-1)} H_j(v)^{p(x)} \xi^{p(x)} \, dx. \end{aligned}$$

Let us derive an upper bound of the right hand side which is independent of j.

Clearly, $G_j \leqslant G$ and $G'_j \leqslant G'$. Further we estimate

$$H_j(x) = \int_R^x G'_j(t)^m \, dt \leqslant \int_R^x G'_j(x)^m \, dt \leqslant x G'_j(x)^m.$$

Thus

$$\begin{aligned} G'_j(v)^{-m(p(x)-1)} H_j(v)^{p(x)} &\leqslant G'_j(v)^{-m(p(x)-1)} (v G'_j(v)^m)^{p(x)} \\ &= G'_j(v)^m v^{p(x)} \leqslant G'(v)^m v^{p(x)}. \end{aligned}$$

Hence we obtain

$$\left(\fint_{B_r} \left(\frac{G_j(v) \eta^{\frac{M}{m}}}{R} \right)^{n'm} dx \right)^{\frac{1}{n'}} \lesssim \fint_{B_r} G(v)^m |\nabla \eta|^m + G'(v)^m \Big[1 + v\xi + v^{p(x)} \xi^{p(x)} \Big] \, dx.$$

Now the right hand side does not depend on j, and we may use monotone convergence on the left hand side, combined with the inequality $1 + v\xi + v^{p(x)}\xi^{p(x)} \lesssim 1 + v^{p(x)}\xi^{p(x)}$, to conclude that

$$\left(\fint_{B_r} \left(\frac{v^\beta \eta^{\frac{M}{m}}}{R} \right)^{n'm} dx \right)^{\frac{1}{n'}} \lesssim \fint_{B_r} v^{\beta m} |\nabla \eta|^m + \beta^m v^{(\beta-1)m} \Big[1 + v^{p(x)} \xi^{p(x)} \Big] \, dx.$$

We choose η such that $\eta = 1$ in B_ϱ, $0 \leqslant \eta \leqslant 1$ and $|\nabla \eta| \lesssim \frac{1}{r-\varrho}$. Thus $|\nabla \eta| \leqslant \xi \lesssim \frac{1}{R} \frac{r}{r-\varrho}$. Multiplying both sides of the inequality in the previous paragraph with R^m, we find that

$$\left(\fint_{B_r} \left(v^\beta \eta^{\frac{M}{m}} \right)^{n'm} dx \right)^{\frac{1}{n'}} \lesssim R^m \fint_{B_r} \beta^m v^{(\beta-1)m+p(x)} \Big[v^{m-p(x)} \xi^m + v^{-p(x)} + \xi^{p(x)} \Big] \, dx.$$

Since $v \geqslant R$, $v^{m-p(x)} \leqslant R^{m-p(x)}$ and $R^m v^{-p(x)} \leqslant R^{m-p(x)}$; note also that $R^{m-p(x)} \leqslant c$, by log-Hölder continuity. Since the measure of B_ϱ is comparable to the measure of B_r, we can change the average on the left hand side to the smaller ball:

$$\left(\fint_{B_\varrho} v^{\beta n'm} \, dx \right)^{\frac{1}{n'}} \lesssim \fint_{B_r} \beta^m v^{(\beta-1)m+p(x)} \Big[R^m \xi^m + 1 + R^m \xi^{p(x)} \Big] \, dx.$$

Finally, since $R^m \lesssim R^{p(x)}$ by log-Hölder continuity, and we conclude the proof by noting that $R\xi \lesssim \frac{r}{r-\varrho}$. □

In what follows, we write

$$\Phi(f, q, B_r) := \left(\fint_{B_r} f^q \, dx \right)^{1/q}$$

for a nonnegative measurable function f and $q \neq 0$.

Lemma 4.3. *Let $p\colon \Omega \to [1,\infty)$ be a bounded* log-*Hölder continuous exponent and let $R \leqslant \varrho < r \leqslant 3R \leqslant 3$. Then*

$$\Phi(v, n'\beta, B_\varrho) \leqslant C^{\frac{1}{\beta}} \beta^{\frac{p^-_{B_{4R}}}{\beta}} \left(\frac{r}{r-\varrho}\right)^{\frac{p^+_{B_{4R}}}{\beta}} \Phi(v, q\beta, B_r)$$

for every $\beta \geqslant p^-_{B_{4R}}$, $1 < q < n'$ and $s > p^+_{B_{4R}} - p^-_{B_{4R}}$. The constant C depends on n, Λ, p^+, $C_{\log}$, q and the $L^{q's}(B_{4R})$-norm of v.

Proof. Replacing β by β/m in Lemma 4.2 we obtain

$$\left(\fint_{B_\varrho} \left(v^{\frac{\beta}{m}}\right)^{n'm} dx\right)^{\frac{1}{n'\beta}} \leqslant \left(C\beta^m \left(\frac{r}{r-\varrho}\right)^M \fint_{B_r} v^{\left(\frac{\beta}{m}-1\right)m+p(x)}\, dx\right)^{\frac{1}{\beta}}.$$

This yields by Hölder's inequality and [23, Lemma 3.4] that

$$\begin{aligned}\left(\fint_{B_\varrho} v^{\beta n'}\, dx\right)^{\frac{1}{n'\beta}} &\leqslant C^{\frac{1}{\beta}} \beta^{\frac{m}{\beta}} \left(\frac{r}{r-\varrho}\right)^{\frac{M}{\beta}} \left(\fint_{B_r} v^{q'(p(x)-m)}\, dx\right)^{\frac{1}{\beta q'}} \left(\fint_{B_r} v^{\beta q}\, dx\right)^{\frac{1}{\beta q}} \\ &\leqslant C^{\frac{1}{\beta}} \beta^{\frac{m}{\beta}} \left(\frac{r}{r-\varrho}\right)^{\frac{M}{\beta}} \left(1 + \|v\|^{q'(M-m)}_{L^{q's}(B_{4R})}\right)^{\frac{1}{\beta q'}} \left(\fint_{B_r} v^{\beta q}\, dx\right)^{\frac{1}{\beta q}}.\end{aligned}$$

To conclude the claim we include the factor $\left(1+\|v\|^{q'(M-m)}_{L^{q's}(B_{4R})}\right)^{\frac{1}{q'}}$ in the constant. □

Proof of Theorem 1.2. By making s slightly smaller if necessary, we may assume that there exists $q \in (1, n')$ such that $\|u\|_{L^{q's}(B_{4R})} < \infty$. Let $R \leqslant \varrho < r \leqslant 3R$. For $j = 0, 1, 2, \ldots$, we denote

$$r_j := \varrho + 2^{-j}(r-\varrho) \quad \text{and} \quad \xi_j := \left(\frac{n'}{q}\right)^j qm.$$

By Lemma 4.3 with $\beta = \frac{\xi_j}{q}$ we obtain

$$\Phi(v, \xi_{j+1}, B_{r_{j+1}}) \leqslant C^{\frac{1}{\xi_j}} \xi_j^{\frac{qm}{\xi_j}} \left(\frac{r_j}{r_j - r_{j+1}}\right)^{\frac{qM}{\xi_j}} \Phi(v, \xi_j, B_{r_j}).$$

Iterating and letting $j \to \infty$ we find that

$$\begin{aligned}\operatorname*{ess\,sup}_{B_\varrho} |v| &= \lim_{j\to\infty} \Phi(v, \xi_{j+1}, B_{r_{j+1}}) \\ &\leqslant \prod_{j=0}^{\infty} C^{\frac{1}{\xi_j}} \xi_j^{\frac{qm}{\xi_j}} \left(2^j \frac{r}{r-\varrho}\right)^{\frac{qM}{\xi_j}} \Phi(v, \xi_0, B_{r_0}) \\ &\lesssim (2n')^{qM\sum\frac{j}{\xi_j}} \left(\frac{r}{r-\varrho}\right)^{qM\sum\frac{1}{\xi_j}} \Phi(v, qm, B_r).\end{aligned}$$

By the root test the sums in the previous estimate are finite and hence

$$\operatorname*{ess\,sup}_{B_\varrho} |v| \lesssim \big(1 - \tfrac{\varrho}{r}\big)^{-\frac{\lambda}{s}} \Phi(v, s, B_r),$$

where $\lambda := \frac{Mn'q}{(n'-q)}$ and $s := qm$. By Hölder's inequality we see that $\Phi(v, s, B_r) \leqslant \Phi(v, t, B_r)$ when $t \geqslant s$, so the claim is clear in this case.

We then consider $t < s$. Set $\sigma := \frac{\varrho}{r} \in (\frac{1}{3}, 1)$. Let us show that

$$\operatorname*{ess\,sup}_{B_{\sigma r}} |v| \lesssim (1 - \sigma)^{-\frac{\lambda}{t}} \Phi(v, t, B_r),$$

for any $t \in (0, s)$. We adapt the argument of [29, Corollary 3.10]. Denote

$$M(\sigma) := \operatorname*{ess\,sup}_{B_{\sigma r}} |v| \quad \text{and} \quad S(\sigma) := (1 - \sigma)^{\frac{\lambda}{t} - \frac{\lambda}{s}} \Phi(v, s, B_{\sigma r}).$$

Set $\sigma' := \frac{1+\sigma}{2}$. We rewrite the conclusion of the previous paragraph as

$$M(\sigma) \lesssim \big(1 - \tfrac{\sigma}{\sigma'}\big)^{-\frac{\lambda}{s}} \Phi(v, s, B_{\sigma' r}) \approx (1 - \sigma)^{-\frac{\lambda}{s}} \Phi(v, s, B_{\sigma' r}).$$

Since $1 - \sigma' = \frac{1-\sigma}{2}$, we further obtain that

$$M(\sigma) \lesssim (1 - \sigma)^{-\frac{\lambda}{t}} S(\sigma'). \tag{4.4}$$

Using this in the second step, we estimate

$$\bigg(\fint_{B_{\sigma r}} v^s \, dx\bigg)^{\frac{1}{s}} \leqslant \bigg(M(\sigma)^{s-t} \fint_{B_{\sigma r}} v^t \, dx\bigg)^{\frac{1}{s}} \lesssim (1 - \sigma)^{\frac{\lambda}{s} - \frac{\lambda}{t}} S(\sigma')^{1-\frac{t}{s}} \bigg(\fint_{B_{\sigma r}} v^t \, dx\bigg)^{\frac{1}{s}}.$$

Dividing both sides by $(1 - \sigma)^{\frac{\lambda}{s} - \frac{\lambda}{t}}$, we obtain

$$S(\sigma) \lesssim S(\sigma')^{1-\frac{t}{s}} \bigg(\fint_{B_{\sigma r}} v^t \, dx\bigg)^{\frac{1}{s}} \lesssim S(\sigma')^{1-\frac{t}{s}} \bigg(\fint_{B_r} v^t \, dx\bigg)^{\frac{1}{s}},$$

where we used $\sigma \approx 1$ in the second step. We start the iteration by recalling the estimate in the form

$$S(\sigma') \lesssim S(\sigma'')^{1-\frac{t}{s}} \bigg(\fint_{B_r} v^t \, dx\bigg)^{\frac{1}{s}}$$

for $\sigma'' = \frac{1+\sigma'}{2}$. At the limit we find that

$$S(\sigma) \lesssim \bigg(\fint_{B_r} v^t \, dx\bigg)^{\frac{1}{s} \sum (1-\frac{t}{s})^j} = \bigg(\fint_{B_r} v^t \, dx\bigg)^{\frac{1}{t}}.$$

We combine this with (4.4) for the claim for the function $v = \max\{u, 0\} + R$.

The same estimate holds also for $-\min\{u, 0\} + R$, since $-u$ is a solution. Thus the claim follows. □

Proof of Corollary 1.3. For $x_0 \in \Omega$ denote $B_r := B(x_0, r)$. If $p(x_0) > 1$, then we can choose r so small that $p^-_{B_r} > 1$. Since u is in particular a solution in B_r, we obtain by well known results the continuity at x_0, e.g. [16]. So it remains to consider the case $x_0 \in Y$.

We define a representative of u such that the essential oscillation equals the oscillation in B_r. By Theorem 1.2 and the Poincaré inequality,

$$\operatorname*{osc}_{B_r} u \leqslant 2 \sup_{B_r} |u - u_{2B_r}| \lesssim \fint_{2B_r} |u - u_{2B_r}|\, dx + r \lesssim r \fint_{2B_r} |\nabla u|\, dx + r.$$

The claim follows by showing that

$$\limsup_{r \to 0} r \fint_{B_r} |\nabla u|\, dx = 0$$

$\mathcal{H}^{n-1}$-a.e., but since $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, this is well known. □

Acknowledgement

We would like to thank the referee's for their comments, especially for noticing the short-coming in the original proof of Theorem 3.2.

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