

UNIFORM CONVEXITY AND ASSOCIATE SPACES

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(Received September 14, 2017)

Abstract. In this note, we prove that the associate space of a generalized Orlicz space $L^{\varphi(\cdot)}$ is given by the conjugate modular φ^* even without the assumption that simple functions belong to the space. Second, we show that every weakly doubling Φ -function is equivalent to a doubling Φ -function. As a consequence, we conclude that $L^{\varphi(\cdot)}$ is uniformly convex if φ and φ^* are weakly doubling.

MSC 2010: 46E30, 46A25

Keywords: Generalized Orlicz space, Musielak–Orlicz spaces, nonstandard growth, variable exponent, double phase, uniform convexity, associate space

1. INTRODUCTION

Generalized Orlicz spaces $L^{\varphi(\cdot)}$ have been studied since the 1940s. A major synthesis of functional analysis in these spaces, based on work, e.g., by Hudzik, Kamińska and Musielak, is given in the monograph [16]. Following ideas by Maeda, Mizuta, Ohno and Shimomura (e.g. [15]) we have studied these spaces from a point-of-view which emphasizes the possibility of choosing the Φ -function generating the norm in the space appropriately [5, 9, 10, 12]. From this perspective, some classical concepts, like convexity of the Φ -function, are too rigid.

Renewed interest in the topic has arisen recently from studies of PDE with non-standard growth, including the variable exponent case $\varphi(x, t) = t^{p(x)}$ and the double phase case $\varphi(x, t) = t^p + a(x)t^q$. Such problems have been studied e.g. in [2, 3, 4, 8, 17]. For a detailed motivation of our context and additional references, we refer to the introduction of [11].

In this note, we tie up some loose ends concerning the basic functional analysis of generalized Orlicz spaces in our monograph [6]. In the book we relied on the assumption that all simple functions belong to our space. This excludes for instance the case $\varphi(x, t) := |x|^{-n}t^2$, where n is the dimension. We can now remove this

assumption from the following result (cf. [6, Theorem 2.7.4]). For simplicity, we consider only the Lebesgue measure on subsets of \mathbb{R}^n . See the next sections for definitions.

Theorem 1.1. *Let $A \subset \mathbb{R}^n$ be measurable. If $\varphi \in \Phi_w(A)$, then $(L^\varphi)' = L^{\varphi^*}$, i.e. for all measurable $f : A \rightarrow \mathbb{R}$*

$$\|f\|_{\varphi(\cdot)} \approx \sup_{\|g\|_{\varphi^*(\cdot)} \leq 1} \int_A |f(x)g(x)| dx.$$

The proof relies among other things on upgrading the weak Φ -function to a strong Φ -function based on our earlier work. The next result is of the same type, upgrading weak doubling to strong doubling.

Theorem 1.2. *Let $A \subset \mathbb{R}^n$ be measurable. If $\varphi \in \Phi_w(A)$ satisfies Δ_2^w and ∇_2^w , then there exists $\psi \in \Phi_w(A)$ with $\varphi \sim \psi$ satisfying Δ_2 and ∇_2 .*

Recall that a vector space X is *uniformly convex* if it has a norm $\|\cdot\|$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ with

$$\|x - y\| \geq \varepsilon \quad \text{or} \quad \|x + y\| \leq 2(1 - \delta)$$

for all unit vectors x and y . In the Orlicz case, it is well known that the space L^φ is reflexive and uniformly convex if and only if φ and φ^* are doubling [18, Theorem 2, p. 297]. Hudzik [13] showed in 1983 that the same conditions are sufficient for uniform convexity (see also [7, 14]). With the equivalence technique, we are able to give a very simple proof of this result.

Theorem 1.3. *Let $A \subset \mathbb{R}^n$ be measurable and $\varphi \in \Phi_w(A)$. If φ satisfies Δ_2^w and ∇_2^w , then $L^{\psi(\cdot)}$ is uniformly convex and reflexive.*

2. Φ -FUNCTIONS

By $A \subset \mathbb{R}^n$ we denote a measurable set. The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq Cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. By c we denote a generic constant whose value may change between appearances. A function f is *almost increasing* if there exists a constant $L \geq 1$ such that $f(s) \leq Lf(t)$ for all $s \leq t$ (abbreviated *L-almost increasing*). *Almost decreasing* is defined analogously.

Definition 2.1. We say that $\varphi : A \times [0, \infty) \rightarrow [0, \infty]$ is a *weak Φ -function*, and write $\varphi \in \Phi_w(A)$, if the following conditions hold

- For every $t \in [0, \infty)$ the function $x \mapsto \varphi(x, t)$ is measurable and for every $x \in A$ the function $t \mapsto \varphi(x, t)$ is non-decreasing and left-continuous.
- $\varphi(x, 0) = \lim_{t \rightarrow 0^+} \varphi(x, t) = 0$ and $\lim_{t \rightarrow \infty} \varphi(x, t) = \infty$ for every $x \in A$.
- The function $t \mapsto \frac{\varphi(x, t)}{t}$ is L -almost increasing for $t > 0$ uniformly in A . "Uniformly" means that L is independent of x .

If $\varphi \in \Phi_w(A)$ is convex, then it is called a Φ -function, and we write $\varphi \in \Phi(A)$. If $\varphi \in \Phi(A)$ is continuous as a function into the extended real line $[0, \infty]$, then it is a *strong Φ -function*, and we write $\varphi \in \Phi_s(A)$.

We say that $\varphi, \psi \in \Phi_w(A)$ are *weakly equivalent*, $\varphi \sim \psi$ if there exist $D > 1$ and $h \in L^1(A)$ such that

$$\varphi(x, t) \leq \psi(x, Dt) + h(x) \quad \text{and} \quad \psi(x, t) \leq \varphi(x, Dt) + h(x).$$

Two functions φ and ψ are *equivalent*, $\varphi \simeq \psi$, if the previous conditions hold with $h \equiv 0$. Note that $\varphi \sim \psi$ if and only if $L^{\varphi(\cdot)} = L^{\psi(\cdot)}$. In the case $\varphi, \psi \in \Phi$ this has been proved in [6, Theorem 2.8.1]. For the weak Φ -functions the proof is the same.

We define the *doubling condition* Δ_2 and the *weak doubling condition* Δ_2^w , by

$$\begin{aligned} \varphi(x, 2t) &\lesssim \varphi(x, t), \\ \varphi(x, 2t) &\lesssim \varphi(x, t) + h(x), \end{aligned}$$

respectively, where $h \in L^1$ and the implicit constant is independent of x . If $\varphi \in \Phi_w(A)$, then we define a conjugate Φ -function by

$$\varphi^*(x, t) := \sup_{s \geq 0} (st - \varphi(x, s)).$$

We say that φ satisfies ∇_2 or ∇_2^w if φ^* satisfies Δ_2 or Δ_2^w , respectively. All these assumptions are invariant under equivalence, \simeq , of Φ -functions.

In some situations, it is useful to have a more quantitative version of the Δ_2 and ∇_2 conditions. It can be shown that (aDec) is equivalent to Δ_2 and (aInc) to ∇_2 (cf. [11, Lemma 2.6] and [5, Proposition 3.6]), where

- (aInc) There exist $\gamma^- > 1$ and $L \geq 1$ such that $t \mapsto \frac{\varphi(x, t)}{t^{\gamma^-}}$ is L -almost increasing in $(0, \infty)$.
- (aDec) There exist $\gamma^+ > 1$ and $L \geq 1$ such that $t \mapsto \frac{\varphi(x, t)}{t^{\gamma^+}}$ is L -almost decreasing in $(0, \infty)$.

Note that the optimal γ^- and γ^+ correspond to the lower and upper Matuszewska–Orlicz indexes, respectively.

Let us start by showing that weak doubling can be upgraded to strong doubling via weak equivalence of Φ -functions. For this we will use the *left-inverse* of a weak Φ -function, defined by the formula

$$\varphi^{-1}(x, \tau) := \inf\{t > 0 \mid \varphi(x, t) \geq \tau\}.$$

We point out that if $\varphi \in \Phi_s(\Omega)$, then by [9, p. 4] we have for every t that

$$(2.2) \quad \varphi(x, \varphi^{-1}(x, t)) = t.$$

Proof of Theorem 1.2. By [10, Proposition 2.3], we may assume without loss of generality that $\varphi \in \Phi_s(A)$. By assumption,

$$\begin{aligned} \varphi(x, 2t) &\leq D\varphi(x, t) + h(x), \\ \varphi^*(x, 2t) &\leq D\varphi^*(x, t) + h(x) \end{aligned}$$

for some $D > 2$, $h \in L^1$ and all $x \in A$ and $t \geq 0$. Using $\varphi = \varphi^{**}$ [6, Corollary 2.6.3] and the definition of the conjugate Φ -function, we obtain from the second inequality that

$$\begin{aligned} \varphi(x, 2t) &= \sup_{u \geq 0} (2tu - \varphi^*(x, u)) \leq \sup_{u \geq 0} (2tu - \frac{1}{D}(\varphi^*(x, 2u) - h(x))) \\ &= \sup_{u \geq 0} (2tu - \frac{1}{D}\varphi^*(x, 2u)) + \frac{1}{D}h(x) = \frac{1}{D} \sup_{u \geq 0} (Dt2u - \varphi^*(x, 2u)) + \frac{1}{D}h(x) \\ &= \frac{1}{D}\varphi(x, Dt) + \frac{1}{D}h(x). \end{aligned}$$

Define $t_x := \varphi^{-1}(x, h(x))$ and suppose that $t \geq t_x$ so that $h(x) \leq \varphi(x, t)$. By convexity, we conclude that $Dh(x) \leq D\varphi(x, t) \leq \varphi(x, Dt)$. Hence in the case $t \geq t_x$ we have

$$\begin{aligned} \varphi(x, 2t) &\leq (D+1)\varphi(x, t), \\ \varphi(x, 2t) &\leq \frac{D+1}{D^2}\varphi(x, Dt). \end{aligned}$$

Let $p := \log_2(D+1)$ and $q := \frac{\log(D^2/(D+1))}{\log(D/2)}$. Note that $q > 1$ since $\frac{D^2}{D+1} > \frac{D}{2}$. Divide the first inequality by $(2t)^p$ and the second one by $(2t)^q$:

$$\begin{aligned} \frac{\varphi(x, 2t)}{(2t)^p} &\leq \frac{D+1}{2^p} \frac{\varphi(x, t)}{t^p} = \frac{\varphi(x, t)}{t^p}, \\ \frac{\varphi(x, 2t)}{(2t)^q} &\leq \frac{(D+1)D^q}{D^{2q}} \frac{\varphi(x, Dt)}{(Dt)^q} = \frac{\varphi(x, Dt)}{(Dt)^q}. \end{aligned}$$

Let $s > t \geq t_x$. Then there exists $k \in \mathbb{N}$ such that $2^k t < s \leq 2^{k+1} t$. Hence

$$\frac{\varphi(x, s)}{s^p} \leq \frac{\varphi(x, 2^{k+1} t)}{(2^{k+1} t)^p} = 2^p \frac{\varphi(x, 2^{k+1} t)}{(2^{k+1} t)^p} \leq 2^p \frac{\varphi(x, 2^k t)}{(2^k t)^p} \leq \dots \leq 2^p \frac{\varphi(x, t)}{t^p},$$

so φ satisfies (aDec) with $\gamma^+ = p$ for $t \geq t_x$. Similarly, we find that φ satisfies (aInc) with $\gamma^- = q$ for $t \geq t_x$.

Define

$$\psi(x, t) := \begin{cases} \varphi(x, t), & \text{for } t \geq t_x \\ c_x t^2 & \text{otherwise,} \end{cases}$$

where c_x is chosen so that the ψ is continuous at t_x . Then ψ satisfies (aDec) on $[0, t_x]$ and $[t_x, \infty)$, hence on the whole real axis with $\gamma^+ = \max\{p, 2\}$, similarly for (aInc) with $\gamma^- = \min\{q, 2\}$.

Furthermore, $\varphi(x, t) = \psi(x, t)$ when $t \geq t_x$, and so it follows that $|\varphi(x, t) - \psi(x, t)| \leq \varphi(x, t_x) = h(x)$, where (2.2) is used for the last step.. Since $h \in L^1$, this means that $\varphi \sim \psi$, so ψ is the required function. \square

Remark 2.3. From the proof of the previous theorem, we see that the two conditions are not interdependent, i.e. if $\varphi \in \Phi_w(A)$ satisfies Δ_2^w , then there exists $\psi \in \Phi_w(A)$ with $\varphi \sim \psi$ satisfying Δ_2 ; similarly for only ∇_2^w and ∇_2 .

3. ASSOCIATE SPACES

We denote by $L^0(A)$ the set of measurable functions in A .

Definition 3.1. Let $\varphi \in \Phi_w(A)$ and define the *modular* $\varrho_{\varphi(\cdot)}$ for $f \in L^0(A)$ by

$$\varrho_{\varphi(\cdot)}(f) := \int_A \varphi(x, |f(x)|) dx.$$

The *generalized Orlicz space*, also called Musielak–Orlicz space, is defined as the set

$$L^{\varphi(\cdot)}(A) := \left\{ f \in L^0(A) \mid \lim_{\lambda \rightarrow 0^+} \varrho_{\varphi(\cdot)}(\lambda f) = 0 \right\}$$

equipped with the (Luxemburg) quasinorm

$$\|f\|_{\varphi(\cdot)} := \inf \left\{ \lambda > 0 \mid \varrho_{\varphi(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

Let us start with a lemma which shows that we can approximate the function 1 with a monotonically increasing sequence of functions in the generalized Orlicz space. Note that the next lemma is trivial if $L^\infty \subset L^{\varphi(\cdot)}$, as was assumed in [6] when dealing with associate spaces.

Lemma 3.2. *Let $\varphi \in \Phi_w(A)$. There exist positive $h_k \in L^{\varphi(\cdot)}(A)$, $k \in \mathbb{N}$, such that $h_k \nearrow 1$ and $\{h_k = 1\} \nearrow A$.*

Proof. For $k \geq 1$, we define

$$E_k := \{x \mid \varphi(x, 2^{-k}) \leq 1\}.$$

Since $\varphi(\cdot, t)$ is assumed to be measurable, E_k is a measurable set. Since $\lim_{t \rightarrow 0^+} \varphi(x, t) = 0$, there exists for every $x \in A$ some index k_x such that $x \in E_{k_x}$. And since φ is non-decreasing, it follows that $E_k \nearrow A$ as $k \rightarrow \infty$. We define

$$h(x) := \sum_{i=0}^{\infty} 2^{-i-1} \chi_{E_i}(x).$$

Then $h(x) \in (0, 1]$ for every x and h is measurable. Suppose that $x \in E_{k+1} \setminus E_k$ for some $k \in \mathbb{N}$. Then

$$h(x) = \sum_{i=k+1}^{\infty} 2^{-i-1} = 2^{-(k+1)}.$$

Hence, by the definition of E_{k+1} , we find that $\varphi(x, h(x)) \leq 1$. Since $A = \bigcup_k E_k$, we have $\varphi(x, h(x)) \leq 1$ in A . (The function h can alternatively be constructed using the left-inverse of φ , as in the previous section.)

Let us define $h_k := \min\{kh\chi_{B(0,k) \cap A}, 1\}$. Then

$$\varrho_{\varphi(\cdot)}\left(\frac{1}{k}h_k\right) \leq \int_{B(0,k) \cap A} \varphi(x, h) dx \leq |B(0, k)| < \infty,$$

so that $h_k \in L^{\varphi(\cdot)}(A)$. Since $h > 0$, it follows that $kh\chi_{B(0,k) \cap A} \nearrow \infty$ for every x , and so $h_k \nearrow 1$, as required. \square

We define the *associate space* by $(L^{\varphi(\cdot)})'(A) := \{f \in L^0(A) \mid \|f\|_{(L^{\varphi(\cdot)})'} < \infty\}$, where

$$\|f\|_{(L^{\varphi})'} := \sup_{\|g\|_{\varphi(\cdot)} \leq 1} \int_A fg dx.$$

If $g \in (L^{\varphi})'$ and $f \in L^{\varphi}$, then $fg \in L^1$ by definition of the associate space. In particular, the integral $\int_A fg d\mu$ is well defined and

$$\left| \int_A fg d\mu \right| \leq \|g\|_{(L^{\varphi})'} \|f\|_{\varphi(\cdot)}.$$

Hölder's inequality holds in generalized Orlicz spaces with a constant 2, without restrictions on the Φ_w -function [6, Lemma 2.6.5]:

$$(3.3) \quad \int_A |f| |g| dx \leq 2 \|f\|_{\varphi(\cdot)} \|g\|_{\varphi^*(\cdot)}.$$

Here φ^* is the conjugate Φ -function defined in the previous section. Furthermore, we can define a conjugate modular on the dual space by the formula

$$(\varrho_{\varphi(\cdot)})^*(J) := \sup_{f \in L^{\varphi(\cdot)}} (J(f) - \varrho_{\varphi(\cdot)}(f))$$

for $J \in (L^{\varphi(\cdot)})^*$, i.e. $J : L^{\varphi(\cdot)} \rightarrow \mathbb{R}$ is a bounded linear functional. By J_f we denote the functional $g \mapsto \int fg \, dx$.

Proof of Theorem 1.1. We follow the outlines of [6, Theorem 2.7.4], but use Lemma 3.2 to get rid of the extraneous assumption that simple functions belong to the space. The inequality $\|f\|_{(L^\varphi)'} \leq 2\|f\|_{\varphi^*(\cdot)}$ follows from (3.3).

Let then $f \in (L^\varphi)'$ and $\varepsilon > 0$. Let $\{q_1, q_2, \dots\}$ be an enumeration of non-negative rational numbers with $q_1 = 0$. For $k \in \mathbb{N}$ and $x \in A$ define

$$r_k(x) := \max_{j=1, \dots, k} q_j |f(x)| - \varphi(x, q_j).$$

The special choice $q_1 = 0$ implies $r_k(x) \geq 0$ for all $x \geq 0$. Since \mathbb{Q} is dense in $[0, \infty)$ and $\varphi(x, \cdot)$ is left-continuous, $r_k(x) \nearrow \varphi^*(x, |f(x)|)$ for every $x \in A$ as $k \rightarrow \infty$.

Since f and $\varphi(\cdot, t)$ are measurable functions, the sets

$$E_{i,k} := \left\{ x \in A \mid q_i |f(x)| - \varphi(x, q_i) = \max_{j=1, \dots, k} (q_j |f(x)| - \varphi(x, q_j)) \right\}$$

are measurable. Let $F_{i,k} := E_{i,k} \setminus (E_{1,k} \cup \dots \cup E_{i-1,k})$. Define

$$g_k := \sum_{i=1}^k q_i \chi_{F_{i,k}}.$$

Then g_k is measurable and bounded and

$$r_k(x) = g_k(x) |f(x)| - \varphi(x, g_k(x))$$

for all $x \in A$.

Let $h_k \in L^{\varphi(\cdot)}(A)$ be as in Lemma 3.2, i.e. $\{h_k = 1\} \nearrow A$ and $0 < h_k \leq 1$. Since g_k is bounded, it follows that $w := \operatorname{sgn} f h_k g_k \in L^{\varphi(\cdot)}$. Denote $E := \{fw \geq \varphi(x, w)\}$.

Since the conjugate modular is defined as a supremum over functions in $L^{\varphi(\cdot)}$, we get a lower bound by using the particular function $w \chi_E$. Thus

$$\begin{aligned} (\varrho_{\varphi(\cdot)})^*(J_f) &\geq J_f(w \chi_E) - \varrho_{\varphi(\cdot)}(w \chi_E) = \int_E fw - \varphi(x, w) \, dx \\ &\geq \int_{\{h_k=1\}} g_k |f| - \varphi(x, g_k) \, dx = \int_A r_k \chi_{\{h_k=1\}} \, dx. \end{aligned}$$

Since $r_k \chi_{\{h_k=1\}} \nearrow \varphi^*(x, |f|)$, it follows by monotone convergence that $(\varrho_{\varphi(\cdot)})^*(J_f) \geq \varrho_{\varphi^*(\cdot)}(f)$. From the definitions of $(\varrho_{\varphi(\cdot)})^*$ and $\varrho_{\varphi^*(\cdot)}$,

$$(\varrho_{\varphi(\cdot)})^*(J_f) = \sup_{g \in L^{\varphi(\cdot)}} \int_A fg - \varphi(x, g) dx \leq \int_A \varphi^*(x, f) dx = \varrho_{\varphi^*(\cdot)}(f).$$

Hence $(\varrho_{\varphi(\cdot)})^*(J_f) = \varrho_{\varphi^*(\cdot)}(f)$

Since $f \mapsto J_f$ is linear, it follows that $(\varrho_{\varphi(\cdot)})^*(\lambda J_f) = \varrho_{\varphi^*(\cdot)}(\lambda f)$ for every $\lambda > 0$ and therefore $\|f\|_{\varphi^*(\cdot)} = \|J_f\|_{(\varrho_{\varphi(\cdot)})^*} \leq \|J_f\|_{(L^{\varphi(\cdot)})^*} = \|f\|_{(L^{\varphi(\cdot)})'}$, where the second step follows from [6, Theorem 2.2.10].

Taking into account that $\varphi^{**} \simeq \varphi$, we have shown that $L^{\varphi(\cdot)} = (L^{\varphi^*(\cdot)})'$. By the definition of the associate space norm, this means that

$$\|f\|_{\varphi(\cdot)} \approx \sup_{\|g\|_{\varphi^*(\cdot)} \leq 1} \int |f| |g| dx$$

for $f \in L^{\varphi(\cdot)}$. In the case $f \in L^0 \setminus L^{\varphi(\cdot)}$, we can approximate $h_k \min\{|f|, k\} \nearrow |f|$ with h_k as before. Since $h_k \min\{|f|, k\} \in L^{\varphi(\cdot)}$, the previous result implies that the formula holds, in the form $\infty = \infty$, when $f \in L^0 \setminus L^{\varphi(\cdot)}$. \square

4. UNIFORM CONVEXITY

The function $\varphi \in \Phi_w(\mathbb{R}^n)$ is *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varphi\left(x, \frac{s+t}{2}\right) \leq (1-\delta) \frac{\varphi(x, s) + \varphi(x, t)}{2}$$

for every $x \in \mathbb{R}^n$ whenever $|s-t| \geq \varepsilon \max\{|s|, |t|\}$.

Theorem 4.1. *The function $\varphi \in \Phi_w(A)$ is equivalent to a uniformly convex Φ -function if and only if it satisfies (aInc).*

Proof. Assume first that φ satisfies (aInc) with $\gamma^- = p > 1$. By [10, Lemma 2.2], there exists $\psi \in \Phi(A)$ such that $\varphi \simeq \psi$ and $\psi^{\frac{1}{p}}$ is convex for some $p > 1$. The claim follows once we show that ψ is uniformly convex. Let $\varepsilon \in (0, 1)$ and $s-t \geq \varepsilon s$, with $s > t > 0$. Since $\psi^{\frac{1}{p}}$ is convex,

$$\psi\left(x, \frac{s+t}{2}\right)^{\frac{1}{p}} \leq \frac{\psi(x, s)^{\frac{1}{p}} + \psi(x, t)^{\frac{1}{p}}}{2}.$$

Since $t \leq (1-\varepsilon)s$ and ψ is convex, we find that $\psi(x, t) \leq \psi(x, (1-\varepsilon)s) \leq (1-\varepsilon)\psi(x, s)$. Therefore, $\psi(x, t)^{\frac{1}{p}} \leq (1-\varepsilon')\psi(x, s)^{\frac{1}{p}}$ for some $\varepsilon' > 0$. Since t^p is uniformly convex, we obtain that

$$\left(\frac{\psi(x, s)^{\frac{1}{p}} + \psi(x, t)^{\frac{1}{p}}}{2}\right)^p \leq (1-\delta) \frac{\psi(x, s) + \psi(x, t)}{2}.$$

Combined with the previous estimate, this shows that ψ is uniformly convex.

Assume now conversely, that $\varphi \simeq \psi$ and ψ is uniformly convex. Choose $\varepsilon = \frac{1}{2}$ and $t = 0$ in the definition of uniform convexity:

$$\psi(x, \frac{s}{2}) \leq \frac{1}{2}(1 - \delta)\psi(x, s)$$

Divide this equation with $(s/2)^p$ where p is chosen such that $2^{p-1}(1 - \delta) = 1$:

$$\frac{\psi(x, \frac{s}{2})}{(s/2)^p} \leq 2^{p-1}(1 - \delta) \frac{\psi(x, s)}{s^p} = \frac{\psi(x, s)}{s^p}.$$

The previous inequality holds for every $s > 0$. If $0 < t < s$ then we can choose $k \in \mathbb{N}$ such that $2^k t \leq s < 2^{k+1}t$. Then by the previous inequality and monotonicity of ψ ,

$$\frac{\psi(x, t)}{t^p} \leq \frac{\psi(x, 2t)}{(2t)^p} \leq \dots \leq \frac{\psi(x, 2^k t)}{(2^k t)^p} \leq 2^p \frac{\psi(x, s)}{s^p}.$$

Hence ψ satisfies (aInc) with $\gamma^- = p$. Since this property is invariant under equivalence, it holds for φ as well. \square

We can now prove the uniform convexity of the space.

Proof of Theorem 1.3. By Theorem 1.2, Δ_2^w and ∇_2^w imply Δ_2 and ∇_2 . If φ satisfies (aInc), then it follows from Theorem 4.1 that it is equivalent to a uniformly convex Φ -function ψ . By (aDec), also ψ is doubling. Hence by [16, Theorem 11.6] (see also [6, Theorem 2.4.14]) $L^{\psi(\cdot)}$ is uniformly convex. Since $\varphi \simeq \psi$, $L^{\varphi(\cdot)} = L^{\psi(\cdot)}$, and hence we have proved $L^{\varphi(\cdot)}$ is uniformly convex. Furthermore, every uniformly convex Banach space is reflexive [1, Chapter 1]. \square

ACKNOWLEDGEMENT

We would like to thank the referee for some valuable comments.

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