# SOBOLEV INEQUALITIES FOR VARIABLE EXPONENTS ATTAINING THE VALUES 1 AND $n$

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*Dedicated to Professor Yoshihiro Mizuta on the occasion of his sixtieth birthday*

Abstract. We study Sobolev embeddings in the Sobolev space $W^{1,p(\cdot)}(\Omega)$ with variable exponent satisfying $1 \leqslant p(x) \leqslant n$. Since the exponent is allowed to reach the values 1 and $n$, we need to introduce new techniques, combining weak- and strong-type estimates, and a new variable exponent target space scale which features a space of exponential type integrability instead of $L^\infty$ at the upper end.

## 1. Introduction

Variable exponent spaces have been studied in many articles over the past decade; for a survey see [10, 27]. These investigations have dealt both with the spaces themselves, with related differential equations, and with applications. One typical feature is that the exponent has to be strictly bounded away from various critical values. In some recent investigations it it has been found that one needs to modify the scales of spaces at the end point to properly deal with such limiting phenomena, see [9, 17].

More concretely, consider the example of the Sobolev embedding theorem. In the constant exponent case it is well-known that the embeddings are qualitatively different according as $p < n$ (Lebesgue space), $p = n$ (exponential Orlicz space) or $p > n$ (Hölder space). In the variable exponent case this has led to theorems assuming either $p^+ < n$, or $p^- > n$, where $p^+$ and $p^-$ denote the greatest and least value of $p$, respectively. In this paper we are concerned with generalizing the former.

Sobolev embeddings and embeddings of Riesz potentials have been studied, e.g., in [1, 2, 4, 6, 8, 10, 11, 12, 13, 15, 16, 20, 24, 28] in the variable exponent setting. Most proofs in the literature are based on the Riesz potential and maximal functions, and thus lead to the additional, unnatural restriction $p^- > 1$.

As far as we know, the only attempt to levy these restrictions is due to Edmunds and Rákosník [11, 12]. Their method is not based on maximal functions, and does

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*Date*: September 25, 2007.

2000 *Mathematics Subject Classification.* 46E35.

*Key words and phrases.* Variable exponent, Sobolev space, Sobolev inequality, limit case.

* Supported by the Academy of Finland.

† Supported in part by the Academy of Finland.

not require the assumption $p^- > 1$; on the other hand, they have to assume especially strong continuity conditions on the exponent. To relax the condition $p^+ < n$, they introduced a weight on the target side and considered embeddings of the type $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}_\omega(\Omega)$. Their weight has the property that $\omega(x) = 0$ whenever $p(x) = n$. This means that the embedding says nothing about the integrability of $u$ in the set $\Omega_n = p^{-1}(n)$. Thus it is not possible to recover the classical results as special cases of their result, and further, there are no prospects of extending the result to $p > n$.

In this paper we introduce a slightly modified scale of variable exponent function spaces, $L^{p(\cdot),*}(\Omega)$, with the property $L^{p(\cdot),*}(\Omega) \cong L^{p^*(\cdot)}(\Omega)$ if $p^+ < n$ and $L^{p(\cdot),*}(\Omega_n) \cong \exp L^{n'}(\Omega_n)$. This space is defined as follows. Denote $\tilde{p} = p^*/n'$ and set

$$M_p^*(t) = \sum_{i=0}^{\lfloor\tilde{p}\rfloor-1} \frac{1}{i!}|t|^{n'(i+1)} + \frac{1}{\lfloor p^*/n\rfloor!}|t|^{p^*}$$

for $1 \leqslant p \leqslant n$, with the understanding that the last term disappears if $p = n$. Using the function $M^*_{p(\cdot)}$ we define a modular by

$$\varrho_{L^{p(\cdot),*}(\Omega)}(u) = \int_\Omega M^*_{p(x)}(u(x))\,dx,$$

where $p$ is a variable exponent satisfying $p^+ \leqslant n$. From this we get a Luxemburg-type norm:

$$\|u\|_{L^{p(\cdot),*}(\Omega)} = \inf\{\lambda > 0 : \varrho_{L^{p(\cdot),*}(\Omega)}(u/\lambda) \leqslant 1\}.$$

The space $L^{p(\cdot),*}(\Omega)$ consists of those functions for which $\|u\|_{L^{p(\cdot),*}(\Omega)} < \infty$.

The following is the main result of this paper:

**Theorem 1.1.** *Suppose that $p$ is* log-*Hölder continuous with $1 \leqslant p(x) \leqslant n$ in the bounded open set $\Omega \subset \mathbb{R}^n$.*

(1) *We have $\|u\|_{L^{p(\cdot),*}(\Omega)} \lesssim \|\nabla u\|_{p(\cdot)}$ for every $u \in W^{1,p(\cdot)}_0(\Omega)$. Here the constant depends only on $n$, $p$ and $|\Omega|$.*
(2) *If $\Omega$ is a John domain, then $\|u - u_\Omega\|_{L^{p(\cdot),*}(\Omega)} \lesssim \|\nabla u\|_{p(\cdot)}$ for every $u \in W^{1,p(\cdot)}(\Omega)$. Here the constant depends additionally on the John constants of $\Omega$.*

The proof is in two parts. First we prove that the lower bound $p^- > 1$ can be relaxed to $p^- \geqslant 1$ by a new kind of weak-type estimate (Section 4). Then we prove the embedding assuming $1 < p^- \leqslant p^+ \leqslant n$ (Section 5). Finally, in Section 6 these results are combined to give the main theorem. Before the main results, we recall some definitions of variable exponent spaces, give pointwise estimates of Riesz potentials with asymptotically (as $p \to n$) optimal constants (Section 2) and weak-type estimates for the Riesz potential, relevant when $p \to 1$ (Section 3).

**Notation and conventions.** We write $f \lesssim g$ if there exists a constant $C$ so that $f \leqslant Cg$. We assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set. For a function $f\colon \Omega \to \mathbb{R}$ we denote the set $\{x \in \Omega : a < f(x) < b\}$ simply by $\{a < f < b\}$, etc. For $f \in L^1(\Omega)$ and $A \subset \mathbb{R}^n$ with positive, finite measure we write

$$f_A = \fint_A f(y)\,dy := |A|^{-1} \int_{A\cap\Omega} f(y)\,dy.$$

By $\mathcal{M}$ we denote the centered Hardy–Littlewood maximal operator, $\mathcal{M}f(x) = \sup_{r>0} |f|_{B(x,r)}$.

Let $p\colon \Omega \to [1,\infty]$ be a measurable function called variable exponent. For $A \subset \Omega$ we write $p_A^+ = \operatorname{ess\,sup}_{x\in A} p(x)$ and $p_A^- = \operatorname{ess\,inf}_{x\in A} p(x)$, and abbreviate $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$. We say that $p \in L^\infty(\Omega)$ is log-*Hölder continuous* in $\Omega$ if

$$|p(x) - p(y)| \leqslant \frac{C}{\log(e + 1/|x-y|)}$$

for every pair of distinct points $x, y \in \Omega$. This is equivalent (with possible different C) to

$$|B|^{p^-_{B\cap\Omega} - p^+_{B\cap\Omega}} \leqslant C.$$

for every ball $B$ with $B \cap \Omega \neq \emptyset$.

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that

$$\varrho_{p(\cdot)}(\lambda u) = \int_{\Omega\setminus\{p=\infty\}} |\lambda u(x)|^{p(x)}\,dx + \operatorname*{ess\,sup}_{x\in\{p=\infty\}} |\lambda u(x)| < \infty$$

for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0\colon \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}.$$

Here $\Omega$ could of course be replaced by some subset as in $\|u\|_{L^{p(\cdot)}(A)}$; we reserve the abbreviation $\|u\|_{p(\cdot)}$ for the norm in the whole set $\Omega$. It is easy to see that $\varrho_{p(\cdot)}(u) \leqslant 1$ if and only if $\|u\|_{L^{p(\cdot)}(\Omega)} \leqslant 1$.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of all $u \in L^{p(\cdot)}(\Omega)$ such that the absolute value of the distributional gradient $\nabla u$ is in $L^{p(\cdot)}(\Omega)$. The norm $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. By $W_0^{1,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For the basic theory of variable exponent spaces see [21].

## 2. Riesz potential estimates with asymptotically optimal constants

Let $\alpha > 0$ be fixed. We consider the Riesz potential

$$\mathcal{I}_\alpha f(x) = \int_\Omega \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy$$

in $\Omega$, and write $p_\alpha^\#(x) = np(x)/(n - \alpha p(x))$. We prove a pointwise estimate for the Riesz potential, based on standard techniques, originally due to Hedberg [19].

**Proposition 2.1.** *Let $p$ be a log-Hölder continuous exponent with $\alpha p^+ < n$. If $k \geqslant \max\left\{\frac{p^+}{n-\alpha p^+}, 1\right\}$, then*

$$\mathcal{I}_\alpha u(x) \lesssim k^{\frac{1}{(p^+)'}} \, [\mathcal{M}u(x)]^{1-\alpha\frac{p(x)}{n}},$$

*for every $u \in L^{p(\cdot)}(\Omega)$ with $\|u\|_{p(\cdot)} \leqslant 1$.*

*Proof.* Let $x \in \Omega$ and let $\delta \in (0, 2\operatorname{diam}\Omega]$. Since $p$ is log-Hölder continuous and bounded, $1/p'$ is also log-Hölder continuous. Let $C_{\log} \geqslant 1$ be the log-Hölder constant of $1/p'$. Then we have

$$\varrho_{L^{p'(\cdot)}(B\cap\Omega)}\Big(C_{\log}^{-1}|B|^{-\frac{1}{p'(x)}}\Big) \leqslant \frac{1}{|B|}\int_{B\cap\Omega}\left(\frac{1}{C_{\log}}|B|^{\frac{1}{p'(y)}-\frac{1}{p'(x)}}\right)^{p'(y)} dy \leqslant \frac{|B\cap\Omega|}{|B|} \leqslant 1.$$

Therefore $\|C^{-1}|B|^{-1/p'(x)}\|_{L^{p'(\cdot)}(B\cap\Omega)} \leqslant 1$, which implies that

$$\|1\|_{L^{p'(\cdot)}(B(x,2^i)\cap\Omega)} \lesssim \left|B(x,2^i)\right|^{\frac{1}{p'(x)}} \lesssim 2^{\frac{in}{p'(x)}}. \tag{2.2}$$

We denote by $A(x,r)$ the annulus $(B(x,2r)\setminus B(x,r))\cap\Omega$. Thus

$$\mathcal{I}_\alpha u(x) \leqslant \sum_{\delta\leqslant 2^i\leqslant 2\operatorname{diam}\Omega} 2^{i(\alpha-n)}\int_{A(x,2^i)}|u(y)|\,dy + \sum_{2^i\leqslant\delta} 2^{i(\alpha-n)}\int_{A(x,2^i)}|u(y)|\,dy.$$

For simplicity we denote by $I$ the set of integers for which $\delta \leqslant 2^i \leqslant 2\operatorname{diam}\Omega$. We note that the second term is dominated by $\delta^\alpha\mathcal{M}u(x)$.

Using Hölder's inequality for the first and third estimates, we conclude that

$$\begin{aligned}\sum_{i\in I} 2^{i(\alpha-n)}\int_{A(x,2^i)}|u(y)|\,dy &\lesssim \sum_{i\in I} 2^{i(\alpha-n)}\|u\|_{L^{p(\cdot)}(A(x,2^i))}\|1\|_{L^{p'(\cdot)}(B(x,2^i)\cap\Omega)}\\ &\lesssim \sum_{i\in I} 2^{i(\alpha-n+\frac{n}{p'(x)})}\|u\|_{L^{p(\cdot)}(A(x,2^i))}\\ &\lesssim \left(\sum_{i\in I} 2^{-in\frac{(p^+)'}{p^\#_\alpha(x)}}\right)^{\frac{1}{(p^+)'}}\left(\sum_{i\in I}\|u\|^{p^+}_{L^{p(\cdot)}(A(x,2^i))}\right)^{\frac{1}{p^+}}\end{aligned} \tag{2.3}$$

for $x \in \Omega$. Note that the exponents from the norm in the second sum would cancel if $p$ were constant. This is the only place where the variability is really a nuance, but also this is easily taken care of: since $\|u\|_{p(\cdot)} \leqslant 1$ we have $\|u\|^{p^+}_{p(\cdot)} \leqslant \varrho_{p(\cdot)}(u)$, and so

$$\sum_{i\in I}\|u\|^{p^+}_{L^{p(\cdot)}(A(x,2^i))} \leqslant \sum_{i\in I}\int_{A(x,2^i)}|u(y)|^{p(y)}\,dy \leqslant \int_\Omega |u(y)|^{p(y)}\,dy \leqslant 1.$$

The first term on the right hand side of (2.3) is a geometric sum and so we find that

$$\left(\sum_{i\in I} 2^{-in\frac{(p^+)'}{p^\#_\alpha(x)}}\right)^{\frac{1}{(p^+)'}} \leqslant \delta^{-\frac{n}{p^\#_\alpha(x)}}\left(1-2^{-n\frac{(p^+)'}{p^\#_\alpha(x)}}\right)^{-\frac{1}{(p^+)'}} \lesssim \delta^{-\frac{n}{p^\#_\alpha(x)}}k^{\frac{1}{(p^+)'}}.$$

The second inequality holds since $n/p^\#_\alpha(x) \geqslant 1/k$ and $k \geqslant 1$ and thus

$$(1-2^{-n\frac{(p^+)'}{p^\#_\alpha(x)}})^{-1} \leqslant (1-2^{-\frac{(p^+)'}{k}})^{-1} \leqslant (1-2^{-(p^+)'})^{-1}k.$$

We have shown that $\mathcal{I}_\alpha u(x) \lesssim \delta^\alpha \mathcal{M}u(x)+k^{1/(p^+)'}\delta^{-n/p^\#_\alpha(x)}$. The claim follows from this by choosing $\delta \leqslant 2\,\mathrm{diam}(\Omega)$ appropriately, following the standard proof. □

## 3. A STRONGER KIND OF WEAK-TYPE ESTIMATE

Weak-type estimates have been used in the context of variable exponent spaces a few times. For instance, Cruz-Uribe, Fiorenza and Neugebauer [5] gave a weak-type estimate of the maximal operator. Their result is very weak: on the positive side, it requires almost no regularity of the exponent; on the negative side it has hardly been put to use in any further proofs.

In this section we propose a new kind of weak-type condition. Like the condition of Cruz-Uribe, Fiorenza and Neugebauer, our condition also reduces to the usual one if the exponent is constant. However, our stronger condition allows us to prove optimal Sobolev embeddings when $p^- = 1$. On the other hand, we need to assume that $p$ is log-Hölder continuous.

The proof of the following lemma is an easy modification of [7, Lemma 3.3] or [18, Lemma 4.2].

**Lemma 3.1.** *Suppose that $p$ is* log-*Hölder continuous with $p^+ < \infty$. Let $f \in L^{p(\cdot)}(\Omega)$ be such that $(1 + |\Omega|)\|f\|_{p(\cdot)} \leqslant 1$. Then*

$$(|f|_B)^{p(x)} \leqslant C(|f|^{p(\cdot)} + \chi_{\{0<|f|<1\}})_B$$

*for every $x \in \Omega$ and every ball $B \subset \mathbb{R}^n$ containing $x$.*

Using this lemma we prove a weak-type estimate for the maximal function.

**Theorem 3.2.** *Suppose that $p$ is* log-*Hölder continuous with $p^+ < \infty$. Let $f \in L^{p(\cdot)}(\Omega)$ be such that $(1 + |\Omega|)\|f\|_{p(\cdot)} \leqslant 1$. Then for every $t > 0$ we have*

$$\int_{\{\mathcal{M}f>t\}} t^{p(x)} dx \lesssim \int_\Omega |f(y)|^{p(y)} dy + \big|\{0 < |f| < 1\}\big|.$$

*Proof.* We write $E = \{\mathcal{M}f > t\}$. For every $x \in E$ we choose $B_x = B(x, r)$ so that $|f|_{B_x} > t$. By the Besicovitch covering theorem there is a countable covering subfamily $(B_i)$ with a bounded overlap-property. We obtain, with Lemma 3.1 for the third inequality, that

$$\begin{aligned}
\int_E t^{p(x)} dx &\leqslant \sum_i \int_{B_i} t^{p(x)} dx \leqslant \sum_i \int_{B_i} \bigg( \fint_{B_i} |f(y)|\, dy \bigg)^{p(x)} dx \\
&\lesssim \sum_i \int_{B_i} \fint_{B_i} |f(y)|^{p(y)} + \chi_{\{0<|f|<1\}}(y)\, dy\, dx \\
&= \sum_i \int_{B_i} |f(y)|^{p(y)} + \chi_{\{0<|f|<1\}}(y)\, dy \\
&\lesssim \int_\Omega |f(y)|^{p(y)} dy + \big|\{0 < |f| < 1\}\big|.
\end{aligned}$$

□

*Remark* 3.3. Pick and Růžička [26] gave an example which shows that the log-Hölder continuity condition is the optimal modulus of continuity for the maximal operator to be bounded. One can show that this example works also for our weak-type estimate. Thus log-Hölder continuity is optimal also for the previous result in so far as moduli of continuity are concerned.

Next we prove the weak-type estimate for the Riesz potential $\mathcal{I}_\alpha$.

**Theorem 3.4.** *Suppose that $p$ is* log-*Hölder continuous with $\alpha p^+ < n$. Let $f \in L^{p(\cdot)}(\Omega)$ be such that $(1 + |\Omega|)\|f\|_{p(\cdot)} \leqslant 1$. Then for every $t > 0$ we have*

$$\int_{\{\mathcal{I}_\alpha f > t\}} t^{p^\#_\alpha(x)} dx \lesssim \int_\Omega |f(y)|^{p(y)} dy + \big|\{0 < |f| < 1\}\big|.$$

*Proof.* For $k = \max\{p^+/(n - \alpha p^+), 1\}$ we obtain by Proposition 2.1 that

$$\{\mathcal{I}_\alpha f(x) > t\} \subset \Big\{C[\mathcal{M}f(x)]^{\frac{p(x)}{p^\#_\alpha(x)}} > t\Big\} =: E.$$

For every $z \in E$ we choose $B_z = B(z, r)$ so that $C(|f|_{B_z})^{\frac{p(z)}{p^\#_\alpha(z)}} > t$. Let $x \in B_z$ and raise this inequality to the power $p^\#_\alpha(x)$. Let us write $q(x) = p(z)p^\#_\alpha(x)/p^\#_\alpha(z)$. Assume first that $q(x) \geqslant p(x)$, i.e. $p(x) \geqslant p(y)$. Since $|f|_{B_z} \leqslant |B_z|^{-1} \int_{B_z} |f|\, dy \lesssim |B_z|^{-1}$, we obtain

$$t^{p^\#_\alpha(x)} \lesssim (|f|_{B_z})^{p(x)}(|f|_{B_z})^{q(x)-p(x)} \lesssim (|f|_{B_z})^{p(x)}|B_z|^{p(x)-q(x)} = (|f|_{B_z})^{p(x)}|B_z|^{\frac{\alpha p(x)(p(z)-p(x))}{n-\alpha p(x)}}.$$

The last term is uniformly bounded since $p$ is log-Hölder continuous. By Lemma 3.1 this yields

$$t^{p^\#_\alpha(x)} \lesssim \big(|f|^{p(\cdot)} + \chi_{\{0<|f|<1\}}\big)_{B_z}$$

for every $x \in B_z$. Assume then that $q(x) < p(x)$. By Lemma 3.1 we obtain

$$t^{p^\#_\alpha(x)} \lesssim \big(|f|^{p(\cdot)} + \chi_{\{0<|f|<1\}}\big)_{B_z}^{\frac{q(x)}{p(x)}} \leqslant \big(|f|^{p(\cdot)} + \chi_{\{0<|f|<1\}}\big)_{B_z},$$

where the last inequality follows since $\big(|f|^{p(\cdot)} + \chi_{\{0<|f|<1\}}\big)_{B_z} \geqslant 1$ and $q(x)/p(x) < 1$.

By the Besicovitch covering theorem there is a countable covering subfamily $(B_i)$, with a bounded overlap-property. Thus we obtain

$$\begin{aligned}\int_E t^{p^\#_\alpha(x)} dx &\leqslant \sum_i \int_{B_i} t^{p^\#_\alpha(x)} dx \lesssim \sum_i \int_{B_i} \fint_{B_i} |f(y)|^{p(y)} + \chi_{\{0<|f|<1\}}(y)\, dy\, dx \\ &= \sum_i \int_{B_i} |f(y)|^{p(y)} + \chi_{\{0<|f|<1\}}(y)\, dy \\ &\lesssim \int_\Omega |f(y)|^{p(y)} dy + |\{0 < |f| < 1\}|. \qquad \square\end{aligned}$$

## 4. Sobolev inequalities based on weak-type estimates

In this section we prove Sobolev embeddings in the variable exponent space without the assumption $p^- > 1$. The proofs are based on the weak-type estimates from the previous section. We denote by $p^*$ the Sobolev conjugate exponent, i.e. $p^* = p_1^\#$ in the notation of the previous section.

The following chain condition is adapted from [14, Section 6].

**Definition 4.1.** We say that $D \subset \Omega$ satisfies the $(N, R, \Omega)$-chain condition if for every $x \in D$ and all $r \in (0, R)$ there exists a sequence of balls $B_0, \ldots, B_k$ belonging to $\Omega$ such that

(1) $B_0 \subset \Omega \setminus B(x, R)$ and $B_k \subset B(x, r)$;
(2) $\frac{1}{N}\operatorname{diam}(B_i) \leqslant \operatorname{dist}(x, B_i) \leqslant N \operatorname{diam}(B_i)$;
(3) the intersection $B_i \cap B_{i+1}$ has measure at least $\frac{1}{N}|B_i|$; and
(4) the family $\{B_i\}$ has overlap at most $N$.

For instance every John domain satisfies the Chain condition, as will be shown in Lemma 6.1.

**Proposition 4.2.** *Suppose that p is* log-*Hölder continuous with* $1 \leqslant p^- \leqslant p^+ < n$.

(1) *We have* $\|u\|_{p^*(\cdot)} \lesssim \|\nabla u\|_{p(\cdot)}$ *for every* $u \in W_0^{1,p(\cdot)}(\Omega)$*. The constant depends only on n, p and* $|\Omega|$.
(2) *Let* $D \subset \Omega$ *satisfy the* $(N, \varepsilon, \Omega)$*-chain condition. Then*
$$\|u - c\|_{L^{p^*(\cdot)}(D)} \lesssim N^{n+1}\|\nabla u\|_{L^{p(\cdot)}(\Omega)} + N^n\varepsilon^{-n}\|u - c\|_{L^1(\Omega)}$$
*for every* $c \in \mathbb{R}$ *and every* $u \in W^{1,p(\cdot)}(\Omega)$.

*Proof.* To prove (1) we first assume that $(1 + |\Omega|)\|\nabla u\|_{p(\cdot)} \leqslant 1$. By the well known point-wise inequality we have for every $v \in W_0^{1,1}(\Omega)$ and for almost every $x \in \Omega$ that $|v(x)| \leqslant C(n)\mathcal{I}_1|\nabla v|(x)$. For $j \in \mathbb{Z}$ we write $\Omega_j = \{2^j < u(x) \leqslant 2^{j+1}\}$ and $v_j = \max\{0, \min\{u - 2^j, 2^j\}\}$. For every $x \in \Omega_{j+1}$ we have $v_j(x) = 2^j$ and thus by the pointwise inequality $\mathcal{I}_1|\nabla v_j|(x) > \frac{1}{2}C(n)2^j$.

We obtain by Theorem 3.4 that
$$\begin{aligned}\int_\Omega |u(x)|^{p^*(x)}dx &= \sum_{j\in\mathbb{Z}}\int_{\Omega_j}|u(x)|^{p^*(x)}dx \leqslant \sum_{j\in\mathbb{Z}}\int_{\Omega_j}2^{(j+1)p^*(x)}dx\\ &\lesssim \sum_{j\in\mathbb{Z}}\int_{\{x\in\Omega_{j+1}:\mathcal{I}_1|\nabla v_j|(x)>C2^j\}}(C2^j)^{p^*(x)}dx\\ &\lesssim \sum_{j\in\mathbb{Z}}\Big(\int_\Omega|\nabla v_j|^{p(y)}dy + \big|\{0<|\nabla v_j|<1\}\big|\Big)\\ &\leqslant \sum_{j\in\mathbb{Z}}\Big(\int_{\Omega_j}|\nabla u(y)|^{p(y)}dy + |\Omega_j|\Big) \leqslant 1 + |\Omega|.\end{aligned}$$

This implies that $\|u\|_{p^*(\cdot)} \leqslant C$ for every $u$ with $(1 + |\Omega|)\|\nabla u\|_{p(\cdot)} \leqslant 1$. Thus we obtain the claim by using this inequality for $u/((1 + |\Omega|)\|\nabla u\|_{p(\cdot)})$.

Now we move on to (2). Let $B(x)$ be the largest ball from the chain associated to $x$. By [14, Lemma 6.2], we conclude that

$$|u(x) - u_{B(x)}| \lesssim N \sum_{i=0}^{k} \operatorname{diam}(B_i) \fint_{B_i} |\nabla u|\, dx \lesssim N^{n+1} \mathcal{I}_1(\nabla u)(x)$$

for almost every $x \in D$. Thus $|u(x) - c| \lesssim N^{n+1} \mathcal{I}_1 |\nabla u|(x) + |u_{B(x)} - c|$. For the second term we have

$$|u_{B(x)} - c| \leqslant \fint_{B(x)} |u(y) - c|\, dy \lesssim \Big(\frac{N}{\varepsilon}\Big)^n \|u - c\|_{L^1(\Omega)}.$$

We write $C' = (\frac{N}{\varepsilon})^n \|u - c\|_{L^1(\Omega)}$. Replacing $u$ by $u - c$ in the proof of claim (1), we obtain

$$\begin{aligned}\int_\Omega |u(x) - c|^{p^*(x)} dx &\lesssim \sum_{j\in\mathbb{Z}} \int_{\{x\in\Omega_{j+1} : \mathcal{I}_1|\nabla v_j|(x) + C' > C2^j\}} 2^{jp^*(x)} dx \\ &\lesssim \sum_{j\in\mathbb{Z}} \int_{\{x\in\Omega_{j+1} : \mathcal{I}_1|\nabla v_j|(x) > C2^{j-1}\}} 2^{(j-1)p^*(x)} dx + \sum_{j\in\mathbb{Z}} \int_{\{x\in\Omega_{j+1} : C' > C2^{j-1}\}} 2^{(j-1)p^*(x)} dx\end{aligned}$$

The first sum on the right hand side can be estimated as before. There is the largest $j$ satisfying $C' > C2^{j-2}$ and hence the second sum on the right hand side is bounded by $C|\Omega|$. The rest of proof is similar to the proof of claim (1). □

## 5. SOBOLEV EMBEDDING OF MIXED EXPONENTIAL TYPE

For simplicity we will use the notation $\tilde{p} = p^*/n'$ throughout this section. Recall that

$$M_p^*(t) = \sum_{i=0}^{\lfloor \tilde{p} \rfloor - 1} \frac{1}{i!} |t|^{n'(i+1)} + \frac{1}{\lfloor p^*/n \rfloor!} |t|^{p^*}$$

for $1 \leqslant p \leqslant n$, with the understanding that the last term disappears if $p = n$. In a bounded domain this expression could equivalently be replaced by the integal

$$\tilde{M}_p^*(t) = \int_1^{p^*} \frac{|t|^q \log^+ |t|}{\Gamma(q/n' + 1)}\, dq,$$

where $\Gamma$ is the gamma function. Note that the function $M_{p(\cdot)}^*$ does not satisfy the $\Delta_2$-condition (see [25] for the definition) if $p^+ = n$. Using the function $M_{p(\cdot)}^*$ we defined in the introduction the Orlicz–Musielak space $L^{p(\cdot),*}(\Omega)$ for a variable exponent satisfying $p^+ \leqslant n$.

This new variable exponent Lebesgue space of exponential type has the following obvious properties in domains with finite measure:

(1) if $p \in [1, n)$ is a constant, then $L^{p,*}(\Omega) = L^{p^*}(\Omega)$;
(2) if $p^+ < n$, then $L^{p(\cdot),*}(\Omega) = L^{p^*(\cdot)}(\Omega)$; and
(3) if $p = n$, then $L^{n,*}(\Omega) = \exp L^{n'}(\Omega)$.

Thus we always have $W^{1,p}(\Omega) \hookrightarrow L^{p,*}(\Omega)$ for a constant exponent $1 \leqslant p \leqslant n$ and by Proposition 4.2 $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot),*}(\Omega)$ for $1 \leqslant p^- \leqslant p^+ < n$. The second main result of this paper is to show that this last embedding holds also if we assume $1 < p^- \leqslant p^+ \leqslant n$.

**Proposition 5.1.** *Suppose that $p$ is* log*-Hölder continuous with $1 < p^- \leqslant p^+ \leqslant n$.*

(1) *We have $\|u\|_{L^{p(\cdot),*}(\Omega)} \lesssim \|\nabla u\|_{p(\cdot)}$ for every $u \in W_0^{1,p(\cdot)}(\Omega)$. The constant depends only on $n$, $p$ and $|\Omega|$.*

(2) *Let $D \subset \Omega$ satisfy the $(N, \varepsilon, \Omega)$-chain condition. Then*
$$\|u - c\|_{L^{p,*(\cdot)}(D)} \lesssim N^{n+1}\|\nabla u\|_{L^{p(\cdot)}(\Omega)} + N^n\varepsilon^{-n}\|u - c\|_{L^1(\Omega)}$$
*for every $c \in \mathbb{R}$ and every $u \in W^{1,p(\cdot)}(\Omega)$.*

*Proof.* We first prove (1). In this proof it is necessary to keep close track on the dependence of constants on various exponents. We will therefore make the dependence on $p^+$ explicit in our constants.

Let $u \in W^{1,p(\cdot)}(\Omega)$ be a function with $(1+|\Omega|)\|\nabla u\|_{p(\cdot)} \leqslant 1$. Then the claim follows if we can prove that $\varrho_{L^{p(\cdot),*}(\Omega)}(\lambda u) \leqslant 4 + |\Omega|$ for some constant $\lambda > 0$ independent of $u$. As before, $|u(x)| \leqslant C(n)\mathcal{I}_1|\nabla u(x)|$ for almost every $x \in \Omega$. Thus
$$\begin{aligned}\varrho_{L^{p(\cdot),*}(\Omega)}(\lambda u) &\leqslant \int_\Omega \sum_{i=1}^{\lfloor\tilde{p}(x)\rfloor} \frac{1}{i!}|\lambda u|^{in'}\,dx + \int_{\{p<n\}} \frac{1}{\lfloor\tilde{p}(x)\rfloor!}|\lambda u|^{p^*(x)}dx\\ &\lesssim \sum_{i=0}^{\infty} \frac{1}{i!}\int_{\{i\leqslant\tilde{p}\}} (\lambda\mathcal{I}_1|\nabla u|)^{in'}\,dx + \int_{\{p<n\}} \frac{1}{\lfloor\tilde{p}(x)\rfloor!}(\lambda\mathcal{I}_1|\nabla u|)^{p^*(x)}dx.\end{aligned}$$

Fix the variable exponent $q$ in such a way that $q^* = \min\{in', p^*\}$ in $\Omega$. Since $q \leqslant p$ we have $\|\nabla u\|_{q(\cdot)} \leqslant (1 + |\Omega|)\|\nabla u\|_{p(\cdot)} \leqslant 1$, and since $q^* = in'$ in $\{i \leqslant \tilde{p}\}$ we have
$$\int_{\{i\leqslant\tilde{p}\}} (\lambda\mathcal{I}_1|\nabla u|)^{in'}\,dx \leqslant \lambda^{in'}\int_\Omega (\mathcal{I}_1|\nabla u|)^{q^*(x)}\,dx.$$
We apply Proposition 2.1 with exponent $q$ and $k = \max\{i/(n-1), 1\} \leqslant i$:
$$(\mathcal{I}_1|\nabla u|(x))^{q^*(x)} \leqslant C^{in'}\,i^{q^*(x)/(q^+)'}\,[\mathcal{M}|\nabla u|(x)]^{q(x)}.$$
Since $q^* \leqslant in'$, we easily derive that $q^*(x)/(q^+)' \leqslant i - 1$. Hence we obtain
$$\begin{aligned}\int_{\{i\leqslant\tilde{p}\}} (\lambda\mathcal{I}_1|\nabla u|)^{in'}\,dx &\leqslant (C\lambda)^{in'}\,i^{i-1}\int_\Omega [\mathcal{M}|\nabla u|]^{q(x)}dx\\ &\lesssim C^i\lambda^{in'}\,i^i\left(\int_\Omega [\mathcal{M}|\nabla u|]^{p(x)}dx + |\Omega|\right).\end{aligned}$$
By [7], the Hardy-Littlewood maximal operator is bounded (we may extend $p$ outside $\Omega$ so that it satisfies the conditions of [7]) and hence $\varrho_{L^{p(\cdot)}(\Omega)}(\mathcal{M}|\nabla u|) \leqslant C$. It follows that
$$\frac{1}{i!}\int_{\{i\leqslant\tilde{p}\}} (\lambda\mathcal{I}_1|\nabla u|)^{in'}\,dx \lesssim \frac{1}{i!}C^i\lambda^{in'}\,i^i \lesssim i^{-i-1/2}e^iC^i\lambda^{in'}\,i^i \lesssim C_1^i\lambda^{in'}$$

where we used Stirling's formula in the second step. We choose $\lambda \leqslant (2C_1)^{-1/n'}$. Then we have an upper bound of $2^{-i}$ for the right-hand-side. Therefore, we have control of the sum in the previous estimate:

$$\sum_{i=0}^{\infty} \frac{1}{i!} \int_{\{i\leqslant \tilde p\}} (\lambda \mathcal{I}_1 |\nabla u|)^{in'}\, dx \lesssim \sum_{i=0}^{\infty} 2^{-i} = 2.$$

It remains to estimate the term

$$\begin{aligned} \int_{\{p<n\}} \frac{1}{\lfloor \tilde p(x) \rfloor !} (\lambda \mathcal{I}_1 |\nabla u|)^{p^*(x)} dx &= \sum_{i=1}^{\infty} \frac{1}{i!} \int_{\{i\leqslant \tilde p < i+1\}} (\lambda \mathcal{I}_1 |\nabla u|)^{p^*(x)} dx \\ &\leqslant \sum_{i=1}^{\infty} \frac{1}{i!} \int_{\Omega} \lambda^{in'} (\mathcal{I}_1 |\nabla u|)^{p_i^*(x)}\, dx, \end{aligned} \tag{5.2}$$

where $p_i(x) = \min\{p(x), \frac{ni+n}{n+i}\}$. Since $p_i \leqslant p$, we note that $\|\nabla u\|_{p_i(\cdot)} \leqslant (1 + |\Omega|)\|\nabla u\|_{p(\cdot)} \leqslant 1$. By Proposition 2.1 we have

$$(\mathcal{I}_1 |\nabla u|)^{p_i^*(x)}(x) \lesssim C^{p_i^*(x)} k^{p_i^*(x)/(p_i^+)'} \left[\mathcal{M}|\nabla u|(x)\right]^{p_i(x)},$$

where $k = \max\{p_i^+/(n-p_i^+), 1\}$. Since $p_i \leqslant \frac{ni+n}{n+i}$ we conclude that $k \leqslant i+1$ and $\frac{p_i^*(x)}{(p_i^+)'} \leqslant \frac{n(p_i^+-1)}{n-p_i^+} \leqslant i$. Therefore we have

$$\int_\Omega [\mathcal{I}_1 |\nabla u|(x)]^{p_i^*(x)} dx \leqslant (Ck)^i \int_\Omega [\mathcal{M}|\nabla u|(x)]^{p_i(x)}\, dx \lesssim (Ci)^i,$$

where we used the same arguments for $\mathcal{M}$ as in the previous paragraph. Using this in (5.2), with Stirling's formula as before, gives

$$\int_{\{p<n\}} \frac{1}{\lfloor \tilde p(x) \rfloor !} (\lambda \mathcal{I}_1 |\nabla u|)^{p^*(x)} dx \lesssim \sum_{i=1}^{\infty} \frac{1}{i!} \lambda^{in'} (Ci)^i \leqslant 2,$$

provided $\lambda$ is chosen small enough. This completes the proof of (1).

As in the proof of Proposition 4.2 (2) we obtain

$$|u(x) - c| \lesssim N^{n+1} \mathcal{I}_1 |\nabla u|(x) + \left(\frac{N}{\varepsilon}\right)^n \|u-c\|_{L^1(\Omega)}$$

and thus

$$\|u-c\|_{L^{p,*(\cdot)}(D)} \leqslant N^{n+1} \|\mathcal{I}_1 |\nabla u|\,\|_{L^{p,*(\cdot)}(D)} + \left(\frac{N}{\varepsilon}\right)^n \|u-c\|_{L^1(\Omega)} \|1\|_{L^{p,*(\cdot)}(D)}.$$

Estimating the first term on the right hand side as before yields the claim. □

## 6. The proof of the main result

We can combine the two results so far proved, allowing the exponent to attain both the value 1 and the value $n$.

Following [23] we say that a domain $\Omega \subset \mathbb{R}^n$ is an $(a, b)$-*John domain*, if there exists a point $x_0 \in \Omega$ such that every point $x \in \Omega$ can be connected to $x_0$ with a rectifiable path $\gamma\colon [0,d] \to \Omega$ parametrized by arc-length from $x = \gamma(0)$ to $x_0 = \gamma(d)$, with

$$d \leqslant b \qquad \text{and} \qquad \operatorname{dist}(\gamma(t), \partial\Omega) \geqslant \tfrac{a}{b} t \quad \text{for all } t \in [0,d].$$

The point $x_0$ is called a John center of $\Omega$. For example every bounded domain with Lipschitz boundary is a John domain; the converse is not true. In a John domain any point can be selected as the John center, possibly with different $a$ and $b$.

We want our final result to be in term of John domains rather than chain conditions, so we need the following lemma, whose proof follows ideas from [14].

**Lemma 6.1.** *Every $(a,b)$-John domain $\Omega$ satisfies the $(N,R,\Omega)$-chain condition for some $N$ and for $R < \operatorname{dist}(x_0, \partial\Omega)/2$.*

*Proof.* For $x \in B(x_0, \operatorname{dist}(x_0, \partial\Omega))$ it is trivial to construct a suitable chain of balls.

For a point $x \in \Omega \setminus B(x_0, \operatorname{dist}(x_0, \partial\Omega))$ define annuli $A^i = (B(x, 2^{i+1}) \setminus B(x, 2^i)) \cap \Omega$. Let $B^i$ be a family of balls of radii $\frac{a}{6b} 2^i$ with overlap $c(n)$ which covers every point in $A^i$ whose distance to the boundary is at least $\frac{a}{2b} 2^i$. Let $\gamma$ be the John path of $x$ and choose all the balls from $B^i$ intersecting $\gamma$ from the annuli with $i = \lfloor \log_2 r \rfloor, \ldots, \lfloor \log_2 R \rfloor + 1$. The $(N,R,\Omega)$-chain consists of the two-fold dilates of these balls. □

We are now ready for the proof of the main theorem.

*Proof of Theorem 1.1.* We choose a Lipschitz function $\phi$ with $0 \leqslant \phi \leqslant 1$, $\phi = 1$ in $p^{-1}([1, \frac{4}{3}])$ and $\operatorname{spt}\phi \subset p^{-1}([1, \frac{5}{3}])$. This can be done since $p^{-1}([1, \frac{4}{3}])$ and $p^{-1}([\frac{5}{3}, n])$ are closed disjoint sets. Let $\psi = 1 - \phi$. We write $\Phi = \{\phi > 0\}$ and $\Psi = \{\psi > 0\}$, and define $p_1 = \min\left\{p, \frac{5}{3}\right\}$ and $p_2 = \max\left\{\frac{4}{3}, p\right\}$. Then $p_1 = p$ in $\Phi$ and $p_2 = p$ in $\Psi$.

To prove (1), we calculate:

$$\|u\|_{L^{p(\cdot),*}(\Omega)} \leqslant \|\phi u\|_{L^{p_1^*(\cdot)}(\Omega)} + \|\psi u\|_{L^{p_2(\cdot),*}(\Omega)} \lesssim \|\nabla(\phi u)\|_{p_1(\cdot)} + \|\nabla(\psi u)\|_{p_2(\cdot)} \lesssim \|u\|_{W^{1,p(\cdot)}(\Omega)},$$

where the second step follows from claims (1) in Propositions 4.2 and 5.1. Finally, we see that $\|u\|_{W^{1,p(\cdot)}(\Omega)} \lesssim \|\nabla u\|_{p(\cdot)}$ by the Poincaré inequality (see e.g. the proof of [15, Theorem 2.6]).

By $\Phi_\varepsilon$ we denote the $\varepsilon$-neighborhood of $\Phi$ in $\Omega$, similarly for $\Psi$. Then $\Phi_\varepsilon$ is satisfies a $(N, \frac{1}{2}\varepsilon, \Phi_{2\varepsilon})$-chain condition, similarly for $\Psi_\varepsilon$. The justification of these claims is as in Lemma 6.1. We assume $\varepsilon$ to be so small that $p^+_{\Phi_{2\varepsilon}} < n$ and $p^-_{\Psi_{2\varepsilon}} > 1$. Choose balls $B_\Phi$ and $B_\Psi$ in $\Phi_\varepsilon$ and $\Psi_\varepsilon$ with diameter $\varepsilon$.

To prove (2), we note that

$$\begin{aligned}
\|u - u_\Omega\|_{L^{p(\cdot),*}(\Omega)} &\lesssim \|u - u_\Omega\|_{L^{p(\cdot),*}(\Phi)} + \|u - u_\Omega\|_{L^{p(\cdot),*}(\Psi)} \\
&\lesssim \|u - u_{B_\Phi}\|_{L^{p^*(\cdot)}(\Phi_\varepsilon)} + |u_{B_\Phi} - u_\Omega|\, \|1\|_{L^{p^*(\cdot)}(\Phi)} \\
&\qquad + \|u - u_{B_\Psi}\|_{L^{p(\cdot),*}(\Psi_\varepsilon)} + |u_{B_\Psi} - u_\Omega|\, \|1\|_{L^{p^*(\cdot)}(\Psi)}.
\end{aligned}$$

Then we use claim (2) in Proposition 4.2:

$$\begin{aligned}
&\|u - u_{B_\Phi}\|_{L^{p^*(\cdot)}(\Phi_\varepsilon)} + |u_{B_\Phi} - u_\Omega| \, \|1\|_{L^{p^*(\cdot)}(\Phi)} \\
&\qquad \lesssim N^n\Big(N\|\nabla u\|_{L^{p(\cdot)}(\Phi_{2\varepsilon})} + \varepsilon^{-n}\|u - u_{B_\Phi}\|_{L^1(\Phi_{2\varepsilon})}\Big) + \varepsilon^{-n}\int_{B_\Phi} |u(x) - u_\Omega| \, dx \\
&\qquad \lesssim N^{n+1}\|\nabla u\|_{p(\cdot)} + N^n\varepsilon^{-n}\|u - u_\Omega\|_{L^1(\Omega)}
\end{aligned}$$

A similar argument with claim (2) in Proposition 5.1 yields that

$$\|u - u_{B_\Psi}\|_{L^{p(\cdot),*}(\Psi_\varepsilon)} + |u_{B_\Psi} - u_\Omega| \, \|1\|_{L^{p^*(\cdot)}(\Psi)} \lesssim N^{n+1}\|\nabla u\|_{p(\cdot)} + N^n\varepsilon^{-n}\|u - u_\Omega\|_{L^1(\Omega)}.$$

Finally, we obtain $\|u - u_\Omega\|_{L^1(\Omega)} \lesssim \|\nabla u\|_{L^1(\Omega)} \leqslant (1 + |\Omega|)\|\nabla u\|_{p(\cdot)}$ by [22, Theorem 3.1]. □

*Remark* 6.2. If $1 \leqslant p^- \leqslant p^+ < n$ or $1 < p^- \leqslant p^+ \leqslant n$ then claim (2) in the previous theorem can be easily derived from Proposition 4.2 (2) or 5.1 (2) either using the Poincaré inequality in $L^1(\Omega)$ [22, Theorem 3.1] or the pointwise inequality $|u - u_\Omega| \lesssim \mathcal{I}_1|\nabla u|$ [3, Chapter 6].

## Acknowledgement

We would like to thank Y. Mizuta and T. Shimomura for pointing out a mistake in one of our proofs, and the referee for some useful suggestions.

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